International Atomic Energy Agency

Data Reduction and Error Propagation

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Data Reduction

Nuclear reaction quantity q (e.g., cross section) is always derived from primary observables x, y, z, … (data reduction) by a function f:

Example: Activation cross section (σ_1, σ_2,..) may be derived from …

- measured counting rate A_1, A_2,…
- detector efficiency ε_1, ε_2,…
- number of sample atoms N_1, N_2,…
- beam flux density ϕ_1, ϕ_2,…

$\rightarrow \sigma_i = f(A_i, \varepsilon_i, N_i, \phi_i) = (A_i/\varepsilon_i)\cdot (1/N_i) \cdot(1/\phi_i)$

Basic Data Reduction

*** Addition $z = x + y$ or subtraction $z = x - y$**

Example:

Background correction to raw count: $N' = N - B$

*** Multiplication $z = x \cdot y$ or division $z = x/y$**

Example:

Efficiency correction to raw count: $N' = N / \varepsilon$

Real data reduction may be a combination of these operations,

e.g., $N' = (N-B)/\varepsilon$

or more complicated, e.g., $N[1-\exp(-\lambda t)]/\varepsilon$

Data Reduction and Error Propagation

Cross sections at n energies σ_i (i=1,n) derived from primary observables A_i, ε_i, N_i, ϕ_i (i=1,n)

Step 1: Measurements of primary observables

Determination of means and covariances for each primary observable $\langle A_i \rangle$, ΔA_i, $\langle \varepsilon_i \rangle$, $\Delta \varepsilon_i$ …

Step 2: Data reduction to cross sections by function f

$\langle A_i \rangle$, ΔA_i, $\langle \varepsilon_i \rangle$, $\Delta \varepsilon_i$ … → $\langle \sigma_i \rangle$ and $\Delta \sigma_i$

Propagation of mean value

$\langle \sigma_i \rangle = (\langle A_i \rangle \langle \varepsilon_i \rangle) \cdot (1/\langle N_i \rangle) \cdot (1/\langle \phi_i \rangle)$

Propagation of standard deviation

$\Delta \sigma_i = (\Delta A_i \Delta \varepsilon_i) \cdot (1/\Delta N_i) \cdot (1/\Delta \phi_i)$? No!

Error Propagation (Linear Combination)

$p = \Sigma_{i=1,n}\ a_i \cdot x_i\ = a_1x_1 + a_2x_2 + \ldots$ (x_i is a random variable).

Mean:

$$\langle p \rangle = \Sigma_{i=1,n}\ a_i \langle x_i \rangle$$

Variance:

$$V_{pp} = \langle p^2 \rangle - \langle p \rangle^2 = \langle (\Sigma_{i=1,n}\ a_i x_i)^2 \rangle - \langle \Sigma_{i=1,n}\ a_i x_i \rangle^2$$

$$= \Sigma_{i,j=1,n}\ a_i a_j \langle x_i x_j \rangle - (\Sigma_{i=1,n,\ j=1,n} a_i \langle x_i \rangle \cdot a_j \langle x_j \rangle)$$

Split i=j term and i≠j term.

$$= \Sigma_{i=1,n}(a_i^2 \langle x_i^2 \rangle - a_i^2 \langle x_i \rangle^2) + \Sigma_{i=1,n;\ j=1,n; i\neq j}(a_i a_j \langle x_i x_j \rangle - a_i a_j \langle x_i \rangle \langle x_j \rangle)$$

$$= \Sigma_{i=1,n} a_i^2 V_{ii} + \Sigma_{i=1,n;\ j=1,n; i\neq j}\ a_i a_j V_{ij}$$

$$= \Sigma_{i=1,n} a_i^2 \underbrace{V_{ii}}_{\text{variance}} + 2\Sigma_{i=1,n;\ j=1,n; i<j}\ a_i a_j \underbrace{V_{ij}}_{\text{covariance}}$$

Error Propagation (Linear Combination, Summary)

$p = \Sigma_{i=1,n}\, a_i \cdot x_i$ (a_i is an exactly known constant).

Variance:

$$V_{pp} = \Sigma_{i=1,n} a_i^2 V_{ii} + 2\Sigma_{i=1,n;\, j=1,n; i<j}\, a_i a_j V_{ij}$$

Covariance:

Derivation of this equation can be simply extended to covariance of two functions $p = \Sigma_{i=1,n}\, a_i \cdot x_i$ and $q = \Sigma_{j=1,m}\, b_j \cdot y_j$:

$$V_{pq} = \langle pq \rangle - \langle p \rangle \langle q \rangle$$
$$= \langle (\Sigma_{i=1,n}\, a_i x_i)\, (\Sigma_{j=1,n}\, b_j y_j) \rangle - \langle \Sigma_{i=1,n}\, a_i x_i \rangle \langle \Sigma_{j=1,n}\, b_j y_j \rangle$$
$$= \Sigma_{i=1,n;\, j=1,n}\, a_i b_j V_{ij}$$

Error Propagation: z=x+y

z=x+y (e.g., background subtraction N'=N-B)

$<z>=<x>+<y>$

$V_{zz}=V_{xx}+V_{yy}+2V_{xy}$

$(\Delta z)^2=(\Delta x)^2+(\Delta y)^2+2c_{xy}\Delta x\Delta y$ ← Correlation coefficient $c_{xy}=V_{xy}/(V_{xx}V_{yy})^{1/2}$

If x and y are independent (i.e., $c_{xy}=0$),

$\Delta z = (V_{zz})^{1/2}=(V_{xx}+V_{yy})^{1/2}=(\Delta x^2+\Delta y^2)^{1/2}$ (**quadrature sum rule**)

$p = \Sigma_{i=1,n}\, a_i \cdot x_i \rightarrow <p>= \Sigma_{i=1,n}\, a_i<x_i>,\ V_{pp}=<p^2>-<p>^2=\Sigma_{i=1,n}a_i^2V_{ii} +2\Sigma_{i=1,n;\ j=1,n;i<j}\, a_ia_jV_{ij}$

Error Propagation for z=x+y (Summary)

$z=x+y$

$\langle z\rangle=\langle x\rangle+\langle y\rangle$

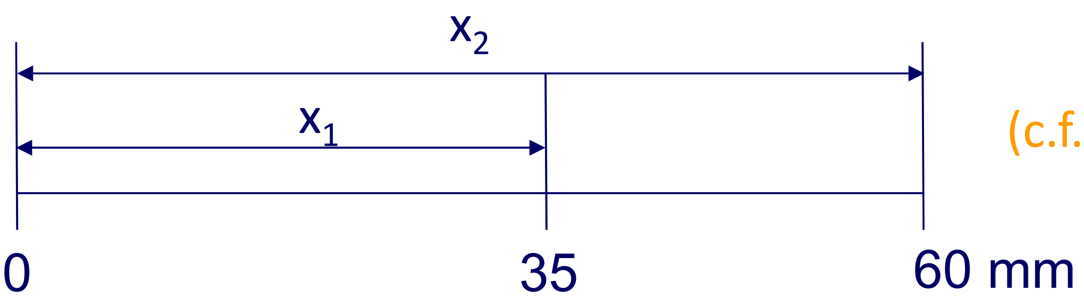
$V_{zz}=V_{xx}+V_{yy}+2V_{xy}$

$(\Delta z)^2=(\Delta x)^2+(\Delta y)^2+2c_{xy}\Delta x\Delta y$

$$
\begin{cases}
=(\Delta x)^2+(\Delta y)^2 & \text{(if x and y are \underline{independent}, } c_{xy}=0) \\
=(\Delta x)^2+(\Delta y)^2+2\Delta x\Delta y = (\Delta x+\Delta y)^2 & \text{(if x and y are \underline{fully correlated}, } c_{xy}=+1) \\
=(\Delta x)^2+(\Delta y)^2-2\Delta x\Delta y = (\Delta x-\Delta y)^2 & \text{(if x and y are \underline{fully anti-correlated}, } c_{xy}=-1)
\end{cases}
$$

Uncertainty becomes maximum when x and y are fully correlated!!

Measurements of Two Lengths by Three Gauge Blocks



(c.f. WM11 D.1).

Measurements of x_1 and x_2 by 3 gauges L_1, L_2 and L_3.

L_1, L_2 and L_3 are independent.

x_1 was measured by L_1 and L_2 ($x_1=L_1-L_2$)
x_2 was measured by L_1 and L_3 ($x_2=L_1+L_3$)

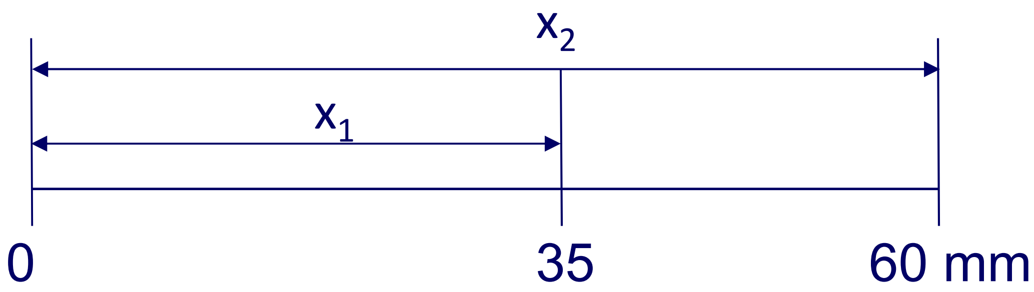
	Length [mm]	SD [mm]	Variance v [mm²]
L_1	50	0.05	0.0025
L_2	15	0.03	0.0009
L_3	10	0.02	0.0004

SD: Standard deviation

How to obtain the covariance $V_{x1,x2}$?

$V_{x1x2}=\langle x1-\langle x1\rangle\rangle\langle x2-\langle x2\rangle\rangle$
$=\langle x1x2\rangle-\langle x1\rangle\langle x2\rangle=\ldots$

Correlation Due to Use of Same Gauge Blocks



$x_1=L_1-L_2,\ x_2=L_1+L_3$

	Length [mm]	SD [mm]	Variance V [mm²]
L_1	50	0.05	0.0025
L_2	15	0.03	0.0009
L_3	10	0.02	0.0004

SD: Standard deviation
L1, L2 and L3 are independent.
(i.e., $V_{L1L2}=V_{L3L1}=V_{L3L2}=0$)

$$\begin{aligned}
V_{x1x2} &= \langle(x_1-\langle x_1\rangle)(x_2-\langle x_2\rangle)\rangle \\
&= \langle x_2\cdot x_1\rangle - \langle x_2\rangle\langle x_1\rangle \\
&= \langle(L_1+L_3)\cdot(L_1-L_2)\rangle - \langle L_1+L_3\rangle\cdot\langle L_1-L_2\rangle \\
&= \langle L_1\cdot L_1\rangle-\langle L_1\cdot L_2\rangle + \langle L_3\cdot L_1\rangle-\langle L_3\cdot L_2\rangle \\
&\quad - \langle L_1\rangle^2+\langle L_1\rangle\langle L_2\rangle - \langle L_3\rangle\langle L_1\rangle + \langle L_3\rangle\langle L_2\rangle \\
&= V_{L1L1} - V_{L1L2} - V_{L3L1} - V_{L3L2} \\
&= V_{L1L1} = 0.0025\ [\mathrm{mm}^2].
\end{aligned}$$

$x_1=\mathbf{L_1}-L_2,$
$x_2=\mathbf{L_1}+L_3$

Use of the same gauge L_1
Introduced correlation!

Error Propagation (General Function)

$p=p(x_1, x_2, \ldots x_n)$: function of n random variables x_n

1^{st} order expansion of f around $\langle p\rangle=p(\langle x_1, \langle x_2\rangle, \ldots \langle x_n\rangle)$:

$p - \langle p\rangle \sim \Sigma_{i=1,n}\, g_i \cdot (x_i - \langle x_i\rangle)$ with $g_i=(\partial p / \partial x_i)_{xi=\langle xi\rangle}$

If we set $p'=p-\langle p\rangle$ and $x_i'=x_i-\langle x_i\rangle$, we obtain the linear combination

$p' = \Sigma_{i=1,n}\, g_i \cdot x_i'$

This is a linear combination, therefore

$V_{pp}=V_{p'p'} \sim \Sigma_{i=1,n} g_i^2 V_{ii} + 2\Sigma_{i=1,n;\, j=1,n; i<j}\, g_i g_j V_{ij}$

(under the condition $|x_i-\langle x_i\rangle| \sim (v_{ii})^{1/2} \ll \langle x_i\rangle$)

Note: $V_{p'p'}=\langle p'^2\rangle-\langle p'\rangle^2=\langle (p-\langle p\rangle)^2\rangle-\langle (p-\langle p\rangle)\rangle^2= \langle (p-\langle p\rangle)^2\rangle = V_{pp.}$

$p = \Sigma_{i=1,n}\, a_i \cdot x_i \rightarrow \langle p\rangle= \Sigma_{i=1,n}\, a_i \langle x_i\rangle$, $V_{pp}=\langle p^2\rangle-\langle p\rangle^2=\Sigma_{i=1,n} a_i^2 V_{ii} + 2\Sigma_{i=1,n;\, j=1,n; i<j}\, a_i a_j V_{ij}$

Error Propagation (General Function)

$p=p(x_1, x_2, \ldots x_n)$: function of n random variables

$$V_{pp} \sim \Sigma_{i=1,n} g_i^2 V_{ii} + 2\Sigma_{i=1,n;\ j=1,n;i<j}\ g_i g_j V_{ij}\ (\text{if } s_i << \langle x_i \rangle),\ g_i = (\partial p/\partial x_i)_{xi=<xi>}$$

This relation can be easily extended to more general case for two functions $p=p(x_1, x_2, \ldots x_n)$, $q=q(y_1, y_2, \ldots y_m)$:

$$V_{pq} \sim \Sigma_{i=1,n;\ j=1,m}\ g_i h_j V_{ij}\quad g_i = (\partial p/\partial x_i)_{xi=<xi>},\ h_i = (\partial q/\partial y_j)_{yj=<yj>}$$

Product Function

For two products of n primary observables,

Hint: $=(\partial p/\partial x_i)=p/x_i$

$$\begin{cases} p=x_{1p}x_{2p},\ldots x_{np} \\ q=x_{1q}x_{2q},\ldots x_{nq} \end{cases}$$

each parameter (e.g., x_{1p}) has the standard deviation (e.g., Δx_{1p}) and correlation between x_{ip} and x_{jq} exists only when p=q ($V_{ip,jq}$).

Prove that

$$v_{pq} \sim \Sigma_{k=1,n} c_{k,pq}\, f_{kp} f_{kq} \;,$$

where $v_{pq}=V_{pq}/(<p><q>)$ (fractional covariance)

$c_{k,pq}=V_{k,pq}/(\Delta x_{kp}\Delta x_{kq})^{1/2}$ (correlation coefficient)

$f_{k,p}=\Delta x_{k,p}/x_{kp}, f_{kq}=\Delta x_{kq}/x_{kq}$ (fractional standard deviation)

$p = p(x_1, x_{2,\ldots,} x_n) \rightarrow V_{pq} \sim \Sigma_{i=1,n;\, j=1,m}\, g_i h_j V_{ij}$ $g_i=(\partial p/\partial x_i)_{xi=<xi>}$, $h_i=(\partial q/\partial y_j)_{yj=<yj>}$

Product Function

$p=x_{1p}x_{2p},\ldots x_{np},\ q=x_{1q}x_{2q},\ldots x_{nq}$

$V_{pq} = \Sigma_{i=1,n;\ j=1,n} g_i h_j V_{ip,\ jq}$

$= \Sigma_{i=1,n;\ j=1,n} [\partial(x_{1p}x_{2p},\ldots x_{np})/\partial x_{ip}]_{x1p=<x1p>}$

$[\partial(x_{1q}x_{2q},\ldots x_{nq})/\partial y_{jq}]_{yjq=<yjq>} V_{ip,jq}$

$= \Sigma_{i=1,n;\ j=1,n} [p/x_{ip}]_{xip=<xip>}\ [q/y_{jq}]_{yjq=<yj,>} V_{ip,jq}$

$=<p><q>\ \Sigma_{i=1,n;\ j=1,n} (1/<x_{ip}>)(1/<y_{jq}>)\ V_{ip,jq}$

$=<p><q>\ \Sigma_{i=1,n}\ (1/<x_{ip}>)(1/<y_{iq}>)\ V_{ip,iq}$

$v_{pq}=V_{pq}/(<p><q>) = \Sigma_{k=1,n}\ V_{k,pq}/(<x_{kp}><y_{kq}>)$

$= \Sigma_{k=1,n}\ c_{ip,jq}\Delta x_{ip}\Delta_{jq}/(<x_{ip}><y_{jq}>)$

$= \boldsymbol{\Sigma_{i=1,n}\ c_{k,pq} f_{ip} f_{jq}}$

Hint: $=(\partial p/\partial x_i)=p/x_i$

Correlation exists only when i=j

Fractional standard deviation $f=x/\Delta x$

$p = p(x_1,x_2,\ldots,x_n) \rightarrow V_{pq} \sim \Sigma_{i=1,n;\ j=1,m}\ g_i h_j V_{ij}\quad g_i=(\partial p/\partial x_i)_{xi=<xi>},\ h_i=(\partial q/\partial y_j)_{yj=<yj>}$

Error Propagation for Product/Quotient Function

For two quantities

$$\begin{cases} p=(x_{1,p}x_{2,p},\ldots x_{m,p})/(x_{m+1,p}x_{m+2,p},\ldots x_{n,p}) \\ q=(x_{1,q}x_{2,q},\ldots x_{m,q})/(x_{m+1,q}x_{m+2,q},\ldots x_{n,q}) \end{cases}$$

fractional covariance between p and q is

$$v_{pq} \sim \Sigma_{k=1,n} c_{k,pq}\, f_{k,p} f_{k,q}$$

if correlation between $x_{k,p}$ and $x_{l,q}$ exists only when k=l ($V_{k,pq}$).

$$\begin{cases} v_{pq}=V_{pq}/(\langle p\rangle\langle q\rangle) & \text{(fractional covariance)} \\ c_{k,pq}=V_{k,pq}/(\Delta x_{k,p}\Delta x_{k,q})^{1/2} & \text{(correlation coefficient)} \\ f_{k,p}=\Delta x_{k,p}/x_{k,p},\ f_{k,q}=\Delta x_{k,q}/x_{k,q} & \text{(fractional standard deviation)} \end{cases}$$

This is probably the most important formula for people who want to construct covariance from EXFOR entries.

Error Propagation of Production Function in EXFOR

$$\mathbf{v_{pq} \sim \Sigma_{k=1,n} c_{k,pq}\, f_{k,p} f_{k,q}}$$

$v_{pq}=V_{pq}/(<p><q>)$ (fractional covariance)

$c_{k,pq}=V_{k,pq}/(\Delta x_{k,p}\Delta x_{k,q})^{1/2}$ (correlation coefficient)

$f_{k,p}=\Delta x_{k,p}/x_{k,p}$, $f_{k,q}=\Delta x_{k,q}/x_{k,q}$ (fractional standard deviation)

$$v_{pq} \sim \Sigma_{k=1,n} c_{k,pq}\, (\mathrm{ERR}\text{-}k)_p (\mathrm{ERR}\text{-}k)_q$$

$c_{k,pq}$ is usually assigned by evaluators. (0 for ERR-S etc.)

EN	…	ERR-1	ERR-2	ERR-3
MEV	…	PER-CENT	PER-CENT	PER-CENT
…				
p		$(\mathrm{ERR}\text{-}1)_p$	$(\mathrm{ERR}\text{-}2)_p$	$(\mathrm{ERR}\text{-}3)_p$
…				
q		$(\mathrm{ERR}\text{-}1)_q$	$(\mathrm{ERR}\text{-}2)_q$	$(\mathrm{ERR}\text{-}3)_q$

Quadrature Sum Rule for Product/Quotient Function

For two quantities

$$\begin{cases} p=(x_{1,p}x_{2,p},\ldots x_{m,p})/(x_{m+1,p}x_{m+2,p},\ldots x_{n,p}) \\ q=(x_{1,q}x_{2,q},\ldots x_{m,q})/(x_{m+1,q}x_{m+2,q},\ldots x_{n,q}) \end{cases}$$

fractional covariance between p and q is

$$v_{pq} \sim \Sigma_{k=1,n} c_{k,pq}\, f_{k,p} f_{k,q}$$

When p=q, we obtain the **quadrature sum rule** for fractional uncertainty:

$$(\Delta p/\langle p\rangle)^2 = v_{pp} = \Sigma_{k=1,n} c_{k,pp}\, (f_{k,p})^2 = \Sigma_{k=1,n} (\Delta x_{k,p}/\langle x_{k,p}\rangle)^2$$

(Note $c_{k,pp}=c_{ip,ip}=1$)

What is "Total Uncertainty" (ERR-T)?

For a quantity $p=(x_{1,p}x_{2,p},\ldots x_{m,p})/(x_{m+1,p}x_{m+2,p},\ldots x_{n,p})$
, fractional total uncertainty is $(\Delta p/\langle p\rangle)^2 = \Sigma_{k=1,n}(\Delta x_{k,p}/\langle x_{k,p}\rangle)^2$.

Source k propagated to the total uncertainty may depend on the authors.

Minor sources could be negligible (e.g., 1.0%+5.0%→5.1%).

Partial uncertainties and their correlation properties for major sources are essential for evaluation.

Product/Quotient Function: Activation Cross Section

Activation cross sections at two energies (σ_p and σ_q) derived from

- counts A (corrected for decay)
- by the same detector (ε) and sample (thickness N)
- under flux ϕ_p and ϕ_q:

  $\sigma_p = A_p/(\varepsilon N \phi_p)$ and $\sigma_q = (A_q/\varepsilon N \phi_q)$

Correlation Correlations in EXFOR entry

$\sigma_p = A_p/(\varepsilon N \phi_p)$ and $\sigma_q = (A_q/\varepsilon N \phi_q)$

$v_{pq} = V_{pq}/(\sigma_p \sigma_q)$
$= c_{A,pq}(\Delta A_p/A_p)(\Delta A_q/A_q)$
$+ c_{\varepsilon,pq}(\Delta\varepsilon_p/\varepsilon_p)(\Delta\varepsilon_q/\varepsilon_q)$
$+ c_{N,pq}(\Delta N_p/N_p)(\Delta N_q/N_q)$
$+ c_{\phi,pq}(\Delta\phi_p/\phi_p)(\Delta\phi_q/\phi_q)$

```
ERR-ANALYS (ERR-S) Counting statistics
           (ERR-1,,F) Detector efficiency
           (ERR-2,,F) Sample thickness
           (ERR-3,,U) Neutron flux
ENDBIB
COMMON
ERR-1      ERR-2      ERR-3
PER-CENT   PER-CENT   PER-CENT
 Δε/ε        ΔN/N      Δφ/φ
ENDCOMMON             3
DATA
EN         DATA       ERR-S
MEV        MB         PER-CENT
 E_p        σ_p         ΔA_p/A_p
 E_q        σ_q         ΔA_q/A_q
ENDDATA
```

New format:

F: Fully correlated ($c_{k,pq}=1$),

U: Uncorrelated ($c_{k,pq}=1$ for p=q, 0 for p≠q).

Product/Quotient Function: Activation Cross Section

For number of products A with the same detector (ε) and sample (thickness N) under flux ϕ, $\sigma_p=A_p/(\varepsilon N\phi_p)$ and $\sigma_q=(A_q/\varepsilon N\phi_q)$

$$v_{pq}=V_{pq}/(\sigma_p\sigma_q)=c_{A,pq}(\Delta A_p/A_p)(\Delta A_q/A_q)+c_{\varepsilon,pq}(\Delta\varepsilon_p/\varepsilon_p)(\Delta\varepsilon_q/\varepsilon_q)$$
$$+c_{N,pq}(\Delta N_p/N_p)(\Delta N_q/N_q)+c_{\phi,pq}(\Delta\phi_p/\phi_p)(\Delta\phi_q/\phi_q)$$

A (Uncorrelated): $c_{A,pq}=1$ if $p=q$, $=0$ if $p\neq q$.

ε, N (Fully correlated): $c_{\varepsilon,pq}=c_{N,pq}=1$ ($\leftarrow \varepsilon_p=\varepsilon_q=\varepsilon$, $N_p=N_q=N$)

ϕ (Uncorrelated?): $c_{\phi,pq}=1$ if $p=q$, $=0$ if $p\neq q$.

Then

$$v_{pq}=V_{pq}/(\sigma_p\sigma_q)=(\Delta\varepsilon_p/\varepsilon_p)^2+(\Delta N_p/N_p)^2 \quad (p\neq q)$$
$$v_{pp}=(\Delta\sigma_p/\sigma_p)^2=(\Delta A_p/A_p)^2+(\Delta\varepsilon_p/\varepsilon_p)^2+(\Delta N_p/N_p)^2+(\Delta\phi_p/\phi_p)^2$$

Typical Assumption on Correlation Coefficients

```
ERR-ANALYS (ERR-S) Counting statistics
           (ERR-1,,F) Detector efficiency
           (ERR-2,,F) Sample thickness
           (ERR-3,,U) Neutron flux
ENDBIB
COMMON
ERR-1      ERR-2      ERR-3
PER-CENT   PER-CENT   PER-CENT
 Δε/ε        ΔN/N       Δφ/φ
ENDCOMMON               3
DATA
EN         DATA       ERR-S
MEV        MB         PER-CENT
 E_p        σ_p        ΔA_p/A_p
 E_q        σ_q        ΔA_q/A_q
ENDDATA
```

Correlation information $c_{k,pq}$ is rarely reported by experimentalists.

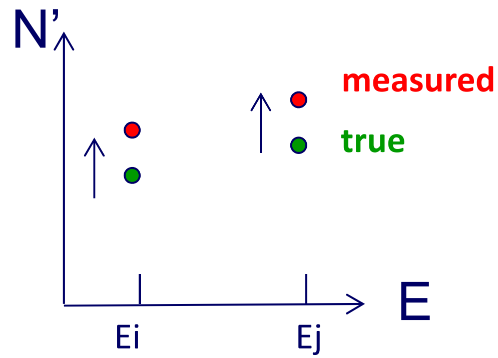
Typical assumption:

F for constant uncertainties, U for energy dependent uncertainties.

Source of Correlation in Primary Observables

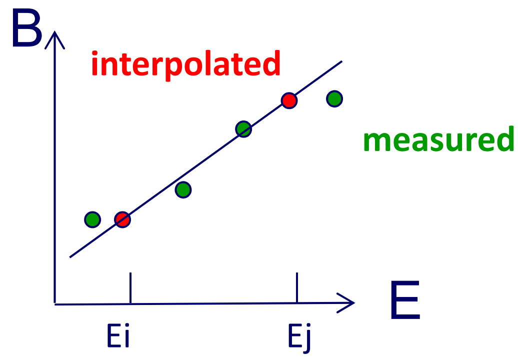
Example 1:

Corrected counts $N'=N/\varepsilon$ at two incident energies E_i and E_j by using the same detector and by assuming that $\varepsilon_i=\varepsilon_j$.



Example 2:

Background B at two incident energies E_i and E_j by based on the interpolation from the same fitting.



Correlation and uncertainty may depend on data reduction procedure.
(c.f. Mean values from two measurements should be unique.)

Correlation: Two Reactions and Two Energies

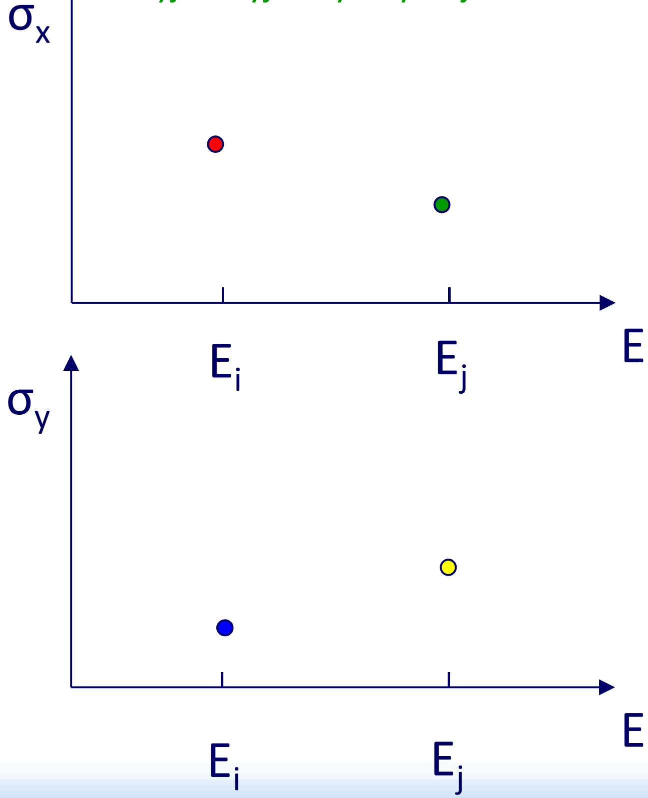
$\sigma = A/(\varepsilon N \phi)$

$\sigma_{xi} = A_{xi}/(\varepsilon_x \cdot N_x \cdot \phi_i)$

$\sigma_{xj} = A_{xj}/(\varepsilon_x \cdot N_x \cdot \phi_j)$

$\sigma_{yi} = A_{yi}/(\varepsilon_y \cdot N_y \cdot \phi_i)$

$\sigma_{yj} = A_{yj}/(\varepsilon_y \cdot N_y \cdot \phi_j)$



A: measured counting rate (independent)

ε: detector efficiency (2 values for x and y)

N: number of sample atoms (2 values for x and y)

ϕ: beam flux density (2 values for i and j)

Source of correlation
between two cross sections
due to assumption of equality

	σ_{xi}	σ_{xj}	σ_{yi}	σ_{yj}
σ_{xi}	All	ε_x, N_x	ϕ_i	(ind.)
σ_{xj}		All	(ind.)	ϕ_j
σ_{yi}			All	ε_y, N_y
σ_{yj}				All

ind.: independent

“Uncertainty due to” or “Uncertainty in”?

$\sigma_p = A_p/(\varepsilon N \phi_p)$

$(\Delta\sigma_p/\sigma_p)^2 = (\Delta A_p/A_p)^2 + (\Delta\varepsilon_p/\varepsilon_p)^2 + (\Delta N_p/N_p)^2 + (\Delta\phi_p/\phi_p)^2$

Thanks to this quadarture sum rule, the following description under the keyword ERR-ANALYS are equivalent:

“Uncertainty <u>due to</u> counting statistics is 5%”

“Uncertainty <u>in</u> counting statistics is 5%”.

Non-Linear Term (Decay Term)

Quadrature sum rule

$\sigma_p = A_p/(\varepsilon N \phi_p)$

$(\Delta\sigma_p/\sigma_p)^2 = (\Delta A_p/A_p)^2 + (\Delta\varepsilon_p/\varepsilon_p)^2 + (\Delta N_p/N_p)^2 + (\Delta\phi_p/\phi_p)^2$

In real activation measurement, number of products is derived from the number of activity count C_p which may depend on decay time during irradiation (t_i), cooling (t_c) and measureing (t_m):

$A_p \rightarrow C_p \lambda t_i / \{ [1-\exp(-\lambda t_m)] \cdot \exp(-\lambda t_c) \cdot [1-\exp(-\lambda t_i)] \}$.

If the uncertainties in decay constant, irradiation/ cooling/ measuring time are not negligible, the quadrature sum rule is not valid.

Non-Linear Term (Decay Term)

For measurement without cooling (i.e., $\exp(-\lambda t_c) \rightarrow 1$) after very long irradiation (i.e., $[1-\exp(-\lambda t_i)] \rightarrow 1$)

$$\sigma_p \sim C_p \lambda t_i \{[(\varepsilon N \phi_p)\ [1-\exp(-\lambda t_m)]\} = C\lambda/[1-\exp(-\lambda t_m)] \quad (C^{-1} = \varepsilon N \phi_p / C_p t_i)$$

If the uncertainties of parameters are negligible except for λ,

$$V_{pp} = (\Delta\sigma_p)^2 \sim (\partial\sigma_p/\partial\lambda_p)^2 (\Delta\lambda)^2$$

$$= \{C[1-\exp(-\lambda t_m)+\lambda t_m \exp(-\lambda t_m)\ ]^2/[1-\exp(-\lambda t_m)]^2\}^2 (\Delta\lambda)^2$$

$$= \sigma_p^2\ \{[1-\exp(-\lambda t_m)+\lambda t_m \exp(-\lambda t_m)]/[1-\exp(-\lambda t_m)]\}^2 (\Delta\lambda/\lambda)^2$$

$$(\Delta\sigma_p/\sigma_p)^2 \sim \{[1-\exp(-\lambda t_m)+\lambda t_m \exp(-\lambda t_m)]/[1-\exp(-\lambda t_m)]\}^2\ (\Delta\lambda/\lambda)^2$$

Non-Linear Term (σ_0 with Am-Be Source)

Resonance integral of ^{170}Er(n,γ)^{171}Er relative to ^{55}Mn(n,γ)^{56}Mn with Am-Be source + paraffin moderator

$$I_0(\alpha)_{\mathrm{Er}} = \left[\frac{(R_{\mathrm{Cd}}-1)_{\mathrm{Mn}}}{(R_{\mathrm{Cd}}-1)_{\mathrm{Er}}}\right] \cdot \left[\frac{\sigma_{0,\mathrm{Er}}}{\sigma_{0,\mathrm{Mn}}}\right] \cdot \left[\frac{G_{epi,\mathrm{Mn}}}{G_{epi,\mathrm{Er}}}\right] \cdot \left[\frac{G_{th,\mathrm{Er}}}{G_{th,\mathrm{Mn}}}\right] \times I_0(\alpha)_{\mathrm{Mn}}. \quad (5)$$

$$I_0(\alpha) = (1\ \mathrm{eV})^\alpha \left[\frac{I_0 - 0.429 g\sigma_0}{(\bar{E}_r)^\alpha} + \frac{0.429\sigma_0}{(2\alpha+1)(E_{\mathrm{Cd}})^\alpha}\right].$$

TABLE IV. Typical experimental uncertainties for the resonance integral cross section measurements.

Uncertainties due to (x_j)	Relative uncertainty, s_j (%)	Error propagation factor, $Z(x_j)$	Relative uncertainty on the resonance integral value, $s_j \times Z(x_j)$ (%)
α-shape parameter	19	0.06	1.2
Cadmium cut-off energy	15	0.03	0.45
Cadmium ratio of ^{56}Mn	1.6	1.10	1.8
Cadmium ratio of ^{171}Er	2.3	1.25	2.9
Thermal neutron self-shielding factor for Mn sample	0.1	1.00	0.10
Thermal neutron self-shielding factor for Er sample	0.5	1.00	0.50
Epithermal neutron self-shielding factor for Mn sample	0.2	1.00	0.20
Epithermal neutron self-shielding factor for Er sample	2.3	1.00	2.30
Reference resonance integral cross section of ^{55}Mn	2.2	1.51	3.3
Reference thermal neutron cross section of ^{55}Mn	0.75	1.51	1.2
Reference thermal neutron cross section of ^{170}Er	6.9	1.00	6.9
Effective resonance energy of ^{55}Mn	11	0.04	0.44
Effective resonance energy of ^{170}Er	2.3	0.07	0.16
Total uncertainty, S_T (%)			8.8

≠1.00!

H.Yücel et al., Phys.Rev.C76(2007)034610 (in compilation)

Summary

- **Error propagation depends on combination of random variables**
  - linear combination
  - non-linear combination (Taylor expansion around mean value)
  - Product/quotient combination (fractional covariance and standard deviation)
- **Quadrature sum rule**

  The formula $(\Delta y/\langle y\rangle)^2=\Sigma_{i=1,n}(\Delta x_i/\langle x_i\rangle)^2$ is applicable when
  - y is a product/quotient combination $(x_1x_2....x_m)/(x_{m+1}x_{m+2}...x_n)$
  - Δx_i is enough smaller than $\langle x_i\rangle$ (for 1st order approximation)
- **Correlation in EXFOR**
  - Proper error propagation requires correlation information on primary observables.

Exercise (~18:00)

Choose the following tasks as you like:

- Paper 1.2 (Error propagation of linear combination)
- Paper 1.3 (Error propagation of product combination)
- Paper 1.4 (Error propagation of product combination)
- Calculation of matrix elements of “macro-correlation coefficient” (i.e., correlation coefficients of cross sections between two incident neutron energies) for a revised EXFOR 23114.002.

  See “sage1.pdf” and “sage2.pdf” available at

  ftp://napc-ftp-ext:qu6uvASWeW4Jasw@ftp.iaea.org/ws2013/.
- Return to hotel. Go for dinner. Etc.