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Centre d'Etudes de Bruyères-le-Châtel

**SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS USING
POWER SERIES WITH DECOMPOSED COEFFICIENTS.**

PART I : MATHEMATICAL ANALYSIS

by

Alois SCHETT, Michel COLLIN, Roger PERRIER

- Octobre 1985 -

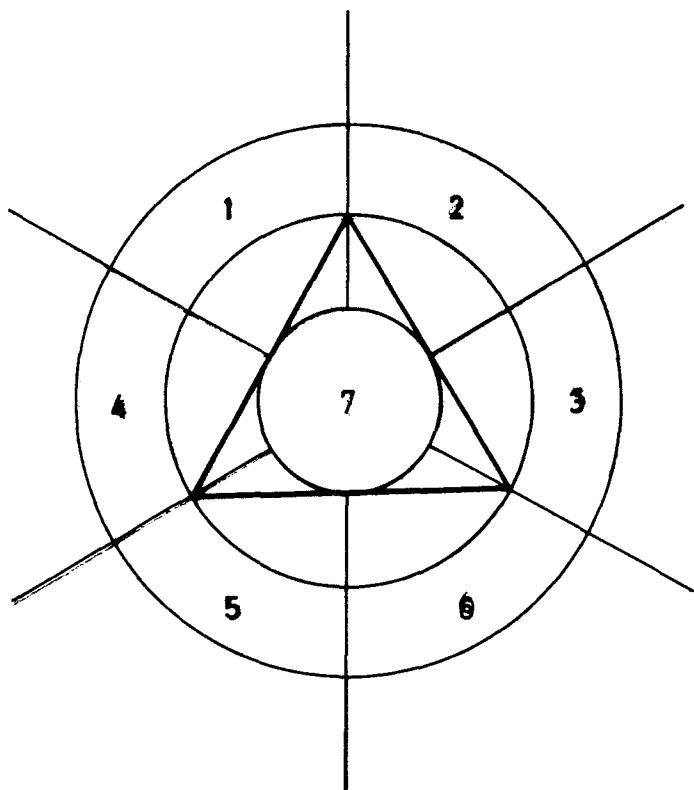
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LA POESIE, COMME L'ART, EST
INSEPARABLE DE LA MERVEILLE.

André Pieyre de Mandiargues.
(L'Âge de craie)

Explanation of the symbolic drawing-the "MANDALA":

THREE basic elements(TRIANGLE) are involved in the generation of the three Jacobian elliptic functions: $sn(u,k)$, $cn(u,k)$ and $dn(u,k)$ as well as the four combinatorial sets:

- the number of permutations with given runs-up,
- the number of permutations with given peaks,
- the number of permutations with given cycle peaks, and
- the total number of permutations.

SYMMETRIES (reflexion, cyclic permutation(CIRCLES, CENTRE)) reveal a MONOGENERATOR of a SEVEN-fold mathematical entity (details in Example 3). This inspires the search for monogenerators of n-fold mathematical entities.

RESUME

Les solutions de systèmes d'équations différentielles ordinaires sont représentées sous forme de séries de puissances. Les coefficients C_n de ces séries sont DECOMPOSÉS en trois parties : $C_n = (A \cdot B \cdot V)_n$: A reflète la forme du système étudié, B exprime le couplage des équations et V contient les valeurs initiales des fonctions. Le terme A et, dans la plupart des cas, le terme B restent constants pendant l'intégration. Des formules de récurrence du terme clé A sont données pour divers systèmes. Elles ont la forme générale $A_{n+1} = f(A_n)$. Connaissant A_n il est possible de calculer directement les termes B_n et V_n . Cette DECOMPOSITION des coefficients C_n présente les avantages suivants :

- Elle permet une intégration économique d'un système (calcul de V seulement au lieu du terme complet $C_n = (A \cdot B \cdot V)_n$ à chaque pas d'intégration).
- Elle permet une réduction du nombre des coefficients à calculer grâce à des identités entre coefficients qui résultent des symétries du système considéré.
- Elle est pratique à cause de sa transparence (un changement de B n'affecte ni A et, dans la plupart des cas, ni V).
- Elle révèle d'intéressantes informations mathématiques (signification mathématique du terme A).

Ce formalisme est illustré par des nombreux exemples.

ABSTRACT

The solution functions of systems of ordinary differential equations are represented in form of power series of which the coefficients C_n are DECOMPOSED into three parts $C_n = (A.B.V)_n$: A reflects the form of the system, B its strength of coupling and V the initial values. The component A and in most practical cases also the component B remain constant during integration.

Recurrence relations for the key-component A are derived for various types of systems. They have the form $A_{n+1} = f(A_n)$. The knowledge of A_n permits a direct, non-recursive computation of the component B and V.

This DECOMPOSITION of the COEFFICIENTS C_n permits economical integration of a system (computation of V only instead of C_n as a whole at each integration step, reduction of coefficients to be computed owing to identities resulting from symmetries of the system), is convenient due to its transparency (intervention on B does not affect A and in most cases also not V), and reveals interesting mathematical information (mathematical significance of A).

The formalism is illustrated by examples.

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1. INTRODUCTION.

In view of applications in NATURAL SCIENCE we study in this work the solution of systems of differential equations of the form

and

$$\begin{aligned} \frac{dy_1}{dt}(t) &= (b_{11}y_1^{r_{111}}y_2^{r_{112}} \dots y_I^{r_{11I}} + b_{12}y_1^{r_{121}}y_2^{r_{122}} \dots y_I^{r_{12I}} + \dots)^{s_{11}} + \dots \\ \frac{dy_2}{dt}(t) &= (b_{21}y_1^{r_{211}}y_2^{r_{212}} \dots y_I^{r_{21I}} + b_{22}y_1^{r_{221}}y_2^{r_{222}} \dots y_I^{r_{22I}} + \dots)^{s_{21}} + \dots \\ &\dots \\ \frac{dy_I}{dt}(t) &= (b_{I1}y_1^{r_{I11}}y_2^{r_{I12}} \dots y_I^{r_{I1I}} + b_{I2}y_1^{r_{I21}}y_2^{r_{I22}} \dots y_I^{r_{I2I}} + \dots)^{s_{I1}} + \dots \end{aligned}$$

where b_{ij} are constants or depend on t , and s_{ij} are fractions.

Many equations in natural science belong to system of the type (1.1) or (1.2), for example:

- Nuclear reaction equations (matter conversion),
 - SU(2) Yang Mills equations /RA82/, p.296,
 - chemical kinetic reaction equations,
 - Euler's equation of motion,
 - the restricted three body problem ,
 - equations in nonlinear optics/B165/.

Usually, such systems have to be solved numerically.

Commonly three different numerical integration methods are used:

- The RUNGE-KUTTA methods,
- the MULTISTEP methods and
- the LIE-series method.

The two former require determination of coefficients for each order of approximation prior to their application. That is a laborious process, especially for higher order approximation.

The most performant explicit RUNGE-KUTTA schemata are published by FEHLBERG /FE68/, /FE69/, /FE70/, by VERNER /VE78/ and HAIRER /HA78/. Efficient implicit RUNGE-KUTTA schemata of the Rosenbrock type are published by GOTTWALD-WANNER /G081/ and KAPS-WANNER /KA81/.

These methods are embodied in the code OSIRIS /C083/.

Concerning the MULTISTEP methods, there exists a standard code called GEAR /HI74/. This Code, since its release in 1971, has continually benefitted from basic mathematical progress as well as from considerable user experience.

In contrast to the above mentioned methods the LIE-series method /WA69/ as well as the INTERVAL ANALYSIS technique /MO66/ permit an automatized approximation of arbitrary order.

The actual status of these numerical integration methods is summarized in TABLE 1.1 .

TABLE 1.1: Status of common numerical integration methods.

Method	Order of approximation for which coefficients are available.	Comment
RUNGE-KUTTA	explicit implicit up to 10 up to 6	Schemata prior to computation have to be established for each order.
Multistep	ADAM-BASHFORTH-MOULTON up to 12, GEAR's method up to 5	Algebraic equations have to be solved, when applying implicit methods.
LIE-series	unlimited	No schemata prior to its usage needed.
Interval- Analysis	(recurrence relation for coefficients).	No algebraic equations to be solved.

All integration methods mentioned in TABLE 1.1 evaluate new function values from known function values. This extrapolation is chiefly governed by the form of the system considered, by the values b_{ij} appearing there and the known function values. These individual elements loose in all these methods their identity in the course of the extrapolation process. These methods act indeed as a black box in the way: Read input data, apply method, determine new function values.

APPARENTLY, ONE CAN EXPECT BENEFITS BY MAKING THE INTEGRATION PROCESS MORE TRANSPARENT, THAT IS TO SAY, BY PRESERVING THE INDIVIDUALITY OF THE VARIOUS ELEMENTS, WHICH DETERMINE THE EXTRAPOLATION, DURING THE INTEGRATION PROCESS.

This would be particularly interesting for applications in natural science, as in this case these elements have a specific meaning:

- The SYSTEM expresses the mathematics of the problem.
- The COEFFICIENTS b_{ij} symbolize physical parameters (interaction, moment of inertia, mass, ...) and
- the THIRD element comprises essentially the initial values of the functions.

An Ansatz of the solution functions Y_i which exhibits and preserves the individuality of these elements would thus have the form

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A(i,H,K) B(H) F(Y_i(t=t_0), K) \right)_n \quad (1.3)$$

A expresses the mathematical aspect of the system. It remains thus constant for a given system.

B indicates the physical aspect of the problem, $B=B(b_{ij})$.

It remains, therefore, also constant for a given problem.

F signifies the initial function values. It is the only part which alters at each integration step.

A priori, the Ansatz (1.3) offers several advantages compared to the numerical methods stated in TABLE 1.1:

- TRANSPARENCY: Alteration of one element does not affect the others, for example, if B is changed neither A nor F is influenced. This is convenient for studying physical problems.
- ECONOMY: The integration process is reduced to the formation of F at each integration step and the summation of the series. A and B remain during the integration constant. Other integration methods, TABLE 1.1, implicitly compute A, B and F at each integration step.
- MATHEMATICAL INFORMATION: As A symbolizes the mathematics of the system, it may have a deeper mathematical meaning.

In this work we are investigating the Ansatz (1.3) for systems of the type (1.1) and (1.2).

The central problem is the derivation of a recurrence relation to compute $A_n(i, H, K)$. Note that if these coefficients are known, then the labels H, K and thus B and F are also known.

The recurrence relations which we derive have the general form

$$A_{n+1} = f(A_n), \text{ n is order of approximation.} \quad (1.4)$$

Several examples illustrate the use of the formalism which has been developed on basis of the DECOMPOSED COEFFICIENT REPRESENTATION (1.3). Its efficiency will be evaluated by comparative calculations using various integration methods and will be published in PART II: Applications.

2. FORMALISM OF THE DECOMPOSED COEFFICIENT REPRESENTATION.

In this Chapter we are first studying systems of the type (1.1) and then systems of the type (1.2).

2.1 SYSTEMS CONTAINING TERMS OF THE TYPE $b Y_1^{r_1} Y_2^{r_2} \dots Y_i^{r_i}$.

In this Section we treat three different cases:

- b is constant,
- $b(t)$ is a polynomial in t , and
- $b(t)$ is an arbitrary, n -times differentiable function.

2.1.1 Case 1: b is constant.

We consider the system

$$dY_i(t)/dt = b_{i1} Y_1^{r_{i11}} Y_2^{r_{i12}} \dots Y_I^{r_{i1I}} + \dots + b_{iJ(i)} Y_1^{r_{iJ(i)1}} \dots Y_I^{r_{iJ(i)I}}$$

(2.1)

where $i=1,2,3,\dots,I$ and b_{ij} are constants.

The solution functions of (2.1) can be represented in the usual way

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} C_n$$

(2.2)

C_n are coefficients, $C_n = C_n(\text{system, } b_{ij}, Y_i(t=t_0))$.

To compute these coefficients recurrence relations are available /M066/, /WA69/. However, as C_n depends on the form of the system, on b_{ij} and on the initial values $Y_i(t=t_0)$, one can try to decompose C_n into three parts

$$C_n = (A \cdot B \cdot F)_n, \text{ where}$$

A symbolizes the influence of the form of the system (2.1) on the coefficients.

B depends only on the b_{ij} , and

F depends on the initial values $Y_i(t=t_0)$ only.

A as well as B remain by nature constant during the integration,

as the system (2.1) and b_{ij} are fixed for a given problem.
Only F changes at each integration step.

In other words, instead of computing $C_n = C_n(A, B, F)$ at each integration step, one has only to compute F when using the decomposed coefficient representation.

Essentially, from such a decomposition one can expect on the one side new mathematical information and on the other side less computational labour for the integration of the system.

It can be shown that (2.2) reads in form of decomposed coefficients

$$y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} AB^H y_1^{k_1} y_2^{k_2} \dots y_I^{k_I} \right)_n \quad (2.3)$$

where $A = a_{i,H,K}$,

$$B^H = b_{11}^{h_{11}} b_{12}^{h_{12}} b_{13}^{h_{13}} \dots b_{IJ(I)}^{h_{IJ(I)}} \quad \text{and} \quad (2.4)$$

$$K = k_1 k_2 k_3 \dots k_I .$$

The labels H and K are automatically known if $(a_{i,H,K})_n$ is known. The knowledge of these labels permit in turn the computation of B^H and $y_1^{k_1} y_2^{k_2} \dots y_I^{k_I}$.

THE COMPUTATION OF $C_n(A, B, F)$ IS THUS REDUCED TO THE COMPUTATION OF A WHICH IN TURN REMAINS CONSTANT IN THE COURSE OF THE INTEGRATION.

To apply representation (2.3) we must compute $a_{i,H,K}$. By means of mathematical induction the following recurrence relation was found for them

$$(a_{i,H,K})_{n+1} = \left(\sum_{j=1}^I \sum_{s=1}^S (k_j + 1 - r_{js}) \cdot a_{i,H_{js}, k'_1, k'_2, k'_3, \dots, k'_I} \right)_n \quad (2.5)$$

where $H_{js} = H - E_{js}$, $E_{js} = 1$ for the element js and =0 else.

$$k'_m = k_m - r_{jsm} + D_m^j, \quad D_m^j = 1 \text{ for } j=m, =0 \text{ else.}$$

$$i=1,2,3,\dots,I, \quad m=1,2,3,\dots,I.$$

Recurrence relation (2.5) starts with the values

$$(a_{i,H,K})_{n=0} = (a_{i,H=0,k_i=1,K=0 \text{ else}})_{n=0} = 1, \text{ for example } \quad (2.6)$$

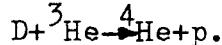
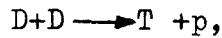
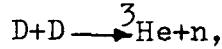
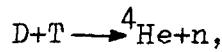
$$(a_{3,H=0,k_1=0,k_2=0,k_3=1,k_4=0,\dots,k_I=0}) = 1.$$

Recurrence relation (2.5) has the following remarkable properties:

- It enables computation of $(a_{i,H,K})_{n+1}$ from $(a_{i,H,K})_n$.
- It generates not only the coefficients $a_{i,H,K}$, but also the labels H and K which one needs for the computation of B^H and $y_1^{k_1} y_2^{k_2} y_3^{k_3} \dots y_I^{k_I}$.

The following examples serve to illustrate the application of the decomposed representation (2.3) and the recurrence relation (2.5).

EXAMPLE 1: We consider the following simultaneous fusion processes



These are processes which can occur below a threshold plasma temperature of 100 kev, as it can be seen from the subsequent TABLE 2.1 which we took from /KE82/, p.312.

TABLE 2.1: Fusion reactions.

Reaction energy (MeV)	Threshold plasma temperature (keV)	Maximum energy gain per fusion
D + T \rightarrow ${}^4\text{He} + N$	17.6	4
D + D \rightarrow ${}^3\text{He} + N$	3.2	50
D + D \rightarrow T + P	4.0	50
D + ${}^3\text{He}$ \rightarrow ${}^4\text{He} + P$	18.3	100
${}^6\text{Li} + P \rightarrow {}^3\text{He} + {}^4\text{He}$	4.0	900
${}^6\text{Li} + D \rightarrow {}^7\text{Li} + P$	5.0	> 900
${}^6\text{Li} + D \rightarrow T + {}^4\text{He} + P$	2.6	> 900
${}^6\text{Li} + D \rightarrow 2({}^4\text{He})$	22.0	> 900
${}^7\text{Li} + P \rightarrow 2({}^4\text{He})$	17.5	> 900
${}^{11}\text{B} + P \rightarrow 3({}^4\text{He})$	8.7	300

Denoting by Y_1, Y_2, Y_3 the amount of D, T, ^3He , respectively, then the following system of differential equations describes the processes of EXAMPLE 1

$$\begin{aligned} \frac{dY_1(t)}{dt} &= -b_1 Y_1 Y_2 - b_2 Y_1^2 - b_3 Y_1^2 - b_4 Y_1 Y_3 \\ \frac{dY_2(t)}{dt} &= -b_1 Y_1 Y_2 + b_3 Y_1^2 \\ \frac{dY_3(t)}{dt} &= b_2 Y_1^2 - b_4 Y_1 Y_3 \end{aligned} \quad (2.1.1)$$

We assume that b_i are constants.

Equ.(2.1.1) reads in the general notation of (2.1)

$$\begin{aligned} \frac{dY_1(t)}{dt} &= b_{11} Y_1^{r_{111}} Y_2^{r_{112}} Y_3^{r_{113}} + b_{12} Y_1^{r_{121}} Y_2^{r_{122}} Y_3^{r_{123}} + \\ &\quad b_{13} Y_1^{r_{131}} Y_2^{r_{132}} Y_3^{r_{133}} + b_{14} Y_1^{r_{141}} Y_2^{r_{142}} Y_3^{r_{143}} \\ \frac{dY_2(t)}{dt} &= b_{21} Y_1^{r_{211}} Y_2^{r_{212}} Y_3^{r_{213}} + b_{22} Y_1^{r_{221}} Y_2^{r_{222}} Y_3^{r_{223}} \\ \frac{dY_3(t)}{dt} &= b_{31} Y_1^{r_{311}} Y_2^{r_{312}} Y_3^{r_{313}} + b_{32} Y_1^{r_{321}} Y_2^{r_{322}} Y_3^{r_{323}} \end{aligned} \quad (2.1.1.1)$$

Equation (2.3) becomes for this system

$$y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{H^K} \left(\sum_{H,K} A \cdot B^H \cdot y_1^{k_1} y_2^{k_2} y_3^{k_3} \right)_n, \quad i=1,2,3. \quad (2.3.1.1)$$

with $H=h_{11} h_{12} h_{13} h_{14} h_{21} h_{22} h_{31} h_{32}$, $K=k_1 k_2 k_3$ and

$$B^H = b_{11}^{h_{11}} b_{12}^{h_{12}} b_{13}^{h_{13}} b_{14}^{h_{14}} b_{21}^{h_{21}} b_{22}^{h_{22}} b_{31}^{h_{31}} b_{32}^{h_{32}} \quad (2.4.1.1)$$

Recurrence relation (2.5) reads for the system (2.1.1.1)

$$(a_{i,H,k_1,k_2,k_3})_{n+1} = ((k_1+1-r_{111}) \cdot a_{i,H_{11},k_1+1-r_{111},k_2-r_{112},k_3-r_{113}} +$$

$$+ (k_1 + 1 - r_{121}) \cdot a_{i, H_{12}, k_1 + 1 - r_{121}, k_2 - r_{122}, k_3 - r_{123}}^+$$

$$+ (k_1 + 1 - r_{131}) \cdot a_{i, H_{13}, k_1 + 1 - r_{131}, k_2 - r_{132}, k_3 - r_{133}}^+$$

$$+ (k_1 + 1 - r_{141}) \cdot a_{i, H_{14}, k_1 + 1 - r_{141}, k_2 - r_{142}, k_3 - r_{143}}^+$$

(2.5.1.1)

$$+ (k_2 + 1 - r_{212}) \cdot a_{i, H_{21}, k_1 - r_{211}, k_2 + 1 - r_{212}, k_3 - r_{213}}^+$$

$$+ (k_2 + 1 - r_{222}) \cdot a_{i, H_{22}, k_1 - r_{221}, k_2 + 1 - r_{222}, k_3 - r_{223}}^+$$

$$+ (k_3 + 1 - r_{313}) \cdot a_{i, H_{31}, k_1 - r_{311}, k_2 - r_{312}, k_3 + 1 - r_{313}}^+$$

$$+ (k_3 + 1 - r_{323}) \cdot a_{i, H_{32}, k_1 - r_{321}, k_2 - r_{322}, k_3 + 1 - r_{323}}^) n$$

where $H_{11} = h_{11}^{-1}, h_{12} h_{13} h_{14} h_{21} h_{22} h_{31} h_{32}$,

$H_{12} = h_{11} h_{12}^{-1}, h_{13} h_{14} h_{21} h_{22} h_{31} h_{32}$,

$H_{13} = h_{11} h_{12} h_{13}^{-1}, h_{14} h_{21} h_{22} h_{31} h_{32}$,

$H_{14} = h_{11} h_{12} h_{13} h_{14}^{-1}, h_{21} h_{22} h_{31} h_{32}$,

$H_{21} = h_{11} h_{12} h_{13} h_{14} h_{21}^{-1}, h_{22} h_{31} h_{32}$,

$H_{22} = h_{11} h_{12} h_{13} h_{14} h_{21} h_{22}^{-1}, h_{31} h_{32}$,

$H_{31} = h_{11} h_{12} h_{13} h_{14} h_{21} h_{22} h_{31}^{-1}, h_{32}$,

$H_{32} = h_{11} h_{12} h_{13} h_{14} h_{21} h_{22} h_{31} h_{32}^{-1}$.

Comparing (2.1.1.1) with (2.1.1) one obtains

$b_1 = -b_{11} = b_{21}$, $b_2 = -b_{12} = b_{31}$, $b_3 = -b_{13} = b_{22}$, $b_4 = -b_{14} = -b_{32}$ and
 $r_{111} = r_{112} = 1$, $r_{121} = r_{131} = 2$, $r_{141} = r_{143} = 1$, $r_{211} = r_{212} = 1$, $r_{221} = 2$,
 $r_{311} = 2$, $r_{321} = r_{323} = 1$, $r_{isj} = 0$ otherwise.

Equ.(2.4.1.1) becomes, therefore,

$$B^H = (-b_1)^{h_{11}} (-b_2)^{h_{12}} (-b_3)^{h_{13}} (-b_4)^{h_{14}} (-b_1)^{h_{21}} b_3^{h_{22}} b_2^{h_{31}} (-b_4)^{h_{32}} \quad (2.4.1)$$

and the recurrence relation (2.5.1.1) reads

$$\begin{aligned} (a_{i,H,K})_{n+1} = & (k_1 \cdot a_{i,H_{11},k_1, k_2-1, k_3} + (k_1-1) \cdot a_{i,H_{12},k_1-1, k_2, k_3} + \\ & + (k_1-1) \cdot a_{i,H_{13},k_1-1, k_2, k_3} + k_1 \cdot a_{i,H_{14},k_1, k_2, k_3-1} + \\ & + k_2 \cdot a_{i,H_{21},k_1-1, k_2, k_3} + (k_2+1) \cdot a_{i,H_{22},k_1-2, k_2+1, k_3} + \\ & + (k_3+1) \cdot a_{i,H_{31},k_1-2, k_2, k_3} + k_3 \cdot a_{i,H_{32},k_1-1, k_2, k_3})_n, \\ i=1,2,3 . \end{aligned} \quad (2.5.1)$$

The starting values of this recurrence relation are

$$\begin{aligned} (a_{1,H=0,k_1=1,k_2=0,k_3=0})_{n=0} &= 1, \\ (a_{2,H=0,k_1=0,k_2=1,k_3=0})_{n=0} &= 1, \\ (a_{3,H=0,k_1=0,k_2=0,k_3=1})_{n=0} &= 1. \end{aligned} \quad (2.6.1)$$

(2.1.1) remains invariant with respect to the following interchanges of the indices i and j:

i: (2,3) and j: (1,2) for $i=2,3$; j: (1,4) for $i=1$.

This yields immediately the identity

$$\begin{aligned} & (a_{2,h_{11}h_{12}h_{13}h_{14}h_{21}h_{22}h_{31}h_{32},k_1k_2k_3})_n = \\ & (a_{3,h_{14}h_{12}h_{13}h_{11}h_{32}h_{31}h_{22}h_{21},k_1k_3k_2})_n \end{aligned} \quad (2.7.1)$$

That is of mathematical, but particularly of numerical interest, as $a_{3,H,K}$ is expressed by $a_{2,H,K}$ and does, therefore, not need be computed. In this context the following general statements are valid:

- SYMMETRIES IN (2.1) YIELD IDENTITIES OF THE COEFFICIENTS :

$$a_{i,H,K} = a_{i_1,H_1,K_1}, \quad (2.7)$$

where $i_1, i=1, 2, 3, \dots, I$, but $i_1 \neq i$; H_1 and K_1 have the same elements as H and K , but those elements are permuted.

- IDENTITIES (2.7) REDUCE THE COMPUTATIONAL LABOUR as in this case the coefficients $a_{i,H,K}$ need not be computed for all $i=1, 2, 3, \dots, I$. That is another particularity of the decomposed coefficient representation.

Explicitely, the first two leading terms of the series (2.3.1.1) read for EXAMPLE 1

$$\begin{aligned} Y_1(t) = & y_1 + (t-t_0)(-b_1 y_1 y_2 - b_2 y_1^2 - b_3 y_1^2 - b_4 y_1 y_3) + \\ & \frac{(t-t_0)^2}{2!} (b_1^2 y_1 y_2^2 + 3b_1 b_2 y_1^2 y_2 + 3b_1 b_3 y_1^2 y_2 + 2b_1 b_4 y_1 y_2 y_3 + 2b_2 y_1^3 + \\ & + 4b_2 b_3 y_1^3 + 3b_2 b_4 y_1^2 y_3 + 2b_3^2 y_1^3 + 3b_3 b_4 y_1^2 y_3 + b_4^2 y_1^2 y_3 + b_1^2 y_1^2 y_2 - \\ & - b_1 b_3 y_1^3 - b_2 b_4 y_1^3 - b_4^2 y_1^2 y_3) + \dots \end{aligned}$$

$$\begin{aligned} Y_2(t) = & y_2 + (t-t_0)(-b_1 y_1 y_2 + b_3 y_1^2) + \frac{(t-t_0)^2}{2!} (b_1^2 y_1 y_2^2 - b_1 b_3 y_1^2 y_2 + \\ & + b_1 b_2 y_1^2 y_2 - 2b_2 b_3 y_1^3 - 2b_3^2 y_1^3 + b_1 b_4 y_1 y_2 y_3 - 2b_3 b_4 y_1^2 y_3 + b_1^2 y_1^2 y_2 - \\ & - b_1 b_3 y_1^3) + \dots \end{aligned}$$

$$\begin{aligned} Y_3(t) = & y_3 + (t-t_0)(b_2 y_1^2 - b_4 y_1 y_3) + \frac{(t-t_0)^2}{2!} (-2b_1 b_2 y_1^2 y_2 + b_1 b_4 y_1 y_2 y_3 - \\ & + 2b_2^2 y_1^3 + b_2 b_4 y_1^2 y_3 - 2b_2 b_3 y_1^3 + b_3 b_4 y_1^2 y_3 - 2b_2 b_4 y_1^2 y_3 + b_4^2 y_1^2 y_3 - \end{aligned}$$

$$+b_2 b_4 y_1^3 + b_4^2 y_1^2 y_3) + \dots \quad (2.3.1)$$

EXAMPLE 2: SU(2)-Yang Mill's equations read /RA82/, p.296

$$\begin{aligned} \frac{dy_1(t)}{dt} &= y_2, & \frac{dy_3(t)}{dt} &= y_4, \\ \frac{dy_2(t)}{dt} &= -y_3^2 y_1 - y_5^2 y_1, & \frac{dy_4(t)}{dt} &= -y_1^2 y_3 - y_5^2 y_3, \\ \frac{dy_5(t)}{dt} &= y_6, & \frac{dy_6(t)}{dt} &= -y_1^2 y_5 - y_3^2 y_5. \end{aligned} \quad (2.1.2)$$

Comparing (2.1.2) with (2.1) yields

$$\begin{aligned} b_{11} &= 1, & b_{21} = b_{22} &= -1, & b_{31} &= 1, & b_{41} = b_{42} &= -1, & b_{51} &= 1, & b_{61} = b_{62} &= -1, \\ b_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

$$\begin{aligned} r_{112} &= 1, & r_{211} &= 1, & r_{213} &= 2, & r_{221} &= 1, & r_{225} &= 2, & r_{314} &= 1, & r_{411} &= 2, \\ r_{413} &= 1, & r_{423} &= 1, & r_{425} &= 2, & r_{516} &= 1, & r_{611} &= 2, & r_{615} &= 1, & r_{623} &= 2, \\ r_{625} &= 1, & r_{isj} &= 0 \text{ otherwise.} \end{aligned}$$

Equ.(2.3) becomes in this case

$$y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_o)^n}{n!} \left(\sum_{H, K} A \cdot B^H \cdot y_1^{k_1} y_2^{k_2} y_3^{k_3} y_4^{k_4} y_5^{k_5} y_6^{k_6} \right)_n \quad (2.3.2)$$

$$\text{where } K = k_1 k_2 k_3 k_4 k_5 k_6, \quad H = h_{21} h_{22} h_{41} h_{42} h_{61} h_{62},$$

$$B^H = (-1)^{h_{21} + h_{22} + h_{41} + h_{42} + h_{61} + h_{62}} \quad (2.4.2)$$

Recurrence relation (2.5) to compute $a_{i,H,K}$ has the form

$$\begin{aligned} (a_{i,K})_{n+1} &= ((k_1+1) \cdot a_{i,k_1+1, k_2-1, k_3 k_4 k_5 k_6} - \\ &(k_2+1) \cdot a_{i,k_1-1, k_2+1, k_3-2, k_4 k_5 k_6} - (k_2+1) \cdot a_{i,k_1-1, k_2+1, k_3 k_4 k_5-2, k_6} + \\ &(k_3+1) \cdot a_{i,k_1 k_2 k_3+1, k_4-1, k_5 k_6} - (k_4+1) \cdot a_{i,k_1-2, k_2 k_3-1, k_4+1, k_5 k_6} - \end{aligned}$$

$$\begin{aligned}
 & (k_4+1) \cdot a_{i, k_1 k_2 k_3^{-1}, k_4+1, k_5-2, k_6} + (k_5+1) \cdot a_{i, k_1 k_2 k_3 k_4 k_5+1, k_6-1} \\
 & (k_6+1) \cdot a_{i, k_1-2, k_2 k_3 k_4 k_5-1, k_6+1} - (k_6+1) \cdot a_{i, k_1 k_2 k_3-2, k_4 k_5-1, k_6+1})_n
 \end{aligned} \tag{2.5.2}$$

with the starting values

$$\begin{aligned}
 (a_{1,100000})_0 &= (a_{2,010000})_0 = (a_{3,001000})_0 = \\
 (a_{4,000100})_0 &= (a_{5,000010})_0 = (a_{6,000001})_0 = 1 .
 \end{aligned} \tag{2.6.2}$$

To reduce the number of coefficients $a_{i,H,K}$ to compute, we make use of the SYMMETRIES of (2.1.2). Indeed, (2.1.2) remains invariant when interchanging simultaneously the indices (1,3) and (2,4) or the indices (1,5) and (2,6). This leads to the following IDENTITIES

$$\begin{aligned}
 a_{1, k_1 k_2 k_3 k_4 k_5 k_6} &= a_{3, k_3 k_4 k_1 k_2 k_5 k_6} = a_{5, k_5 k_6 k_3 k_4 k_1 k_2} , \\
 a_{2, k_1 k_2 k_3 k_4 k_5 k_6} &= a_{4, k_3 k_4 k_1 k_2 k_5 k_6} = a_{6, k_5 k_6 k_3 k_4 k_1 k_2} .
 \end{aligned} \tag{2.7.2}$$

Furthermore, one can show that

$$(a_{1, k_1 k_2 k_3 k_4 k_5 k_6})_{n+1} = (a_{2, k_1 k_2 k_3 k_4 k_5 k_6})_n . \tag{2.8.2}$$

From (2.7.2) and (2.8.2) follows that ONE HAS ONLY TO COMPUTE $a_{1,K}$, because the $a_{i,K}$ ($i=2,3,4,5,6$) are expressed by $a_{1,K}$.

In addition, the coefficients $a_{i,K}$ satisfy the following interesting relations

$$\sum_K |(a_{i,K})_n| = n! , \quad i=1,3,5 ; \quad \sum_K |(a_{i,K})_n| = (n-1)! , \quad i=2,4,6 . \tag{2.9.2}$$

THE COEFFICIENTS $a_{i,K}$ CORRESPOND THUS TO ENUMERATIONS IN COMBINATORICS, THAT IS TO SAY, THEY DO NOT ONLY SERVE TO INTEGRATE THE SYSTEM, BUT HAVE IN ADDITION A MATHEMATICAL MEANING.

Combinatorial interpretations have for a similar case already been given in /DU79/, /SC76/. In /DU79/ a combinatorial interpretation is among others given for the coefficients of the power series representation of the Jacobian elliptic functions: sn, cn, dn . These functions are also solutions of Yang-Mill's equations /AC79/, /RA82/.

With regard to (2.3.2) one can conclude that the combinatorial enumeration is the more complex the larger t becomes.

It would appear that nature is playing combinatorial games rather than probability games.

For the sake of completeness we give below the leading terms of (2.3.2) for $Y_1(t)$. Those for Y_i ($i=2,3,4,5,6$) are then also known thanks to the identities (2.7.2) and (2.8.2) .

$$Y_1(t) = y_1 + (t-t_0)y_2 + \frac{(t-t_0)^2}{2!}(-y_1y_3^2 - y_1y_5^2) + \frac{(t-t_0)^3}{3!}(-y_2y_3^2 - 2y_1y_3y_4 -$$

$$y_2y_5^2 - 2y_1y_5y_6) + \frac{(t-t_0)^4}{4!}(y_1y_3^4 + 6y_1y_3^2y_5^2 - 4y_2y_3y_4 - 2y_1y_4^2 +$$

$$2y_1^3y_3^2 + y_1y_5^4 - 4y_2y_5y_6 - 2y_1y_6^2 + 2y_1^3y_5^2) + \frac{(t-t_0)^5}{5!}(y_2y_3^4 + 8y_1y_3^3y_4 +$$

$$20y_1y_3y_4y_5^2 - 6y_2y_4^2 + 10y_1^2y_2y_3^2 + 14y_2y_3^2y_5^2 + 8y_1^3y_3y_4 + y_2y_5^4 +$$

$$8y_1y_5^3y_6 + 20y_1y_3^2y_5y_6 - 6y_2y_6^2 + 10y_1^2y_2y_5^2 + 8y_1^3y_5y_6) + \dots$$

Recall that $y_i = Y_i(t=t_0)$.

From this relation one can immediately see that (2.9.2) is satisfied. Indeed,

$$\sum_K |(a_{1,K})_n| = 1, 2!, 3!, 4! \text{ and } 5! \text{ for } n=1,2,3,4 \text{ and } 5,$$

respectively.

We can conclude, formulae (2.5.2), (2.7.2) and (2.8.2) permit economical computation of the solution functions of (2.1.2) and formula (2.9.2) yields additional mathematical interdisciplinary information.

EXAMPLE 3: We consider the system

$$\begin{aligned} \frac{dy_1(t)}{dt} &= b_1 y_2 y_3 \\ \frac{dy_2(t)}{dt} &= b_2 y_1 y_3 \\ \frac{dy_3(t)}{dt} &= b_3 y_1 y_2 \end{aligned} \quad (2.1.3)$$

This is an example by excellency which demonstrates the mathematical interest of the use of decomposed coefficients, representation (2.3).

Comparing (2.1.3) with (2.1) one obtains for b_{ij} and r_{ijs}

$$b_{11}=b_1, \quad b_{21}=b_2, \quad b_{31}=b_3, \quad b_{ij}=0 \text{ otherwise ;}$$

$$r_{112}=r_{113}=1, \quad r_{211}=r_{213}=1, \quad r_{311}=r_{312}=1, \quad r_{ijs}=0 \text{ otherwise.}$$

Equ.(2.3) reads for the system (2.1.3)

$$y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A \cdot B^H \cdot y_1^{k_1} y_2^{k_2} y_3^{k_3} \right)_n \quad \text{with } i=1,2,3; \quad (2.3.3)$$

$$H=h_{11} h_{21} h_{31}, \quad K=k_1 k_2 k_3, \quad \text{and}$$

$$B^H = b_1^{h_{11}} b_2^{h_{21}} b_3^{h_{31}}. \quad (2.4.3)$$

Recurrence relation (2.5) reads for this example

$$\begin{aligned} (a_{i,H,K})_{n+1} &= ((k_1+1) \cdot a_{i,H_{11},k_1+1,k_2-1,k_3-1} + \\ &+ (k_2+1) \cdot a_{i,H_{21},k_1-1,k_2+1,k_3-1} + (k_3+1) \cdot a_{i,H_{31},k_1-1,k_2-1,k_3+1})_n \end{aligned} \quad (2.5.3)$$

$$\text{with } H_{11}=h_{11}-1, h_{21}, h_{31}; \quad H_{21}=h_{11} h_{21}-1, h_{31}; \quad H_{31}=h_{11} h_{21} h_{31}-1,$$

and the starting values

$$(a_{1,H=0,k_1=1,k_2=k_3=0})_0 = (a_{2,H=0,k_1=k_3=0,k_2=1})_0 = (a_{3,H=0,k_1=k_2=0,k_3=1})_0 \\ = 1. \quad (2.6.3)$$

Remarkable are the IDENTITIES of the coefficients $a_{i,H,K}$ which one obtains directly from the SYMMETRIES of (2.1.3) by cyclic replacement of the indices (1,2,3):

$$(a_{1,h_{11}h_{21}h_{31},k_1k_2k_3})_n = (a_{2,h_{21}h_{11}h_{31},k_2k_1k_3})_n = \\ (a_{3,h_{31}h_{21}h_{11},k_3k_2k_1})_n \quad (2.7.3)$$

One has thus only to compute $(a_{1,H,K})_n$, as $(a_{2,H,K})_n$ and $(a_{3,H,K})_n$ are expressed by $(a_{1,H,K})_n$.

The Jacobian elliptic functions sn, cn, and dn are solutions of (2.1.3). Recurrence relation (2.5.3) generates thus among others the Taylor series coefficients of these functions /SC76/. In addition, the coefficients $a_{i,H,K}$ or combinations of them are identical to the following combinatorial sets :

- The number of permutations of n natural numbers with given CYCLE PEAKS /DU79/ ,
- The number of permutations of n natural numbers with given RUNS-UP /SC76/ ,
- the number of permutations of n natural numbers with given PEAKS /SC76/ .
- The sum of all $a_{i,H,K}$ is for a given n equal to $n!$.

THE COEFFICIENTS $a_{i,H,K}$ PERMIT THUS THE CALCULATION OF THREE FUNCTIONS AND FOUR COMBINATORIAL SETS.

The coefficients $(a_{1,H,K})_n$, $n=1,2,3,\dots,15$, are already tabulated in form of microfiche in /SC76/ , and parts of them, for $n=16,17,\dots,50$, are tabulated in the same form in /SC77/ .

The following tables contain the coefficients $(a_{1,H,K})_n$ for $n=1,2,3,\dots,25$. (Those given in /SC76/ and /SC77/ are included). They are here tabulated in a more consistent way than in /SC76/ and /SC77/. We repeat here for the sake of convenience some useful relations from /SC76/.

$$h_{11} + h_{21} + h_{31} = n, \quad k_1 + k_2 + k_3 = n+1.$$

For $(a_{1,H,K})_n = (a_{1,h_{11},h_{21},h_{31}}, k_1 k_2 k_3)_n$ h_{ij} can be expressed by n and k_s as follows

$$h_{11} = (n+1-k_1)/2, \quad h_{21} = (n-k_2)/2, \quad h_{31} = (n-k_3)/2.$$

The coefficients $(a_{1,H,K})_n$ can thus uniquely be tabulated using two indices only, for example, the indices k_2, k_3 . Recall that we need only tabulate the coefficients $a_{1,H,K}$ as those of $a_{2,H,K}$ and $a_{3,H,K}$ are then thanks to the identities (2.7.3) also given.

In the subsequent TABLE 2.2 are tabulated the coefficients $(a_{1,H,K})_n$ and at the same time the number of permutations of n natural numbers with even and odd CYCLE PEAKS. Indeed, DUMOND/DU79/ has shown that the COEFFICIENTS $(a_{1,H,k_1 k_2 k_3})_n$ ARE IDENTICAL TO THE NUMBER OF PERMUTATIONS $(N(CP))_n$ of n natural numbers with

$CP = (k_1 - 1)/2$ odd and $k_2/2$ even CYCLE PEAKS for n -even, but

$CP = k_1/2$ odd and $(k_2 - 1)/2$ even CYCLE PEAKS for n -odd.

Therefore, the IDENTITY holds

$$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n \quad (2.10.3)$$

Examples of CYCLE PEAKS:

The sequence of the six numbers

123456 becomes 324165 by the permutations or cycles :

$\begin{array}{c} \swarrow \\ 1 \end{array} \begin{array}{c} \searrow \\ 3 \end{array}$ then $\begin{array}{c} \swarrow \\ 1 \end{array} \begin{array}{c} \searrow \\ 4 \end{array}$ which is symbolized with (134);

$\begin{array}{c} \swarrow \\ 5 \end{array} \begin{array}{c} \searrow \\ 6 \end{array}$ which is symbolized with (56).

Cycles (134) have the peak 4, i.e., an even cycle peak ;
cycle (56) has the peak 6, i.e., an even cycle peak.

For $n=3$ one has

Permutation	Cycles with the peak	CYCLE PEAKS
123	(1)(2)(3)	0 zero
132	(1)(23)	3 one odd
213	(12)(3)	2 one even
231	(123)	3 one odd
312	(132)	3 one odd
321	(13)(2)	3 one odd

Indeed, there is one permutation with zero cycle peak,
one permutation with one even cycle peak and
four permutations with one odd cycle peak.

TABLE 2.2: The coefficients $(a_{1,k_1 k_2 k_3})_n$ and the number of permutations $(N(CP))_n$ of n natural numbers with even and odd CYCLE PEAKS CP.

$n=1, 2, 3, \dots, 25$.

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cycle odd	Cycle even	Counter
0	1	1	0	0	0	0	1
1	1	0	1	1	0	0	1
2	1	1	0	2	0	0	1
2	1	1	2	0	0	1	2
3	1	0	1	3	0	0	1
3	1	0	3	1	0	1	2
3	4	2	1	1	1	0	3
4	14	1	2	2	0	1	1
4	1	1	0	4	0	0	2
4	4	3	0	2	1	0	3
4	4	1	4	0	0	2	4
4	4	3	2	0	1	1	5
5	14	0	3	3	0	1	1
5	1	0	1	5	0	0	2
5	44	2	1	3	1	0	3
5	1	0	5	1	0	2	4
5	44	2	3	1	1	1	5
5	16	4	1	1	2	0	6
6	135	1	2	4	0	1	1
6	135	1	4	2	0	2	2
6	328	3	2	2	1	1	3
6	1	1	0	6	0	0	4
6	44	3	0	4	1	0	5
6	16	5	0	2	2	0	6
6	1	1	6	0	0	3	7
6	44	3	4	0	1	2	8
6	16	5	2	0	2	1	9
7	135	0	3	5	0	1	1
7	135	0	5	3	0	2	2
7	2064	2	3	3	1	1	3
7	1	0	1	7	0	0	4
7	408	2	1	5	1	0	5
7	912	4	1	3	2	0	6
7	1	0	7	1	0	3	7
7	408	2	5	1	1	2	8
7	912	4	3	1	2	1	9
7	64	6	1	1	3	0	10

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic Peaks		Counter
					odd	even	
8	5478	1	4	4	0	2	1
8	1228	1	2	6	0	1	2
8	11880	3	2	4	1	1	3
8	1228	1	6	2	0	3	4
8	11880	3	4	2	1	2	5
8	5856	5	2	2	2	1	6
8	1	1	0	8	0	0	7
8	408	3	0	6	1	0	8
8	912	5	0	4	2	0	9
8	64	7	0	2	3	0	10
8	1	1	8	0	0	4	11
8	408	3	5	0	1	3	12
8	912	5	4	0	2	2	13
8	64	7	2	0	3	1	14
9	5478	0	5	5	0	2	1
9	1228	0	3	7	0	1	2
9	64920	2	3	5	1	1	3
9	1228	0	7	3	0	3	4
9	64920	2	5	3	1	2	5
9	124320	4	3	3	2	1	6
9	1	0	1	9	0	0	7
9	3688	2	1	7	1	0	8
9	30768	4	1	5	2	0	9
9	15808	6	1	3	3	0	10
9	1	0	9	1	0	4	11
9	3688	2	7	1	1	3	12
9	30768	4	5	1	2	2	13
9	15808	6	3	1	3	1	14
9	256	8	1	1	4	0	15
10	165826	1	4	6	0	2	1
10	165826	1	6	4	0	3	2
10	1146480	3	4	4	1	2	3
10	11069	1	2	8	0	1	4
10	343648	3	2	6	1	1	5
10	621648	5	2	4	2	1	6
10	11069	1	8	2	0	4	7
10	343648	3	6	2	1	3	8
10	621648	5	4	2	2	2	9
10	96896	7	2	2	3	1	10
10	1	1	0	10	0	0	11
10	3688	3	0	8	1	0	12
10	30768	5	0	6	2	0	13
10	15808	7	0	4	3	0	14
10	256	9	0	2	4	0	15
10	1	1	10	0	0	5	16
10	3688	3	8	0	1	4	17
10	30768	5	6	0	2	3	18
10	15808	7	4	0	3	2	19
10	256	9	2	0	4	1	20

TABLE 2.2 (cont.)

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n	$(a_1, k_1 k_2 k_3)_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic		Peaks	Counter
					odd	even		
11	165826	0	5	7	0	2	1	1
11	165826	0	7	5	0	3	2	2
11	5429352	2	5	5	1	2	3	3
11	11069	0	3	9	0	1	4	4
11	1782800	2	3	7	1	1	5	5
11	9756048	4	3	5	2	1	5	6
11	11069	0	9	3	0	4	7	7
11	1782800	2	7	3	1	3	8	8
11	9756048	4	5	3	2	2	9	9
11	5651456	6	3	3	3	1	10	10
11	1	0	1	11	0	0	11	11
11	33212	2	1	9	1	0	12	12
11	870640	4	1	7	2	0	13	13
11	1538560	6	1	5	3	0	14	14
11	259328	8	1	3	4	0	15	15
11	1	0	11	1	0	5	16	16
11	33212	2	9	1	1	4	17	17
11	870640	4	7	1	2	3	18	18
11	1538560	6	5	1	3	2	19	19
11	259328	8	3	1	4	1	20	20
11	1024	10	1	1	5	0	21	21
12	13180268	1	6	6	0	3	1	1
12	4494351	1	4	8	0	2	2	2
12	78650552	3	4	6	1	2	3	3
12	4494351	1	8	4	0	4	4	4
12	78650552	3	6	4	1	3	5	5
12	131469216	5	4	4	2	2	6	6
12	99642	1	2	10	0	1	7	7
12	9129868	3	2	8	1	1	8	8
12	44593984	5	2	6	2	1	9	9
12	26721792	7	2	4	3	1	10	10
12	99642	1	10	2	0	5	11	11
12	9129868	3	8	2	1	4	12	12
12	44593984	5	6	2	2	3	13	13
12	26721792	7	4	2	3	2	14	14
12	1566208	9	2	2	4	1	15	15
12	1	1	0	12	0	0	16	16
12	33212	3	0	10	1	0	17	17
12	870640	5	0	8	2	0	18	18
12	1538560	7	0	6	3	0	19	19
12	259328	9	0	4	4	0	20	20
12	1024	11	0	2	5	0	21	21
12	1	1	12	0	0	6	22	22
12	33212	3	10	0	1	5	23	23
12	870640	5	8	0	2	4	24	24
12	1538560	7	6	0	3	3	25	25
12	259328	9	4	0	4	2	26	26
12	1024	11	2	0	5	1	27	27

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counter
13	13180268	0	7	7	0	3	1
13	4494351	0	5	9	0	2	2
13	350988072	2	5	7	1	2	3
13	4494351	0	9	5	0	4	4
13	350988072	2	7	5	1	3	5
13	1601152704	4	5	5	2	2	6
13	99642	0	3	11	0	1	7
13	46363428	2	3	9	1	1	8
13	610611072	4	3	7	2	1	9
13	980493312	0	3	5	3	1	10
13	99642	0	11	3	0	5	11
13	46363428	2	9	3	1	4	12
13	610611072	4	7	3	2	3	13
13	980493312	6	5	3	3	2	14
13	227870208	8	3	3	4	1	15
13	1	0	1	13	0	0	16
13	298932	2	1	11	1	0	17
13	22945056	4	1	9	2	0	18
13	106923008	6	1	7	3	0	19
13	65008896	8	1	5	4	0	20
13	4180992	10	1	3	5	0	21
13	1	0	13	1	0	6	22
13	298932	2	11	1	1	5	23
13	22945056	4	9	1	2	4	24
13	106923008	6	7	1	3	3	25
13	65008896	8	5	1	4	2	26
13	4180992	10	3	1	5	1	27
13	4096	12	1	1	6	0	28
14	834687179	1	6	8	0	3	1
14	834687179	1	8	6	0	4	2
14	11318443824	3	6	6	1	3	3
14	116294673	1	4	10	0	2	4
14	4614655500	3	4	8	1	2	5
14	18163000896	5	4	6	2	2	6
14	116294673	1	10	4	0	5	7
14	4614655500	3	8	4	1	4	8
14	18163000896	5	6	4	2	3	9
14	11627894784	7	4	4	3	2	10
14	896803	1	2	12	0	1	11
14	234158760	3	2	10	1	1	12
14	2679876768	5	2	8	2	1	13
14	4210012160	7	2	6	3	1	14
14	1050465024	9	2	4	4	1	15
14	896803	1	12	2	0	6	16
14	234158760	3	10	2	1	5	17
14	2679876768	5	8	2	2	4	18
14	4210012160	7	6	2	3	3	19
14	1050465024	9	4	2	4	2	20
14	25135104	11	2	2	5	1	21
14	1	1	0	14	0	0	22
14	298932	3	0	12	1	0	23
14	22945056	5	0	10	2	0	24
14	106923008	7	0	8	3	0	25
14	65008896	9	0	6	4	0	26
14	4180992	11	0	4	5	0	27
14	4096	13	0	2	6	0	28
14	1	1	14	0	7	0	29
14	298932	3	12	0	1	6	30
14	22945056	5	10	0	2	5	31
14	106923008	7	8	0	3	4	32
14	65008896	9	6	0	4	3	33
14	4180992	11	4	0	5	2	34
14	4096	13	2	0	6	1	35

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cyclic odd	Peaks even	Counter
15	834687179	0	7	9	0	3	1
15	834687179	0	9	7	0	4	2
15	47310326336	2	7	7	1	3	3
15	116294673	0	5	11	0	2	4
15	20015036304	2	5	9	1	2	5
15	195642911424	4	5	7	2	2	6
15	116294673	0	11	5	0	5	7
15	20015036304	2	9	5	1	4	8
15	195642911424	4	7	5	2	3	9
15	299351274240	6	5	5	3	2	10
15	896803	0	3	13	0	1	11
15	1178416608	2	3	11	1	1	12
15	34199593440	4	3	9	2	1	13
15	123561102848	6	3	7	3	1	14
15	81225837312	8	3	5	4	1	15
15	896803	0	13	3	0	6	16
15	1178416608	2	11	3	1	5	17
15	34199593440	4	9	3	2	4	18
15	123561102848	6	7	3	3	3	19
15	81225837312	8	5	3	4	2	20
15	8680206336	10	3	3	5	1	21
15	1	0	1	15	0	0	22
15	2690416	2	1	13	1	0	23
15	586629984	4	1	11	2	0	24
15	6337665152	6	1	9	3	0	25
15	9860488448	8	1	7	4	0	26
15	2536974336	10	1	5	5	0	27
15	67047424	12	1	3	6	0	28
15	1	0	15	1	0	7	29
15	2690416	2	13	1	1	6	30
15	586629984	4	11	1	2	5	31
15	6337665152	6	9	1	3	4	32
15	9860488448	8	7	1	4	3	33
15	2536974336	10	5	1	5	2	34
15	67047424	12	3	1	6	1	35
15	16384	14	1	1	7	0	36
16	109645021894	1	8	8	0	4	1
16	47152124264	1	6	10	0	3	2
16	1293879256784	3	6	5	1	3	3
16	47152124264	1	10	6	0	5	4
16	1293879256784	3	8	6	1	4	5
16	4535108405376	5	6	6	2	3	6
16	2949965020	1	4	12	0	2	7
16	249836137968	3	4	10	1	2	8
16	2027377515168	5	4	6	2	2	9
16	3011490789632	7	4	6	3	2	10
16	2949965020	1	12	4	0	6	11
16	249836137968	3	10	4	1	5	12
16	2027377515168	5	8	4	2	4	13
16	3011490789632	7	6	4	3	3	14
16	899060436480	9	4	4	4	2	15
16	8071256	1	2	14	0	1	16
16	5916745168	3	2	12	1	1	17
16	147077701056	5	2	10	2	1	18
16	506606202496	7	2	8	3	1	19
16	338070674432	9	2	6	4	1	20
16	39530059776	11	2	4	5	1	21

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic Peaks odd even	Counter
16	8071256	1	14	2	0 7	22
16	5016745198	3	12	2	1 6	23
16	147077701056	5	10	2	2 5	24
16	506606202496	7	8	2	3 4	25
16	338070674432	9	6	2	4 3	26
16	39530059776	11	4	2	5 2	27
16	402513920	13	2	2	6 1	28
16	1	1	0	16	0 0	29
16	2690416	3	0	14	1 0	30
16	586629984	5	0	12	2 0	31
16	6337665152	7	0	10	3 0	32
16	9860488448	9	0	8	4 0	33
16	2536974336	11	0	6	5 0	34
16	67047424	13	0	4	6 0	35
16	16384	15	0	2	7 0	36
16	1	1	16	0	0 8	37
16	2690416	3	14	0	1 7	38
16	586629984	5	12	0	2 6	39
16	6337665152	7	10	0	3 5	40
16	9860488448	9	8	0	4 4	41
16	2536974336	11	6	0	5 3	42
16	67047424	13	4	0	6 2	43
16	16384	15	2	0	7 1	44
17	109645021894	0	9	9	0 4	1
17	47152124264	0	7	11	0 3	2
17	5230319188144	2	7	9	1 3	3
17	47152124264	0	11	7	0 5	4
17	5230319188144	2	9	7	1 4	5
17	43377610135424	4	7	7	2 3	6
17	2949965020	0	5	13	0 2	7
17	1067820739728	2	5	11	1 2	8
17	20398524496224	4	5	9	2 2	9
17	64510106081024	6	5	7	3 2	10
17	2949965020	0	13	5	0 6	11
17	1067820739728	2	11	5	1 5	12
17	20398524496224	4	9	5	2 4	13
17	64510106081024	6	7	5	3 3	14
17	44229433403904	8	5	5	4 2	15
17	8071256	0	3	15	0 1	16
17	29663093168	2	3	13	1 1	17
17	1805733999168	4	3	11	2 1	18
17	13126530488704	6	3	9	3 1	19
17	19141448848384	8	3	7	4 1	20
17	6059496450048	10	3	5	5 1	21
17	8071256	0	15	3	0 7	22
17	29663093168	2	13	3	1 6	23
17	1805733999168	4	11	3	2 5	24
17	13126530488704	6	9	3	3 4	25
17	19141448848384	8	7	3	4 3	26
17	6059496450048	10	5	3	5 2	27
17	321473159168	12	3	3	6 1	28
17	1	0	1	17	0 0	29
17	24213776	2	1	15	1 0	30
17	14804306080	4	1	13	2 0	31
17	345558617984	6	1	11	3 0	32

TABLE 2.2 (cont.)

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n	$(a_{1,k_1 k_2 k_3})_n = (\mathbb{X}(\text{CP}))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counted
17	1165333452544	8	1	9	4	0	33
17	782931974144	10	1	7	5	0	34
17	95153582080	12	1	5	6	0	35
17	1073463296	14	1	3	7	0	36
17	1	0	17	1	0	8	37
17	24213776	2	15	1	1	7	38
17	14804306080	4	13	1	2	6	39
17	345558617984	6	11	1	3	5	40
17	1165333452544	8	9	1	4	4	41
17	782931974144	10	7	1	5	3	42
17	95153582080	12	5	1	6	2	43
17	1073463296	14	3	1	7	1	44
17	65536	16	1	1	8	0	45
18	11966116940238	1	8	10	0	4	1
18	11966116940238	1	10	8	0	5	2
18	267656185928288	3	8	8	1	4	3
18	2504055894564	1	6	12	0	3	4
18	129952360438912	3	6	10	1	3	5
18	874290627900128	5	6	8	2	3	6
18	2504055894564	1	12	5	0	6	7
18	129952360438912	3	10	6	1	5	8
18	874290627900128	5	8	6	2	4	9
18	1256976952365568	7	6	6	3	3	10
18	74197080276	1	4	14	0	2	11
18	12947659906496	3	4	12	1	2	12
18	200614879404192	5	4	10	2	2	13
18	593820895590528	7	4	8	3	2	14
18	415732273458688	9	4	6	4	2	15
18	74197080276	1	14	4	0	7	16
18	12947659906496	3	12	4	1	6	17
18	200614879404192	5	10	4	2	5	18
18	593820895590528	7	8	4	3	4	19
18	415732273458688	9	6	4	4	3	20
18	64452642410496	11	4	4	5	2	21
18	72641337	1	2	16	0	1	22
18	148569710464	3	2	14	1	1	23
18	7683009684448	5	2	12	2	1	24
18	52503403884288	7	2	10	3	1	25
18	75741667359488	9	2	8	4	1	26
18	24800856154112	11	2	6	5	1	27
18	1455215874048	13	2	4	6	1	28
18	72641337	1	16	2	0	8	29
18	148569710464	3	14	2	1	7	30
18	7683009684448	5	12	2	2	6	31
18	52503403884288	7	10	2	3	5	32
18	75741667359488	9	8	2	4	4	33
18	24800856154112	11	6	2	5	3	34
18	1455215874048	13	4	2	6	2	35
18	6441828352	15	2	2	7	1	36
18		1	0	18	0	0	37

TABLE 2.2 (cont.)

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n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counter
18	24213776	3	0	16	1	0	38
18	14804306080	5	0	14	2	0	39
18	345558617984	7	0	12	3	0	40
18	1165333452544	9	0	10	4	0	41
18	782931974144	11	0	8	5	0	42
18	95153582080	13	0	6	6	0	43
18	1073463296	15	0	4	7	0	44
18	65536	17	0	2	8	0	45
18	1	1	18	0	0	9	46
18	24213776	3	16	0	1	8	47
18	14804306080	5	14	0	2	7	48
18	345558617984	7	12	0	3	6	49
18	1165333452544	9	10	0	4	5	50
18	782931974144	11	8	0	5	4	51
18	95153582080	13	6	0	6	3	52
18	1073463296	15	4	0	7	2	53
18	65536	17	2	0	8	1	54
19	11966116940238	0	9	11	0	4	1
19	11966116940238	0	11	9	0	5	2
19	1042290896589624	2	9	9	1	4	3
19	2504055894564	0	7	13	0	3	4
19	515634687573408	2	7	11	1	3	5
19	7812226231316064	4	7	9	2	3	6
19	2504055894564	0	13	7	0	6	7
19	515634687573408	2	11	7	1	5	8
19	7812226231316064	4	9	7	2	4	9
19	22787488712961024	6	7	7	3	3	10
19	74197080276	0	5	15	0	2	11
19	54906074210736	2	5	13	1	2	12
19	1938160478532384	4	5	11	2	2	13
19	11408638830576384	6	5	9	3	2	14
19	16034019340045824	8	5	7	4	2	15
19	74197080276	0	15	5	0	7	16
19	54906074210736	2	13	5	1	6	17
19	1938160478532384	4	11	5	2	5	18
19	11408638830576384	6	9	5	3	4	19
19	16034019340045824	8	7	5	4	3	20
19	5697766348019712	10	5	5	5	2	21
19	72641337	0	3	17	0	1	22
19	743659713888	2	3	15	1	1	23
19	92285663994720	4	3	13	2	1	24
19	1262179461020160	6	3	11	3	1	25
19	3581992627440384	8	3	9	4	1	26
19	2541671850405888	10	3	7	5	1	27
19	425533512929280	12	3	5	6	1	28
19	72641337	0	17	3	0	8	29
19	743659713888	2	15	3	1	7	30
19	92285663994720	4	13	3	2	6	31
19	1262179461020160	6	11	3	3	5	32
19	3581992627440384	8	9	3	4	4	33

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(C_i))_n$	k_1	k_2	k_3	Cyclic Peaks odd even	Counte
19	2541671850405888	10	7	3	5 3	34
19	425533512929280	12	5	3	6 2	35
19	11738354417664	14	3	3	7 1	36
19	1	0	1	19	0 0	37
19	217924020	2	1	17	1 0	38
19	371548371744	4	1	15	2 0	39
19	17992189979904	6	1	13	3 0	40
19	119641512257280	8	1	11	4 0	41
19	171748920960000	10	1	9	5 0	42
19	57102164668416	12	1	7	6 0	43
19	3497455190016	14	1	5	7 0	44
19	17178624000	16	1	3	8 0	45
19	1	0	19	1	0 9	46
19	217924020	2	17	1	1 8	47
19	371548371744	4	15	1	2 7	48
19	17992189979904	6	13	1	3 6	49
19	119641512257280	8	11	1	4 5	50
19	171748920960000	10	9	1	5 4	51
19	57102164668416	12	7	1	6 3	52
19	3497455190016	14	5	1	7 2	53
19	17178624000	16	3	1	8 1	54
19	262144	18	1	1	9 0	55
20	2347836365864484	1	10	10	0 5	1
20	1171517154238290	1	8	12	0 4	2
20	46301504557878360	3	8	10	1 4	3
20	1171517154238290	1	12	8	0 6	4
20	46301504557878360	3	10	8	1 5	5
20	277345004441455296	5	8	8	2 4	6
20	128453495887560	1	6	14	0 3	7
20	12075863691882960	3	6	12	1 3	8
20	144457181866526976	5	6	10	2 3	9
20	390462325186281216	7	6	8	3 3	10
20	128453495887560	1	14	6	0 7	11
20	12075863691882960	3	12	6	1 6	12
20	144457181866526976	5	10	6	2 5	13
20	390462325186281216	7	8	6	3 4	14
20	281453934240838656	9	6	6	4 3	15
20	1659539731885	1	4	16	0 2	16
20	654827922740880	3	4	14	1 2	17
20	18463592790714240	5	4	12	2 2	18
20	99583109243626752	7	4	10	3 2	19
20	137824748851251456	9	4	8	4 2	20
20	51386936848091136	11	4	6	5 2	21
20	1859539731885	1	16	4	0 8	22
20	654827922740880	3	14	4	1 7	23
20	18463592790714240	5	12	4	2 6	24
20	99583109243626752	7	10	4	3 5	25
20	137824748851251456	9	8	4	4 4	26
20	51386936848091136	11	6	4	5 3	27
20	4419672091140096	13	4	4	6 2	28

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cyclic odd	Peaks even	Counter
20	653772070	1	2	18	0	1	29
20	3720877336980	3	2	16	1	1	30
20	390383357439744	5	2	14	2	1	31
20	4977568950857472	7	2	12	3	1	32
20	13779523726751232	9	2	10	4	1	33
20	9855981815878656	11	2	8	5	1	34
20	1725280064126976	13	2	6	6	1	35
20	52977197187072	15	2	4	7	1	36
20	653772070	1	18	2	0	9	37
20	3720877336980	3	16	2	1	8	38
20	390383357439744	5	14	2	2	7	39
20	4977568950857472	7	12	2	3	6	40
20	13779523726751232	9	10	2	4	5	41
20	9855981815878656	11	8	2	5	4	42
20	1725280064126976	13	6	2	6	3	43
20	52977197187072	15	4	2	7	2	44
20	103076462592	17	2	2	8	1	45
20	1	1	0	20	0	0	46
20	217924020	3	0	18	1	0	47
20	371548371744	5	0	16	2	0	48
20	17992189979904	7	0	14	3	0	49
20	119641512257280	9	0	12	4	0	50
20	171748920960000	11	0	10	5	0	51
20	57102164668416	13	0	8	6	0	52
20	3497455190016	15	0	6	7	0	53
20	17178624000	17	0	4	8	0	54
20	262144	19	0	2	9	0	55
20	1	1	20	0	0	10	56
20	217924020	3	18	0	1	9	57
20	371548371744	5	16	0	2	8	58
20	17992189979904	7	14	0	3	7	59
20	119641512257280	9	12	0	4	6	60
20	171748920960000	11	10	0	5	5	61
20	57102164668416	13	8	0	6	4	62
20	3497455190016	15	6	0	7	3	63
20	17178624000	17	4	0	8	2	64
20	262144	19	2	0	9	1	65
21	2347836365864484	0	11	11	0	5	1
21	1171517154238290	0	9	13	0	4	2
21	176441083183139400	2	9	11	1	4	3
21	1171517154238290	0	13	9	0	6	4
21	176441083183139400	2	11	9	1	5	5
21	2312755113364843680	4	9	9	2	4	6
21	128453495887560	0	7	15	0	3	7
21	47398077251981040	2	7	13	1	3	8
21	1237608310098257280	4	7	11	2	3	9
21	6396568130500880640	6	7	9	3	3	10
21	128453495887560	0	15	7	0	7	11
21	47398077251981040	2	13	7	1	6	12
21	1237608310098257280	4	11	7	2	5	13

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counter
21	6396568130500880640	6	9	7	3	4	14
21	8780482611148047360	8	7	7	4	3	15
21	1859539731885	0	5	17	0	2	16
21	2764957379258160	2	5	15	1	2	17
21	173940737023241280	4	5	13	2	2	18
21	1785387969393120000	6	5	11	3	2	19
21	4579027783215217920	8	5	9	4	2	20
21	3356577901584046080	10	5	7	5	2	21
21	1859539731885	0	17	5	0	8	22
21	2764957379258160	2	15	5	1	7	23
21	173940737023241280	4	13	5	2	6	24
21	1785387969393120000	6	11	5	3	5	25
21	4579027783215217920	8	9	5	4	4	26
21	3356577901584046080	10	7	5	5	3	27
21	674098979361914880	12	5	5	6	2	28
21	653772070	0	3	19	0	1	29
21	18612558835740	2	3	17	1	1	30
21	4630762515553920	4	3	15	2	1	31
21	114162720823015680	6	3	13	3	1	32
21	582078977925557760	8	3	11	4	1	33
21	797510032647183360	10	3	9	5	1	34
21	306824242753044480	12	3	7	6	1	35
21	28825026707128320	14	3	5	7	1	36
21	653772070	0	19	3	0	9	37
21	18612558835740	2	17	3	1	8	38
21	4630762515553920	4	15	3	2	7	39
21	114162720823015680	6	13	3	3	6	40
21	582078977925557760	8	11	3	4	5	41
21	797510032647183360	10	9	3	5	4	42
21	306824242753044480	12	7	3	6	3	43
21	28825026707128320	14	5	3	7	2	44
21	425569877360640	16	3	3	8	1	45
21	1	0	1	21	0	0	46
21	1961316220	2	1	19	1	0	47
21	9303419165040	4	1	17	2	0	48
21	912656818686720	6	1	15	3	0	49
21	11283802171749120	8	1	13	4	0	50
21	30883983731149824	10	1	11	5	0	51
21	22171780982046720	12	1	9	6	0	52
21	3959839273451520	14	1	7	7	0	53
21	127231162122240	16	1	5	8	0	54
21	274872401920	18	1	3	9	0	55
21	1	0	21	1	0	10	56
21	1961316220	2	19	1	1	9	57
21	9303419165040	4	17	1	2	8	58
21	912656818686720	6	15	1	3	7	59
21	11283802171749120	8	13	1	4	6	60
21	30883983731149824	10	11	1	5	5	61
21	22171780982046720	12	9	1	6	4	62
21	3959839273451520	14	7	1	7	3	63
21	127231162122240	16	5	1	8	2	64
21	274872401920	18	3	1	9	1	65
21	1048576	20	1	1	10	0	66

TABLE 2.2(cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cyclic odd	Peaks even	Counter
22	393938089395885894	1	10	12	0	5	1
22	393938089395885894	1	12	10	0	6	2
22	13132724283488441520	3	10	10	1	5	3
22	107266611330420090	1	8	14	0	4	4
22	7154577993317037240	3	8	12	1	4	5
22	72807896214369707040	5	8	10	2	4	6
22	107266611330420090	1	14	8	0	7	7
22	7154577993317037240	3	12	8	1	6	8
22	72807896214369707040	5	10	8	2	5	9
22	185382087238200230400	7	8	8	3	4	10
22	6460701405171285	1	6	16	0	3	11
22	1069023849545704800	3	6	14	1	3	12
22	21636815568348657600	5	6	12	2	3	13
22	1 104746684255227840	7	6	10	3	3	14
22	136240407342813753600	9	6	8	4	3	15
22	6460701405171285	1	16	6	0	8	16
22	1069023849545704800	3	14	6	1	7	17
22	21636815568348657600	5	12	6	2	6	18
22	1 104746684255227840	7	10	6	3	5	19
22	136240407342813753600	9	8	6	4	4	20
22	55081278374519623680	11	6	6	5	3	21
22	46535238000235	1	4	18	0	2	22
22	32664250458714060	3	4	16	1	2	23
22	1624141447787609280	5	4	14	2	2	24
22	15067687041069265920	7	4	12	3	2	25
22	37273107999729058560	9	4	10	4	2	26
22	27642370714781414400	11	4	8	5	2	27
22	5921814969980682240	13	4	6	6	2	28
22	46535238000235	1	18	4	0	9	29
22	32664250458714060	3	16	4	1	8	30
22	1624141447787609280	5	14	4	2	7	31
22	15067687041069265920	7	12	4	3	6	32
22	37273107999729058560	9	10	4	4	5	33
22	27642370714781414400	11	8	4	5	4	34
22	5921814969980682240	13	6	4	6	3	35
22	295059385109053440	15	4	4	7	2	36
22	5883948671	1	2	20	0	1	37
22	93088618175560	3	2	18	1	1	38
22	19526386584587760	5	2	16	2	1	39
22	446448432123340800	7	2	14	3	1	40
22	2201766199320910080	9	2	12	4	1	41
22	2998315290768758784	11	2	10	5	1	42
22	1175456506925875200	13	2	8	6	1	43
22	116229653629501440	15	2	6	7	1	44
22	1917813145927680	17	2	4	8	1	45
22	5883948671	1	20	2	0	10	46
22	93088618175560	3	18	2	1	9	47
22	19526386584587760	5	16	2	2	8	48
22	446448432123340800	7	14	2	3	7	49
22	2201766199320910080	9	12	2	4	6	50
22	2998315290768758784	11	10	2	5	5	51
22	1175456506925875200	13	8	2	6	4	52
22	116229653629501440	15	6	2	7	3	53
22	1917813145927680	17	4	2	8	2	54
22	1649255383040	19	2	2	9	1	55
22	1	1	0	22	0	0	56
22	1961316220	3	0	20	1	0	57

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cyclic odd	Peaks even	Counter
22	9303419165040	5	0	18	2	0	58
22	912656818686720	7	0	16	3	0	59
22	11283802171749120	9	0	14	4	0	60
22	30883983731149824	11	0	12	5	0	61
22	22171780982046720	13	0	10	6	0	62
22	3959839273451520	15	0	8	7	0	63
22	127231162122240	17	0	6	8	0	64
22	274872401920	19	0	4	9	0	65
22	1048576	21	0	2	10	0	66
22	1	1	22	0	0	11	67
22	1901316220	3	20	0	1	10	68
22	9303419165040	5	18	0	2	9	69
22	912656818686720	7	16	0	3	8	70
22	11283802171749120	9	14	0	4	7	71
22	30883983731149824	11	12	0	5	6	72
22	22171780982046720	13	10	0	6	5	73
22	3959839273451520	15	8	0	7	4	74
22	127231162122240	17	6	0	8	3	75
22	274872401920	19	4	0	9	2	76
22	1048576	21	2	0	10	1	77
23	393938089395885894	0	11	13	0	5	1
23	393938089395885894	0	13	11	0	6	2
23	48852686995966586016	2	11	11	1	5	3
23	107266611330420090	0	9	15	0	4	4
23	26904847432535851920	2	9	13	1	4	5
23	581221659826537397280	4	9	11	2	4	6
23	107266611330420090	0	15	9	0	7	7
23	26904847432535851920	2	13	9	1	6	8
23	581221659826537397280	4	11	9	2	5	9
23	2753832534954795753600	6	9	9	3	4	10
23	6460701405171285	0	7	17	0	3	11
23	4168575661763215680	2	7	15	1	3	12
23	180387035681919453120	4	7	13	2	3	13
23	1549437224433007142400	6	7	11	3	3	14
23	3719695032416447904000	8	7	9	4	3	15
23	6460701405171285	0	17	7	0	8	16
23	4168575661763215680	2	15	7	1	7	17
23	180387035681919453120	4	13	7	2	6	18
23	1549437224433007142400	6	11	7	3	5	19
23	3719695032416447904000	8	9	7	4	4	20
23	2785740579604735918080	10	7	7	5	3	21
23	46535238000235	0	5	19	0	2	22
23	137594594091174120	2	5	17	1	2	23
23	15057478343551700160	4	5	15	2	2	24
23	258032682966603336960	6	5	13	3	2	25
23	1122555017545706085120	8	5	11	4	2	26
23	1494239601916768665600	10	5	9	5	2	27
23	628610230575117926400	12	5	7	6	2	28
23	46535238000235	0	19	5	0	9	29
23	137594594091174120	2	17	5	1	8	30
23	15057478343551700160	4	15	5	2	7	31
23	258032682966603336960	6	13	5	3	6	32
23	1122555017545706085120	8	11	5	4	5	33
23	1494239601916768665600	10	9	5	5	4	34
23	628610230575117926400	12	7	5	6	3	35
23	75487670416403986400	14	5	5	7	2	36

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even
23	5883948671	0	3	21	0	1
23	465524485501040	2	3	19	1	1
23	229964529864955120	4	3	17	2	1
23	9934127001367226880	6	3	15	3	1
23	86336922007892025600	8	3	13	4	1
23	208495094589223501824	10	3	11	5	1
23	155833570356849623040	12	3	9	6	1
23	34834356739772252160	14	3	7	7	1
23	1910218285693992960	16	3	5	8	1
23	5883948671	0	21	3	0	10
23	465524485501040	2	19	3	1	9
23	229964529864955120	4	17	3	2	8
23	9934127001367226880	6	15	3	3	7
23	86336922007892025600	8	13	3	4	6
23	208495094589223501824	10	11	3	5	5
23	155833570356849623040	12	9	3	6	4
23	34834356739772252160	14	7	3	7	3
23	1910218285693992960	16	5	3	8	2
23	15373841019699200	18	3	3	9	1
23	1	0	1	23	0	0
23	17651846024	2	1	21	1	0
23	232733558500720	4	1	19	2	0
23	45608832444953280	6	1	17	3	0
23	1009053592891411200	8	1	15	4	0
23	4901229450088955904	10	1	13	5	0
23	6655471539077922816	12	1	11	6	0
23	2632028412773990400	14	1	9	7	0
23	266300951202693120	16	1	7	8	0
23	4604235840225280	18	1	5	9	0
23	4398022393856	20	1	3	10	0
23	1	0	23	1	0	11
23	17651846024	2	21	1	1	10
23	232733558500720	4	19	1	2	9
23	45608832444953280	6	17	1	3	8
23	1009053592891411200	8	15	1	4	7
23	4901229450088955904	10	13	1	5	6
23	6655471539077922816	12	11	1	6	5
23	2632028412773990400	14	9	1	7	4
23	266300951202693120	16	7	1	8	3
23	4604235840225280	18	5	1	9	2
23	4398022393856	20	3	1	10	1
23	4194304	22	1	1	11	0
24	107947764316226205276	1	12	12	0	6
24	59752013018382750024	1	10	14	0	5
24	3212029212884748110256	3	10	12	1	5
24	59752013018382750024	1	14	10	0	7
24	3212029212884748110256	3	12	10	1	6
24	29309871725912597261760	5	10	10	2	5
24	9412382749388124015	1	8	16	0	4
24	1026220404546948714960	3	8	14	1	4
24	16872649748901832320480	5	8	12	2	4
24	71585862542687823580800	7	8	10	3	4
24	9412382749388124015	1	16	8	0	8
24	1026220404546948714960	3	14	8	1	7
24	16872649748901832320480	5	12	8	2	6
24	71585862542687823580800	7	10	8	3	5
24	94811916379543021452800	9	8	8	4	4

TABLE 2.2 (cont.)

n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counter
24	321298267540551700	1	6	18	0	3	16
24	91749051106099270440	3	6	16	1	3	17
24	3036767522726331696000	5	6	14	2	3	18
24	23180925589962542058240	7	6	12	3	3	19
24	53328366439085588920320	9	6	10	4	3	20
24	40491663241385484533760	11	6	8	5	3	21
24	321298267540551700	1	18	6	0	9	22
24	91749051106099270440	3	16	6	1	8	23
24	3036767522726331696000	5	14	6	2	7	24
24	23180925589962542058240	7	12	6	3	6	25
24	53328366439085588920320	9	10	6	4	5	26
24	40491663241385484533760	11	8	6	5	4	27
24	9857370613881306808320	13	6	6	6	3	28
24	1163848723925346	1	4	20	0	2	29
24	1616676055220210840	3	4	18	1	2	30
24	138801550734006099120	5	4	16	2	2	31
24	2129870695916661292800	7	4	14	3	2	32
24	8820106019723361776640	9	4	12	4	2	33
24	11634646894347497324544	11	4	10	5	2	34
24	5033234280444047769600	13	4	8	6	2	35
24	651842341831529594880	15	4	6	7	2	36
24	1163848723925346	1	20	4	0	10	37
24	1616676055220210840	3	18	4	1	9	38
24	138801550734006099120	5	16	4	2	8	39
24	2129870695916661292800	7	14	4	3	7	40
24	8820106019723361776640	9	12	4	4	6	41
24	11634646894347497324544	11	10	4	5	5	42
24	5033234280444047769600	13	8	4	6	4	43
24	651842341831529594880	15	6	4	7	3	44
24	19378911995294515200	17	4	4	8	2	45
24	52955538084	1	2	22	0	1	46
24	2327878379272504	3	2	20	1	1	47
24	967968521936098720	5	2	18	2	1	48
24	38650159898797176000	7	2	16	3	1	49
24	323158864417936803840	9	2	14	4	1	50
24	769066925087762006016	11	2	12	5	1	51
24	577559295779241885696	13	2	10	6	1	52
24	132452141153525760000	15	2	8	7	1	53
24	7677637760624885760	17	2	6	8	1	54
24	69230662708101120	19	2	4	9	1	55
24	52955538084	1	22	2	0	11	56
24	2327878379272504	3	20	2	1	10	57
24	967968521936098720	5	18	2	2	9	58
24	38650159898797176000	7	16	2	3	8	59
24	323158864417936803840	9	14	2	4	7	60
24	769066925087762006016	11	12	2	5	6	61
24	577559295779241885696	13	10	2	6	5	62
24	132452141153525760000	15	8	2	7	4	63
24	7677637760624885760	17	6	2	8	3	64
24	69230662708101120	19	4	2	9	2	65
24	26386226637824	21	2	2	10	1	66
24	1	1	0	24	0	0	67
24	17651846024	3	0	22	1	0	68
24	232733558500720	5	0	20	2	0	69
24	45608832444953280	7	0	18	3	0	70
24	1009053592691411200	9	0	16	4	0	71
24	4901229450088955904	11	0	14	5	0	72
24	6655471539077922816	13	0	12	6	0	73

TABLE 2.2 (cont.)

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n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k_1	k_2	k_3	Cyclic odd	Peaks even	Counter
24	2632028412773990400	15	0	10	7	0	74
24	266300951202693120	17	0	8	8	0	75
24	4604235840225280	19	0	6	9	0	76
24	4398022393856	21	0	4	10	0	77
24	4194304	23	0	2	11	0	78
24	1	1	24	0	0	12	79
24	17651846024	3	22	0	1	11	80
24	232733558500720	5	20	0	2	10	81
24	45608832444953280	7	18	0	3	9	82
24	1009053592891411200	9	16	0	4	8	83
24	4901229450088955904	11	14	0	5	7	84
24	6655471539077922816	13	12	0	6	6	85
24	2632028412773990400	15	10	0	7	5	86
24	266300951202693120	17	8	0	8	4	87
24	4604235840225280	19	6	0	9	3	88
24	4398022393856	21	4	0	10	2	89
24	4194304	23	2	0	11	1	90
25	1 7947764316226205276	0	13	13	0	6	1
25	59752013018382750024	0	11	15	0	5	2
25	11767988992706317294416	2	11	13	1	5	3
25	59752013018382750024	0	15	11	0	7	4
25	11767988992706317294416	2	13	11	1	6	5
25	223638059738796940954944	4	11	11	2	5	6
25	9412382749388124015	0	9	17	0	4	7
25	3826779467814883629360	2	9	15	1	4	8
25	130850626537013924714400	4	9	13	2	4	9
25	996671552044762725528960	6	9	11	3	4	10
25	9412382749388124015	0	17	9	0	8	11
25	3826779467814883629360	2	15	9	1	7	12
25	130850626537013924714400	4	13	9	2	6	13
25	996671552044762725528960	6	11	9	3	5	14
25	2285024498269647264691200	8	9	9	4	4	15
25	321298267540551700	0	7	19	0	3	16
25	356329584129132734040	2	7	17	1	3	17
25	24861585667704836526720	4	7	15	2	3	18
25	339762422439121096715520	6	7	13	3	3	19
25	1330813305372823393628160	8	7	11	4	3	20
25	1737187291082443590696960	10	7	9	5	3	21
25	321298267540551700	0	19	7	0	0	22
25	356329584129132734040	2	17	7	1	8	23
25	24861585667704836526720	4	15	7	2	7	24
25	339762422439121096715520	6	13	7	3	6	25
25	1330813305372823393628160	8	11	7	4	5	26
25	1737187291082443590696960	10	9	7	5	4	27
25	776012429842624741048320	12	7	7	6	3	28
25	1163848723925346	0	5	21	0	2	29
25	6801094745382449640	2	5	19	1	2	30
25	1273602229300589913360	4	5	17	2	2	31
25	35350524819518716811520	6	5	15	3	2	32
25	248284697460118766436400	8	5	13	4	2	33
25	553792586709016345411584	10	5	11	5	2	34
25	424728494037560501452800	12	5	9	6	2	35
25	109187733054313166929920	14	5	7	7	2	36

TABLE 2.2 (cont.)

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n	$(a_{1,k_1 k_2 k_3})_n = (N(CP))_n$	k ₁	k ₂	k ₃	Cyclic odd	Peaks even	Counter
25	1163848723925346	0	21	5	0	10	37
25	6801094745382449640	2	19	5	1	9	38
25	1273602229300589913360	4	17	5	2	8	39
25	35350524819518716811520	6	15	5	3	7	40
25	248284697460118766438400	8	13	5	4	6	41
25	553792586709016345411584	10	11	5	5	5	42
25	424728494037560501452800	12	9	5	6	4	43
25	109187733054313166929920	14	7	5	7	3	44
25	8151549605898361696960	16	5	5	8	2	45
25	52955538084	0	3	23	0	1	46
25	11640195055356744	2	3	21	1	1	47
25	11353104398146787040	4	3	19	2	1	48
25	843180755622454405440	6	3	17	3	1	49
25	12046315121808831221760	8	3	15	4	1	50
25	48264384356709944426496	10	3	13	5	1	51
25	63275661523573277884416	12	3	11	6	1	52
25	27895312196871496335360	14	3	9	7	1	53
25	3797506338484947517440	16	3	7	8	1	54
25	124896857136381296640	18	3	5	9	1	55
25	52955538084	0	23	3	0	11	56
25	11640195055356744	2	21	3	1	10	57
25	11353104398146787040	4	19	3	2	9	58
25	843180755622454405440	6	17	3	3	8	59
25	12046315121808831221760	8	15	3	4	7	60
25	48264384356709944426496	10	13	3	5	6	61
25	63275661523573277884416	12	11	3	6	5	62
25	27895312196871496335360	14	9	3	7	4	63
25	3797506338484947517440	16	7	3	8	3	64
25	124896857136381296640	18	5	3	9	2	65
25	554399454424203264	20	3	3	10	1	66
25	1	0	1	25	0	0	67
25	158866614264	2	1	23	1	0	68
25	5819812891661136	4	1	21	2	0	69
25	2259853542156884800	6	1	19	3	0	70
25	87202761117626211840	8	1	17	4	0	71
25	716376110273114701824	10	1	15	5	0	72
25	1693272192484782391296	12	1	13	6	0	73
25	1274464676219028701184	14	1	11	7	0	74
25	295751682605237207040	16	1	9	8	0	75
25	17573163611835596800	18	1	7	9	0	76
25	166179098927824896	20	1	5	10	0	77
25	70368639320064	22	1	3	11	0	78
25	1	0	25	1	0	12	79
25	158866614264	2	23	1	1	11	80
25	5819812891661136	4	21	1	2	10	81
25	2259853542156884800	6	19	1	3	9	82
25	87202761117626211840	8	17	1	4	8	83
25	716376110273114701824	10	15	1	5	7	84
25	1693272192484782391296	12	13	1	6	6	85
25	1274464676219028701184	14	11	1	7	5	86
25	295751682605237207040	16	9	1	8	4	87
25	17573163611835596800	18	7	1	9	3	88
25	166179098927824896	20	5	1	10	2	89
25	70368639320064	22	3	1	11	1	90
25	16777216	24	1	1	12	0	91

It was shown /SC76/ that

$$(a_{1,k_1=0,k_2k_3})_{n-\text{odd}} \quad \text{or} \quad (a_{1,k_1=1,k_2k_3})_{n-\text{even}}$$

ARE THE TAYLOR SERIES COEFFICIENTS OF THE JACOBIAN ELLIPTIC
FUNCTION $\text{sn}(u, k)$:

$$\text{sn}(u, k) = (-1)^{n/2} \sum_{n=0}^{\infty} \frac{u^{n+1}}{(n+1)!} \left(\sum_{\substack{k_1=0 \\ k_2=0}}^n a_{1,k_1=1,k_2k_3} \cdot k^{k_2} \right), \quad n-\text{even},$$

or

$$\text{sn}(u, k) = (-1)^{(n-1)/2} \sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\sum_{k_2=1}^n a_{1,k_1=0,k_2k_3} \cdot k^{k_2-1} \right), \quad n-\text{odd}.$$

(2.11.3)

We tabulate these coefficients for n -even in the
subsequent TABLE 2.3.

TABLE 2.3: Taylor series coefficients $(a_{1,k_1=1,k_2k_3})_n$ of the Jacobian elliptic functions $\text{sn}(u,k)$, (2.11.3), n=odd.
 $n=1, 3, 5, \dots, 49.$

$$n \ (a_{1,k_1=1,k_2k_3})_n \ k_2 \ k_3$$

$$1 \quad 1 \quad 1$$

$$3 \quad 1 \quad 3 \quad 1$$

$$1 \quad 1 \quad 3$$

$$5 \quad 1 \quad 5 \quad 1$$

$$14 \quad 3 \quad 3$$

$$1 \quad 1 \quad 5$$

$$7 \quad 1 \quad 7 \quad 1$$

$$135 \quad 5 \quad 3$$

$$135 \quad 3 \quad 5$$

$$1 \quad 1 \quad 7$$

$$9 \quad 1 \quad 9 \quad 1$$

$$1228 \quad 7 \quad 3$$

$$5478 \quad 5 \quad 5$$

$$1228 \quad 3 \quad 7$$

$$1 \quad 1 \quad 9$$

$$11 \quad 1 \quad 11 \quad 1$$

$$11069 \quad 9 \quad 3$$

$$165826 \quad 7 \quad 5$$

$$165826 \quad 5 \quad 7$$

$$11069 \quad 3 \quad 9$$

$$1 \quad 1 \quad 11$$

$$13 \quad 1 \quad 13 \quad 1$$

$$99642 \quad 11 \quad 3$$

$$4494351 \quad 9 \quad 5$$

$$13180268 \quad 7 \quad 7$$

$$4494351 \quad 5 \quad 9$$

$$99642 \quad 3 \quad 11$$

$$1 \quad 1 \quad 13$$

$$n \ (a_{1,k_1=1,k_2k_3})_n \ k_2 \ k_3$$

$$15 \quad 1 \quad 15 \quad 1$$

$$896803 \quad 13 \quad 3$$

$$116294673 \quad 11 \quad 5$$

$$834687179 \quad 9 \quad 7$$

$$834687179 \quad 7 \quad 9$$

$$116294673 \quad 5 \quad 11$$

$$896803 \quad 3 \quad 13$$

$$1 \quad 1 \quad 15$$

$$-----$$

$$5 \quad 1 \quad 17 \quad 1$$

$$8071256 \quad 15 \quad 3$$

$$2949965020 \quad 13 \quad 5$$

$$47152124264 \quad 11 \quad 7$$

$$109645021894 \quad 9 \quad 9$$

$$47152124264 \quad 7 \quad 11$$

$$2949965020 \quad 5 \quad 13$$

$$8071256 \quad 3 \quad 15$$

$$1 \quad 1 \quad 17$$

$$-----$$

$$9 \quad 1 \quad 19 \quad 1$$

$$72641337 \quad 17 \quad 3$$

$$74197080276 \quad 15 \quad 5$$

$$2504055894564 \quad 13 \quad 7$$

$$11966116940238 \quad 11 \quad 9$$

$$11966116940238 \quad 9 \quad 11$$

$$2504055894564 \quad 7 \quad 13$$

$$74197080276 \quad 5 \quad 15$$

$$72641337 \quad 3 \quad 17$$

$$1 \quad 1 \quad 19$$

$$-----$$

$$11 \quad 1 \quad 21 \quad 1$$

$$653772070 \quad 19 \quad 3$$

$$1859539731885 \quad 17 \quad 5$$

$$128453495887560 \quad 15 \quad 7$$

$$1171517154238290 \quad 13 \quad 9$$

$$2347836365864484 \quad 11 \quad 11$$

$$1171517154238290 \quad 9 \quad 13$$

$$128453495887560 \quad 7 \quad 15$$

$$1859539731885 \quad 5 \quad 17$$

$$653772070 \quad 3 \quad 19$$

$$1 \quad 1 \quad 21$$

$$-----$$

TABLE 2.3 (cont.)

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n	$(a_{1,k_1=1,k_2k_3})_n$	k_2	k_3
---	--------------------------	-------	-------

23

	1	23	1
5883948671	21	3	
46535238000235	19	5	
6460701405171285	17	7	
107266611330420090	15	9	
393938089395885894	13	11	
393938089395885894	11	13	
107266611330420090	9	15	
6460701405171285	7	17	
46535238000235	5	19	
5883948671	3	21	
	1	1	23

25

	1	25	1
52955538084	23	3	
1163848723925346	21	5	
321298267540551700	19	7	
9412382749388124015	17	9	
59752013018382750024	15	11	
107947764316226205276	13	13	
59752013018382750024	11	15	
9412382749388124015	9	17	
321298267540551700	7	19	
1163848723925346	5	21	
52955538084	3	23	
	1	1	25

27

	1	27	1
476599842805	25	3	
29100851707716150	23	5	
15875718186751193446	21	7	
803475280086029066515	19	9	
8470841585571575617239	17	11	
25835579116799316507780	15	13	
25835579116799316507780	13	15	
8470841585571575617239	11	17	
803475280086029066515	9	19	
15875718186751193446	7	21	
29100851707716150	5	23	
476599842805	3	25	
	1	1	27

TABLE 2.3 (cont.)

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n	$(a_1, k_1=1, k_2 k_3)_n$	k_2	k_3
29			
	1	29	1
	4289398585298	27	3
	727566807977891803	25	5
	781562415106660985428	23	7
	67362921649153881472361	21	9
	1146456994425541774291534	19	11
	5632500127524872577252027	17	13
	9424979520638053300516632	15	15
	5632500127524872577252027	13	17
	1146456994425541774291534	11	19
	67362921649153881472361	9	21
	781562415106660985428	7	23
	727566807977891803	5	25
	4289398585298	3	27
	1	1	29

31		1	31	1
	38604587267739	29	3	
	18189614152200873621	27	5	
	38396599486084770569951	25	7	
	5581153512072331417781229	23	9	
	150221961163114696686151695	21	11	
	1149330973559307337432235521	19	13	
	3051808875538951440990525939	17	15	
	3051808875538951440990525939	15	17	
	1149330973559307337432235521	13	19	
	150221961163114696686151695	11	21	
	5581153512072331417781229	9	23	
	38396599486084770569951	7	25	
	18189614152200873621	5	27	
	38604587267739	3	29	
	1	1	31	

33		1	33	1
	347441285409712	31	3	
	454744658216502193656	29	5	
	1884152729554433297404688	27	7	
	458814920174904775826257436	25	9	
	19239380962379456298762250416	23	11	
	223559382769795167319093086664	21	13	
	906467723949073501017465886864	19	15	
	1429953329302734392093044646982	17	17	
	906467723949073501017465886864	15	19	
	223559382769795167319093086664	13	21	
	19239380962379456298762250416	11	23	
	458814920174904775826257436	9	25	
	1884152729554433297404688	7	27	
	454744658216502193656	5	29	
	347441285409712	3	31	
	1	1	33	

TABLE 2.3 (cont.)

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n	$(a_{1, k_1=1, k_2=k_3})_n$	k_2	k_3
35	1	35	1
	3126971568687473	33	3
	11368657974646161302248	31	5
	92396925087242863212482504	29	7
	37524907781760654616571819884	27	9
	2424371762015227695363084225932	25	11
	41982964485265754951017173213880	23	13
	252583298644057469403578416269848	21	15
	602297594518030428986818986545686	19	17
	602297594518030428986818986545686	17	19
	252583298644057469403578416269848	15	21
	41982964485265754951017173213880	13	23
	2424371762015227695363084225932	11	25
	37524907781760654616571819884	9	27
	92396925087242863212482504	7	29
	11368657974646161302248	5	31
	3126971568687473	3	33
	1	1	35

37

	1	37	1
	28142744118187326	35	3
	284216848055029040209305	33	5
	4529421792220618780953132624	31	7
	3058692313447287528959880082164	29	9
	301977301501927982712251650296648	27	11
	7681155057059283727400087851836804	25	13
	6707798573761183985048056248053296	23	15
	234170438234669757816987374536542702	21	17
	35251357167933458085553139395470836	19	19
	234170438234669757816987374536542702	17	21
	67077985737611839850488056248053296	15	23
	7681155057059283727400087851836804	13	25
	301977301501927982712251650296648	11	27
	3058692313447287528959880082164	9	29
	4529421792220618780953132624	7	31
	284216848055029040209305	5	33
	28142744118187326	3	35
	1	1	37

39

	1	39	1
	253284697063686007	37	3
	7105425014717554019615631	35	5
	221994390052130259394532925609	33	7
	248766472286660081843970414904068	31	9
	37303324488483426954302995423715292	29	11
	1378203273696399945207173716059020652	27	13
	17173078391956624011742130717002163700	25	15
	85635607007228962104291998560813839198	23	17
	187377221472810770345920109207417275058	21	19
	187377221472810770345920109207417275058	19	21
	85635607007228962104291998560813839198	17	23
	17173078391956624011742130717002163700	15	25
	1378203273696399945207173716059020652	13	27
	37303324488483426954302995423715292	11	29
	248766472286660081843970414904068	9	31
	221994390052130259394532925609	7	33
	7105425014717554019615631	5	35
	253284697063686007	3	37
	1	1	39

TABLE 2.3 (cont.)

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n	$(a_{1,k_1=1,k_2k_3})_n$	k ₂	k ₃
41			
	1	41	1
	2279562273573174140	39	3
	177635661714292879129333150	37	5
	10879128434075642651641785959580	35	7
	20203253868959518771392559825506285	33	9
	4580796878616173620118408244176914608	31	11
	243689884438907962985939130480387999720	29	13
	4274452522959262726615911031357902157680	27	15
	29863765165573633115609534911253825271570	25	17
	92446695058285716391958652429341086990280	23	19
	1339695764875440902791749638211749031668	21	21
	92446695058285716391958652429341086990280	19	23
	29863765165573633115609534911253825271570	17	25
	4274452522959262726615911031357902157680	15	27
	243689884438907962985939130480387999720	13	29
	4580796878616173620118408244176914608	11	31
	20203253868959518771392559825506285	9	33
	10879128434075642651641785959580	7	35
	177635661714292879129333150	5	37
	2279562273573174140	3	39
	1	1	41
43			
	1	43	1
	20516060462158567341	41	3
	4440891888211006424569211370	39	5
	533114507941647087221696108146570	37	7
	1639243235717722648852313037046453305	35	9
	560128160549135541529462577248201125213	33	11
	42615064610639130927440580694098016933848	31	13
	1040957982950195520590134594765664197681560	29	15
	10033617862597411302371670253845686246467650	27	17
	43017289543421872952097748472726300184788090	25	19
	87728918732198931944006930116683301862240028	23	21
	87728918732198931944006930116683301862240028	21	23
	43017289543421872952097748472726300184788090	19	25
	10033617862597411302371670253845686246467650	17	27
	1040957982950195520590134594765664197681560	15	29
	42615064610639130927440580694098016933848	13	31
	560128160549135541529462577248201125213	11	33
	1639243235717722648852313037046453305	9	35
	533114507941647087221696108146570	7	37
	4440891888211006424569211370	5	39
	20516060462158567341	3	41
	1	1	43

TABLE 2.3 (cont.)

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n	$(a_{1,k_1=1,k_2k_3})_n$	k_2	k_3
45			
	1	45	1
	184644544159427106154	43	3
	111022300477586804328521775591	41	5
	26123594546702044085526699031503100	39	7
	132923444218451509189072119405846654355	37	9
	68283013125971106451240903324125753716898	35	11
	7390289780812377609071588455917782806706037	33	13
	249230724309924443773931882861214431260678608	31	15
	3273247158340961478628421988032806047303042330	29	17
	19106900875568186798772740152583987267862674420	27	19
	53584249785424102912573513119405869087185148918	25	21
	75239204631157522675631000601051937933922764392	23	23
	53584249785424102912573513119405869087185148918	21	25
	19106900875568186798772740152583987267862674420	19	27
	3273247158340961478628421988032806047303042330	17	29
	249230724309924443773931882861214431260678608	15	31
	7390289780812377609071588455917782806706037	13	33
	68283013125971106451240903324125753716898	11	35
47			
	132923444218451509189072119405846654355	9	37
	26123594546702044085526699031503100	7	39
	111022300477586804328521775591	5	41
	184644544159427106154	3	43
	1	1	45
47			
	1661800897434843955475	47	1
	2775557542867631254917084671065	45	3
	1280082056495642083638458387900885491	43	5
	10774309551783598613466933578712446737055	41	7
	8306051633303508006760284701965046518394653	39	9
	1273545261977469819494923690008367374545329695	37	11
	58883620568814372676383312320742625862669958405	35	13
	1043080227289567521787610710182510189123900602538	33	15
	.8175785866750908617220315276737911020940067654750	31	17
	30949875721911960369001953290125975409197457599738	29	19
	1273545261977469819494923690008367374545329695	27	21
	59521218780319064216950833748175975316172942823870	25	23
	59521218780319064216950833748175975316172942823870	23	25
	30949875721911960369001953290125975409197457599738	21	27
	8175785866750908617220315276737911020940067654750	19	29
	1043080227289567521787610710182510189123900602538	17	31
	58883620568814372676383312320742625862669958405	15	33
	1273545261977469819494923690008367374545329695	13	35
	8306051633303508006760284701965046518394653	11	37
	10774309557783598613466933578712446737055	9	39
	1280082056495642083638458387900885491	7	41
	2775557542867631254917084671065	5	43
	1661800897434843955475	3	45
	1	1	47

TABLE 2.3 (cont.)

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$$(a_{1,k_1=1,k_2k_3})_n \quad k_2 \quad k_3$$

49

1	49	1
14956208076913595599368	47	3
69388938863336838872742230963028	45	5
62724702167664030806105463320712967928	43	7
873107580262755356033649662814680367103746	41	9
1008798346891106069674845062004217099062126168	39	11
218418024200718459604948339142805127847887061764	37	13
13767561449229500950631899040013363589097919031464	35	15
326218687075211486962131649800062757253999650634479	33	17
3394443742229202076972991369333038170312239508202448	31	19
17084515829874885039358166642716434050375758695548328	29	21
44110314824135914973179189027929995303934228678921264	27	23
60310604513008106191189250732031934051971919257179164	25	25
44110314824135914973179189027929995303934228678921264	23	27
17084515829874885039358166642716434050375758695548328	21	29
3394443742229202076972991369333038170312239508202448	19	31
326218687075211486962131649800062757253999650634479	17	33
13767561449229500950631899040013363589097919031464	15	35
218418024200718459604948339142805127847887061764	13	37
1008798346891106069674845062004217099062126168	11	39
873107580262755356033649662814680367103746	9	41
62724702167664030806105463320712967928	7	43
69388938863336838872742230963028	5	45
14956208076913595599368	3	47
1	1	49

It was shown /SC76/ that the coefficients

$(a_{1,k_1,k_2=0,k_3})_{n\text{-even}}$ or $(a_{1,k_1,k_2=1,k_3})_{n\text{-odd}}$ are

THE TAYLOR SERIES COEFFICIENTS OF THE JACOBIAN ELLIPTIC FUNCTIONS $\text{cn}(u,k)$ and $\text{dn}(u,k)$:

$$\text{cn}(u,k) = 1 + (-1)^{n/2} \sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\sum_{k_1=1}^{n-1} a_{1,k_1,k_2=0,k_3} \cdot k^{k_1-1} \right), \quad n\text{-even},$$

or

$$\text{cn}(u,k) = 1 + (-1)^{(n+1)/2} \sum_{n=0}^{\infty} \frac{u^{n+1}}{(n+1)!} \left(\sum_{k_1=0}^{n-1} a_{1,k_1,k_2=1,k_3} \cdot k^{k_1} \right), \quad n\text{-odd}. \quad (2.12.3)$$

$$\text{dn}(u,k) = 1 + (-1)^{n/2} \sum_{n=0}^{\infty} \frac{u^n}{n!} \left(\sum_{k_1}^{n-1} a_{1,k_1,k_2=0,k_3} \cdot k^{n-k_1-1} \right), \quad n\text{-even},$$

or

$$\text{dn}(u,k) = 1 + (-1)^{(n+1)/2} \sum_{n=0}^{\infty} \frac{u^{n+1}}{(n+1)!} \left(\sum_{k_1}^{n-1} a_{1,k_1,k_2=1,k_3} \cdot k^{n-1-k_1} \right), \quad n\text{-odd}. \quad (2.13.3)$$

The coefficients $(a_{1,k_1,k_2=0,k_3})_{n\text{-even}}$ are tabulated in the following TABLE 2.4.

TABLE 2.4: Taylor series coefficients $(a_{1,k_1 k_2=0, k_3})_n$
of the Jacobian elliptic functions $\text{cn}(u, k)$,
(2.12.3), and $\text{dn}(u, k)$, (2.13.3), n-even.
 $n=0, 2, 4, 6, \dots, 50.$

$(a_{1,k_1 k_2=0, k_3})_n$			k_1	k_3	$(a_{1,k_1 k_2=0, k_3})_n$			k_1	k_3
N = 0	1	1	0		N = 16			1	16
N = 2	1	1	2			2690416		3	14
						586629984		5	12
						6337665152		7	10
						9860488448		9	8
						2536974336		11	6
N = 4	1	1	4			67047424		13	4
	4	3	2			16384		15	2
					N = 18				
N = 6	1	1	6				1	1	18
	44	3	4			24213776		3	16
	16	5	2			14804306080		5	14
						345558617984		7	12
						1165333452544		9	10
						782931974144		11	8
						95153582080		13	6
N = 8	1	1	8			1073463296		15	4
	408	3	6			65536		17	2
	912	5	4		N = 20				
	64	7	2				1	1	20
						217924020		3	18
N = 10	1	1	10			371548371744		5	16
	3688	3	8			17992189979904		7	14
	30768	5	6			119641512257280		9	12
	15808	7	4			171748920960000		11	10
	256	9	2			57102164668416		13	8
						3497455190016		15	6
						17178624000		17	4
						262144		19	2
N = 12	1	1	12		N = 22				
	33212	3	10				1	1	22
	870640	5	8			1961316220		3	20
	1538560	7	6			9303419165040		5	18
	259328	9	4			912656818686720		7	16
	1024	11	2			11283802171749120		9	14
						30883983731149824		11	12
						22171780982046720		13	10
N = 14	1	1	14			3959839273451520		15	8
	298932	3	12			127231162122240		17	6
	22945056	5	10			274672401920		19	4
	106923008	7	8			1048576		21	2
	65008897	9	6						
	1180992	11	4						
	4046	13	2						

TABLE 2.4 (cont.)

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$$(a_{1, k_1 k_2 = 0, k_3})_n \quad k_1 \quad k_3$$

N = 24

	1	24
17651846024	3	22
232733558500720	5	20
45608832444953280	7	18
1009053592891411200	9	16
4901229450088955904	11	14
6655471539077922816	13	12
2632028412773990400	15	10
266300951202693120	17	8
4604235840225280	19	6
4398022393856	21	4
4194304	23	2

N = 26

	1	26
158866614264	3	24
5819812891661136	5	22
2259853542156884800	7	20
87202761117626211840	9	18
716376110273114701824	11	16
1693272192484782391296	13	14
1274464676219028701184	15	12
295751682605237207040	17	10
17573163611835596800	19	8
166179098927824896	21	6
70368639320064	23	4
16777216	25	2

N = 28

	1	28
1429799528428	3	26
145509858586733712	5	24
111428311714927846144	7	22
7370225067120481047040	9	20
99020093727436106102784	11	18
385190678334759891320832	13	16
502186837918249197109248	15	14
225195504620352576997376	17	12
32032609142507284725760	19	10
1146127667313268228096	21	8
5989977008224862208	23	6
1125899453857792	25	4
67198864	27	2

TABLE 2.4 (cont.)

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$(a_{1, k_1 k_2 = 0, k_3})_n$	k_1	k_3
n = 30		
12868195755908	1	30
3637888729721421568	3	28
5479122038971541617408	5	26
613887494691792209993216	7	24
13167638443349313860675584	9	22
81021396112222611059515392	11	20
171737069436034825499246592	13	18
133923514868455741736681472	15	16
37616624304591039218581504	17	14
3383855154588868186996736	19	12
74208892231402953637888	21	10
215770902543537799168	23	8
18014396563324928	25	6
268435456	27	4
	29	2

N = 32	1	1	32
115813761803232	3	30	
90948601574079299520	5	28	
268999019240499899029760	7	26	
50643171484933322929049088	9	24	
1704114648317720171436085248	11	22	
16111208467816945884560171008	13	20	
53082196221488518115144663040	15	18	
67083998960526192929674690560	17	16	
33093884724665138608054730752	19	14	
6036884965559854643231588352	21	12	
351400311330636141063831552	23	10	
4783248052841380409507840	25	8	
7770040319983516385280	27	6	
288230367830212608	29	4	
1073741824	31	2	

N = 34	1	1	34
1042323856229152	3	32	
2273728415841470761536	5	30	
13195094474226710295619328	7	28	
4151473793790552383625069056	9	26	
216338605052759260110060220416	11	24	
3075140830049570551391879225344	13	22	
15235746354655353693186036957184	15	20	
2968492017472944761653821243392	17	18	
23795728083567594767162297810944	19	16	
7733649264326537084367655665664	21	14	
941239114566906745772662849536	23	12	
36064401405578205314485846016	25	10	
30745660660798921890172928	27	8	
277760939080040920907776	29	6	
461168598299397712	31	4	
4294967296	33	2	

TABLE 2.4(cont.)

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N = 36

$(a_{1,k_1 k_2=0, k_3})_n$	k_1	k_3
1	1	36
9380914706062436	3	34
56843339123033013338944	5	32
646940478992848398769505792	7	30
338902250192956077259381785088	9	28
27090381753547809667495429068800	11	26
569463705621852446817394895577088	13	24
4137664126710959136708569940623360	15	22
11979671960555455836081749131460608	17	20
14772066646249520296394239624609792	19	18
7854692012922976629024898867527680	21	16
1733682568283424050289211617574912	23	14
143698285034150052313250618408960	25	12
3671364086557678901281907802112	27	10
19728979597377021147304951808	29	8
10072071724722072768741376	31	6
73786976144514351104	33	4
17179869184	35	2

N = 38

1	1	38
84428232354561996	3	36
1421084711666109180684240	5	34
31710286852534775338902034944	7	32
27590510890252567973883416062464	9	30
3359226587091202365713022396745728	11	28
103105527898665249997416155693727744	13	26
1077575892661766160087537691827634176	15	24
4508199947037195612812437596290875392	17	22
8213198407224912948771515725452148736	19	20
6730586739070899453483296762203471872	21	18
245487963702705024874203322486982656	23	16
376671812239455963009975310325121024	25	14
21602793603019069780917959742455808	27	12
371656448792038205624261504139264	29	10
1264659054767625642880536674304	31	8
362606166645254273410007040	33	6
1180591620081756143616	35	4
68719476736	37	2

N = 40

1	1	40
759854091191058040	3	38
35527129569391142978504784	5	36
1554076193521951078715984426304	7	34
2242172681710993093854403794762240	9	32
413633110157770969153195026080292864	11	30
18355333798828085859088261915740389376	13	28
271771274272422005673581640752765140992	15	26
1607573040942217275141404912122015973376	17	24
4196314548141550483898166722837182152704	19	22
5072500387619619140133334801808161243136	21	20
2865401789693082889199622747870015258624	23	18
735653682900469128576605472027372945408	25	16
79916193330140683047810724293481857024	27	14
3211031448946775664761876531323600896	29	12
37478102283360831622656139135549440	31	10
81015414949189133932894057660416	33	8
13054019158029729761146699776	35	6
18889465928798521262080	37	4
274877906944	39	2

TABLE 2.4 (cont.)

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N = 42

$(a_{1,k_1 k_2=0, k_3})_n$	k_1	k_3
1	1	42
6838686820719522440	3	40
888178351313257025143680880	5	38
76156963253699305983254596098240	7	36
18200631462621471757727169621507840	9	34
50676682260926711267065429218103861248	11	32
3226406811308513441348904983961972326400	13	30
6686482443324079866102620631010504540160	15	28
549655459584178887026205670472905596272640	17	26
2007175890517934827347043797591394313830400	19	24
3461746543872273295629705772362853308694528	21	22
2888509627201748899454363280169778649497600	23	20
1156900013936353413004719503992852567818240	25	18
213365886480869009476348096152757610741760	27	16
16650134150475140633667970379806251417600	29	14
473313290826732859527682707174979534848	31	12
3769282849306521057374563454932746240	33	10
5187949825037714354992289824112640	35	8
469948033124539767970468986880	37	6
302231454892387299491840	39	4
1099511627776	41	2

N = 44

1	1	44
61548181386475702044	3	42
22204459846247226250477761840	5	40
3731882601868539148753052935413760	7	38
14762118799251366388049087559124243200	9	36
6186409283287913495828286189212018039808	11	34
561700496824122745104735474437399335944192	13	32
16137271790726031164409154841185453251624960	15	30
181818713540547860220809520735474813170810880	17	28
911314581749394580309238395408887166812815360	19	26
2185040342172233710165234767254828423738032128	21	24
2598198976577956008906582718608285482103078912	23	22
1542748630930845649763595926201613357091389440	25	20
447942780207967442232026959822357515070341120	27	18
60319581105778980133848290550977453610762240	29	16
3420925865312361702236352709165217111605248	31	14
69338256094993148207373392986014096556032	33	12
378396618006570530209562133484535808000	35	10
332142046384479973470922576274391040	37	8
16918185709765496963166539612160	39	6
4835703278411237698830336	41	4
4398046511104	43	2

TABLE 2.4 (cont.)

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N = 46

$(a_{1,k_1 k_2=0, k_3})_n$	k_1	k_3
1	1	46
553933632478281318484	3	44
555111506219308312950721330656	5	42
182867299006198972411243623305612800	7	40
1196774692027741337458276064976132954880	9	38
753273205458785098408026910884229593781248	11	36
97081192922862770832513786109739685582385152	13	34
3836496456821784336545689394570223630103216128	15	32
58584443107770934537026416765266177701364039680	17	30
396893404850221215363669647377163452645049630720	19	28
1295889048385403236487838306304442118929716871168	21	26
2135406861791294847364452836494584827005561208832	23	24
1808808490184014996264447407162420023476185202688	25	22
782854981070912529160409120502384573448265400320	27	20
167748474242489578267042445929216676933696225280	29	18
16711980843193286984925085854360626931966148608	31	16
695350559704315021906853317286928676187799552	33	14
10111569728034439612939221453176437701869568	35	12
3793964232360579990146648427684199137280	37	10
21261405107846214930806302772612300800	39	8
609055638185103739627360607010816	41	6
77371252455138355088195584	43	4
17592186044416	45	2

N = 48

	1	1	48
4985402692304531866448	3	46	
13877787750482325793793279386912	5	44	
8960630600509742095126393619994481024	7	42	
96992787565420470409268222155766171022080	9	40	
91552427666396226323941500863262517526099968	11	38	
16686930436954409253274600219009763354321793024	13	36	
901417261406938813128944246221776947974544080896	15	34	
18485145262286424220302767300566978094165665185792	17	32	
167155353069083467241305514678438222343914431447040	19	30	
730879726448587317116505055259223663864040388558648	21	28	
1632287596022932727006300352610124658173028161552384	23	26	
1914297376556192079044598182741777579348647819083776	25	24	
1185218137633402420536387717355337617702902638313472	27	22	
381179143993492935228604617760913225631377873960960	29	20	
61159554161926705776754609181598161054566006652928	31	18	
4556770909047899393086780219557232488270281048064	33	16	
140172250689010538205600238840727050174737678336	35	14	
1469617520097726258834794837106661584642506752	37	12	
3800746851345946072400789846481207579115520	39	10	
1360693762902099187033274710414397014016	41	8	
21926018990512992848739103154372608	43	6	
1237940039284553442155036672	45	4	
70368744177664	47	2	

TABLE 2.4 (cont.)

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v = 50

$(a_{1,k_1 k_2=0, k_3})_n$	k_1	k_3
1	1	50
4486862423074 786798128	3	48
346944694656937928113495454700768	5	46
439074389688599103381208754135111431808	7	44
7859207035411502926933862876578364093112576	9	42
11112716177133122635886137586044237213431072768	11	40
2856333555970319131931643060311011660193918930944	13	38
209846236077380510238102845038841144246819761373184	15	36
5735394349126752048413545122760857937981576056274944	17	34
68511180556682536979177585823499102882064166381682688	19	32
395685301961571212094489594681794774651645222587465728	21	30
1176270283440967429483865894836884460087001604679532544	23	28
1865049203648341867422570498887991598257490278045712384	25	26
1600044624113181164403368099014082930123488014687535104	27	24
73923053940203995319079236138549244414142499889594368	29	22
179469548944650017027516393823179932736901691681538048	31	20
21821377973386222967465690562465846449264812395331584	33	18
1226790611350600556041894789530964327773586518441984	35	16
28076050587784453871272184363763524969959002210304	37	14
213065906761901989517908261022517959865252446208	39	12
380533315516953963808053037989608365014122496	41	10
87103405889693589815667735238088629157888	43	8
789336952291456267339929726870355968	45	6
19807040628562636329921282048	47	4
281474976710656	49	2

It can be shown /SC76/ that

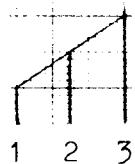
$$\sum_{k_3}^n (a_{1,k_1 k_2 k_3})_n = (N(RU(k_2)))_n , \quad (2.14.3)$$

summation over k_3 for a fixed k_2 .

$(N(RU(k_2)))_n$ is THE NUMBER OF PERMUTATIONS OF n -NATURAL NUMBERS WITH $RU(k_2) = (n-k_2)/2$ RUNS-UP.

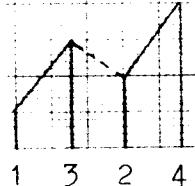
Examples of RUNS-UP (more details in /DA66/):

The sequence 123 has one RUN-UP:



i.e., there is only one successive sequence of numbers which increases: 123.

The sequence 1324 has two RUNS-UP:



i.e., there are two successive sequences of numbers which increase: 13 and 24.

For $n=3$ we have

Permutations RUN-UP

123	1
132	1
213	1
231	1
312	1
321	0

Tables of $(N(RU))_n$ are for $n=0, 1, 2, 3, \dots, 15$ already given in /DA66/, p.260.

In the subsequent TABLE 2.5 we tabulate them for $n=0, 1, 2, \dots, 50$.

TABLE 2.5: Number of permutations $(N(RU(k_2)))_n$ of n natural numbers with $RU(k_2) = (n-k_2)/2$ RUNS-UP.
 $n=0, 1, 2, 3, \dots, 50.$

$(N(RU(k_2)))_n$ $RU(k_2)$			$(N(RU(k_2)))_n$ $RU(k_2)$		
$N = 0$	1	0	$N = 9$	1	0
	-----			4916	1
$N = 1$	1	0		101160	2
	-----			206276	3
	1	0		50521	4
$N = 2$	1	1	$N = 10$	1	0
	-----			14757	1
$N = 3$	5	1		540242	2
	-----			1949762	3
	1	0		1073517	4
$N = 4$	18	1		50521	5
	-----			-----	
$N = 5$	58	0	$N = 11$	1	0
	61	1		44281	1
	61	2		2819266	2
	-----			16889786	3
	179	0		17460701	4
$N = 6$	479	1		2702765	5
	61	2	$N = 12$	1	0
	-----			132854	1
$N = 7$	1	0		14494859	2
	543	1		137963364	3
	3111	2		241595239	4
	1385	3		82112518	5
	-----			2702765	6
$N = 8$	1	0	$N = 13$	1	0
	1636	1		398574	1
	18270	2		73802835	2
	19028	3		1081702420	3
	1385	4		3002137335	4
	-----			1869618654	5
				199360081	6

TABLE 2.2 (cont.)

- 54 -

$(N(RU(k_2)))_n$	$RU(k_2)$	$(N(RU(k_2)))_n$	$RU(k_2)$
$n = 14$		$n = 19$	
1	0	1	0
1195735	1	290565357	1
373398489	2	1139405165906	2
8236142455	3	167687984079924	3
34591152955	4	3847582256323470	4
35576491869	5	24028863623822694	5
8200548715	6	49750647042865188	6
199360981	7	55562596236584612	7
<hr/>			
$n = 15$		$n = 20$	
1	0	1	0
3587219	1	871696090	1
1881341265	2	5951965440609	2
61386982075	3	1191656966048088	3
377209516235	4	36808184099950242	4
598888328289	5	306640904681607804	5
248913100771	6	863018184171651690	6
19391512145	7	881693472848825496	7
<hr/>			
$n = 16$		$n = 21$	
1	0	1	0
10761672	1	2615088290	1
9453340172	2	29775517732665	2
450403628440	3	8436830209386360	3
3947368484790	4	347956854424225410	4
9228238224824	5	3814748160697088748	5
6230311951468	6	14285645441047550010	6
1037611984488	7	20129547566105595960	7
19391512145	8	10600752445365780765	8
<hr/>			
$n = 17$		$n = 22$	
1	0	1	0
32285032	1	7845264891	1
47417364268	2	148927275340835	2
3266265481144	3	59563995267159825	3
40030352647510	4	3258164142958824090	4
133089568351384	5	46485668875182906558	5
136363484718028	6	227675620501286126358	6
40485427573192	7	430514022461011889490	7
2404879675441	8	321119624533057444725	8
<hr/>			
$n = 18$			
1	0	87858810432172127735	9
96855113	1	69348874393137901	10
237571096820	2	5959755026257457591	11
23480284103492	3		
396202094120174	4		
1824258425692814	5		
2704352279794052	6		
1289098837188020	7		
162339237202073	8		

TABLE 2.5 (cont.)

- 55 -

$(N(RU(k_2)))_n$	$RU(k_2)$
$n = 23$	
1	0
23535794695	1
744793282001995	2
419628657326253805	3
30276501210903974010	4
556956655628435509398	5
3517611093018914521350	6
8734466541928039605930	7
8903145796765371676245	8
3596035145415015095315	9
497590097280095120351	10
15514534163557086905	11
<hr/>	
$n = 24$	
1	0
70607384108	1
3724460661698570	2
2951551677141814540	3
279622198081182080775	4
6580670730076350213528	5
52969380732415550399724	6
169710720152128653823800	7
229963677422363674949535	8
13064668834024288544700	9
28429567769957073003946	10
1849604577602098359868	11
15514534163557086905	12
<hr/>	
$n = 25$	
1	0
211822152348	1
18623856670943226	2
20735350953226673180	3
2569727712919191388695	4
76861333200138765641208	5
780731339742471058185804	6
3181293371070916412153688	7
5606489717701469012380095	8
4321996497843524281945580	9
1380901053210555864351066	10
156259176364676554292748	11
4087072509293123892361	12
<hr/>	
$n = 26$	
1	0
635466457069	1
93124155264220134	2
145538557662676520006	3
23521521084384029288675	4
889160036321152675661103	5
11302427414654205241033572	6
57868907982715869938720772	7
130304552282705053744152183	8
132576340918340182468386875	9
59252897602326343124991446	10
10498466322440340070488534	11
570954341826357760187269	12
4087072509293123892361	13

TABLE 2.5 (cont.)

- 56 -

$(N(RU(k_2)))_n$	$RU(k_2)$
$N = 27$	
1	0
1906399371233	1
465636027516070326	2
1020818635054548482990	3
214604460912709793298195	4
10204147779051591959468283	5
161158116971643110944014084	6
1026267603545896922455281588	7
2909604284598576352915236375	8
3821996000275514004340872455	9
2304921576995574665371915366	10
596982111030085880371184958	11
56267723835420304286635861	12
1252259641403629865468285	13
<hr/>	
$N = 28$	
1	0
5719198113726	1
2328227797564632455	2
7156440074014708998428	3
1952877339550533664126545	4
116323110326908997640116818	5
2268526032875237505583143903	6
17811385807763100500989435080	7
62804751684272457991477679019	8
104623571135819105964544176770	9
82801317119386694011878074781	10
29865039592660997906140661596	11
4391603651035937009021821315	12
202614181824158919227551278	13
1252259641403629865468285	14
<hr/>	
$N = 29$	
1	0
17157594341206	1
11641287686974119151	2
50150957985244514167916	3
17733337737583126575104321	4
1318611760387009647323815898	5
31584654413262449530102973463	6
303467203642450307604171828648	7
1317040179941315192868972634443	8
2741504871791832509224071506858	9
2785063370865311633894881338101	10
1349306447586296503936259814956	11
288980328831864412662389502451	12
2303099751339603885523169766	13
441543893249023104553682821	14
<hr/>	

TABLE 2.5 (cont.)

- 57 -

$(N(RU(k_2)))_n$	$RU(k_2)$
$n = 30$	
1	0
51472783023647	1
58206901689917808317	2
351347738088885952154187	3
160753511671908763001800957	4
14877129456746351778639165619	5
435654130819765027190491157081	6
5088947179662216256074327978591	7
26941601113039112892635112215251	8
69210114903281915182554002878061	9
8864284377881701913257294675559	10
56099618632272624295587907786897	11
16669653353900685844113556265967	12
2066900577021015112403189095937	13
8191576544409786597750311107	14
441543893249023104553682821	15

 $n = 31$

1	0
154418349070971	1
291035949687513703701	2
2460947546066130528095551	3
1455213950761312129867909101	4
167185001280991862351070442863	5
5961046269791872389049168354433	6
84175982049689014330543760506323	7
539431903806460379275386155316723	8
1692175858753303968968217625696673	9
2692021950774898722369051222723471	10
2176720072321087377931094825854221	11
865538282905698140467542168944351	12
155824235702971523099567443186101	13
10643159505971944260947515405851	14
177519391579539289436664789665	15

 $n = 32$

1	0
463255047212944	1
1455184226569691576664	2
17234490793104539566668784	3
13158449245503462657013570684	4
1872504934958420664848736780816	5
81004486794195170167011667907752	6
1375899610251380790350090606329072	7
10601334059554539691300808568991782	8
40242819873409681099526927217987632	9
78530747130065824766336904811249640	10
79676803121908895638474744444605264	11
41228937723532239013068407656296764	12
102660223432011816961116148635184	13
108777280418804399865315162700184	14
37432579656881550755379154697168	15
177519391579539289436664789665	16

TABLE 2.5 (cont.)

- 58 -

	$(N(RU(k_2)))_n$	$RU(k_2)$
$n = 33$		
	1	0
1389765141638864	1	1
7275935030499874271640	2	
120682180710075728330828080	3	
118874139970151881941855524540	4	
20913357066434710417104430285392	5	
1094253436893622466797823891978728	6	
22258583889654615258591592453091120	7	
204988871996952028978415376586783590	8	
934234922547656575951824554245633520	9	
2212545167959117855486451982088069288	10	
2774935437364694496880961979960916752	11	
1827491474307394954211457635853471740	12	
607014105084901108192497397263524080	13	
93141545387373984614660836610116440	14	
5511501186115504069678014446412944	15	
80723299235887898062168247453281	16	
<hr/>		
$n = 34$		
	1	0
4169295424916625	1	
36379718235218762162984	2	
844986267086414594669674120	3	
1073125678610538982141632079020	4	
233018781230035611636695121252812	5	
14706301892145090407965112492287480	6	
356858080519585300681628188527920088	7	
3907723917851622182546301658584052310	8	
21235274352353659435717727932642357910	9	
60476972365356323604492859937533957848	10	
92586602242856505549586001306245986040	11	
7621157666869651332097702267590687772	12	
3283680410605884509100548448796395820	13	
695020355182815331172646042538045320	14	
636564263706450549233322630889383464	15	
19198372433130812845085595505197105	16	
80723299235887898062168247453281	17	
<hr/>		
$n = 35$		
	1	0
12507886274749909	1	
181898724593547408146920	2	
5915995261151958725552608360	3	
9681790722973270447925439586540	4	
2591107861174265741539328767835452	5	
196774375347407029982827145309804728	6	
5676409849420971499199655302749125880	7	
73568468213869283116919691966487291030	8	
473809243216048728564470260574717741870	9	
1609780809310141346665833705610490841368	10	
2976169464700688158103378069169133088792	11	
3016328643631690899619457582572623776780	12	
1648709477550555014955485034876571464860	13	
464250335851487522096811122823974480840	14	
6143471348586886893268877812799159304	15	
317980345119119020921115175229038321	16	
41222060339517702122347079671259045	17	
<hr/>		

TABLE 2.5 (cont.)

- 59 -

 $(N(RU(k_2)))_n$ RU(k_2)

N = 36

37523658824249762	1	0
909494035727984107481597	1	1
41417605688526111048520813040	2	2
87307680369332840834369981921300	3	3
28763594822437201459026603315026552	5	5
2622844576045648033315236108223347764	6	6
89671958374304934177599853883362396944	7	7
1369868566473618214470827524788015590990	8	8
10400176517168442221946409098282895625100	9	9
41860154130185796665578502247590509280518	10	
92598609827767947836365201175047423662736	11	
114098419131901218545830354463514324573796	12	
77694770973813585299612129349966291095800	13	
28301645037648133275406887875784403128100	14	
5154228469022348148369013071964595304304	15	
412107077818275362153441189846554061113	16	
10982182147240476636745493314181181538	17	
41222060339517702122347079671259045	18	

N = 37

112570976472749322	1	0
4547471454444320561899893	1	1
289952343628826072831085102384	2	2
787011651494651350840785461682900	3	3
318844158097150535592654995959088472	4	4
34844832953976791671032761093094211284	5	5
1408027645439669565463563474847796300496	6	6
25260548714286218197911264706830237779598	7	7
225000725155672766506398323363135328696700	8	8
1066266414042933689972183910968492816142678	9	9
2799530492121635546885655662987538892731216	10	10
4148841015886281733354871678038522045623204	11	11
3466939845875781425639491746011261754472152	12	12
1597695415829931717982921041897410601672900	13	13
386194242840877858802694508237177679458224	14	14
44524904382137175841277637696723855842553	15	15
2032804686426518130899857025382557598282	16	16
23489580527043108252017828576198947741	17	17

N = 38

337712929418248003	1	0
22737361212205779355725735	1	1
2029816471959779172396138413157	2	2
7092093386104355765824832793320004	3	3
3530109076962000780693587733938777292	4	4
461591620670321356184427579101120135436	5	5
21991535505444463273729271150044299789540	6	6
461813963987978109370153459937613357164574	7	7
4805485300957793145777704702743006238608858	8	8
26666608472859390053037430274237920384233538	9	9
82515730357527490307897206735177772407243494	10	10
145713982778781576537156626895776134531548340	11	11
147542309045167761025879608956804853963849756	12	12
84469505363701615503539119421148786747707772	13	13
26351280270536599084729819132429203478261044	14	14
4172681544496671814381023601652130999011817	15	15
293772685935614013787883184372008795152635	16	16
6967528538780149398024230733467033861263	17	17
23489580527043108252017828576198947741	18	18

TABLE 2.5 (cont.)

- 60 -

$(N(RU(k_2)))_n$	$RU(k_2)$
N = 39	
1	0
1013138788254744047	1
113686818218694355835556783	2
14209488373999669203271063567089	3
63893794602041914825940171569101060	4
39043962648165139260604210057126150332	5
6099534122869113652256978984864847524844	6
341874414719095304366734184307293620364436	7
8378634239926294977862111326540490266746718	8
101464127925933588175919765470744612391188930	9
656108483949203054029340129813856452841081458	10
2377860750734601298036309499845371332282804046	11
4963101255194979258355270980157247721804604404	12
6023638103125271619218942218374596940465620172	13
4220123364089359981913189770694973063249722460	14
1661584742023650726662015587316793175303170084	15
348508733132682962552412331913953950793478313	16
35318133274726521368862053062933093824413127	17
1432889299677321582878429274626315433477271	18
14851150718114980017877156781405826684425	19
<hr/>	
N = 40	
1	0
3039416364764232180	1
568434128579606944603313654	2
99470397656635338725351689457028	3
575513064534719222517169489219623477	4
431464296762479831226250455947029786512	5
8 426218514095266517898248894899676182600	6
5292803642103895634111951197200755188637328	7
150983642446724397232824247158870675043805106	8
2120527018111042959833304104454578911567764184	9
15909024849377869486310457800976622369877678148	10
67156858461930687881392580962906813246485040760	11
164501164142362703525499036001302505693922778882	12
237084747612307022594240504598472833219640810704	13
200690872899219970525328752189023979060295013576	14
97930484007716132327567570684465292130145219664	15
26455050871591394304187747239011618953913315085	16
3675695796544208985777058180600335939408807636	17
229607570461693505410812148475839140160724662	18
4877862777038448969332496938353773541124388	19
14851150718114980017877156781405826684425	20
<hr/>	

TABLE 2.5(cont.)

- 61 -

N = 41

$(N(RU(k_2)))_n$	$RU(k_2)$
1	0
9118249094292696580	1
2842170758395856584057391110	2
696313247225076236927467545490740	3
5182999574332798604171187360418150245	4
4764523682452389158609304439072355602896	5
1058484769586112859669464749312106683969160	6
81643988749953101974180418927068518762672720	7
2704334816289016039444922932828021110649257330	8
43913620762831201770420559916449895520838842040	9
380741116235378204328852304118509805821922053156	10
1862788241611963210998238518166389152066709500440	11
5321352555873819970002542357364885280784800205730	12
9033306811810092866452478200179606588033066351120	13
9139221780649677461553900877860315057823526743560	14
5444135479029839748458539717486711804758041972496	15
1852321518839677335313871365732036346780591594445	16
340289759851778468935699014233104709510614787940	17
30549654886347913614862398577208063822399658310	18
1108666930151273531447215974499153728746749780	19
10364622733519612119397957304745185976310201	20

N = 42

1	0
27354747282878089781	1
14210854147590997597702122170	2
4874297890893594305185882941906250	3
46671367132648065105833147607855528105	4
525807994929263098639998012689710589941	5
13908002238775491239619930178668629915288856	6
1255355889567293802543120761636078875275196440	7
48178079573162006423866561169107208887629538050	8
901967164901018234624113711733248542662169432010	9
9005576718488059931625571264567053519239656483196	10
50839692998018096143865384304315656419794681626396	11
168426790487472800259030090779283525908887485651610	12
33436227736872746884260131480052427650234394977650	13
400537033815992039381850298460643235497378270830040	14
287578082998370839202415442654272161699205148813656	15
121012100391037589298401691961511029296097984314141	16
28581035264369342430574307789746991953896841927905	17
3512365549757322086299801846988431328003090873050	18
195986284707639235800753415891507314533121532970	19
375095032252812491236964172992013811268967581	20
10364622733519612119397957304745185976310201	21

TABLE 2.5 (cont.)

- 62 -

$$(N(RU(k_2)))_n \quad RU(k_2)$$

N = 43

		1	0
82064241848634269385		1	
71054271832144879303634202090		2	
34120625248712768594209893273986210		3	
420217778917904755347485020256608377945		4	
579975620904731928298638305158253904444921		5	
182486614687855122534215572259098259637633240		6	
19247578410672671775335409329901242026586612280		7	
854177317651638335676938921200632759597407647170		8	
18390006202021558624878691113328509741659587197490		9	
210764323045873696195115725637506088927924852515356		10	
1367435626761153529804666406819735275078550120037420		11	
5227463622147181929353 59955568401276118080773818170		12	
12059463717730151470537565183988519012916303406125530		13	
16965370418563408292221820759039492671827720173713560		14	
14522439046373384566620782900731442309638655404843896		15	
7444336308884690517276241146581129907161695268130525		16	
2210457238163302878054117692256255011347369310618085		17	
358605807455975656637687130656547894767289097726090		18	
28717658402141862714028194301699373234810285024130		19	
937734102054210055543729194658701824394513802701		20	
7947579422597592703608040510088070619519273805		21	

N = 44

		1	0
246192725545902808198		1	
355271362525358312312176055235		2	
238847147857590833809762094651784980		3	
3783222473395345170565350948360612891275		4	
6394439452214177877722183332449774242122206		5	
2391465186431972746578657503438499754135914513		6	
294370761215413585428591822688550676447565814640		7	
15079194173987359187992688530977892931926941758010		8	
372472905415003848935972482025658769600662163225900		9	
4885800939013886585719397516220840611027911582759726		10	
.36298598845561626197994989046516551372148924368713848		11	
159402738715663772359724493432424472679601571866240070		12	
424927329199510546362222399123489637594983726667934540		13	
697006625339751415473571410139950110702581042941827250		14	
704676166716026045948571581308267101676214120155864176		15	
434454805796048786436186135546686036961638464111277973		16	
159253702733447196421932771841361354375936573821068750		17	
33162530019340825198081483064598567208516020411428095		18	
3630229329875362242310909492362110819528624800023700		19	
182035390194931925847433868489503640974226491031391		20	
315494822133432665288633325909892509822870181718		21	
7947579422597592703608040510088070619519273805		22	

TABLE 2.5 (cont.)

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$(N(RU(k_2)))_n$	$RU(k_2)$
1	0
738578176637708424638	1
177635682296686034488798220491	2
1671944245857636851000827149604704260	3
34058078452176694986772929494842283850715	4
70475029983398189081084369291088498727430166	5
31306458364990927753365101778003789127999043673	6
4492088304197026909319394380438292138845836484016	7
265177423794247513758733459707280700136184984325370	8
7499202639756719187047272437354897624506535470516380	9
112286115260081718372442632373304780841203359481827646	10
952126995984250680611150288459180856224095138466651928	11
4783637642493950085349002094833975947177315632767706406	12
14661092662700060198974494644982709668656592057359033980	13
27861884060443980883253574078281366687084557325335811970	14
32997067173632830071982861582795481923203934411901025456	15
24202474925293974595674144611356378643201066997854271573	16
10787337265223237312001880641007879846697441653072741926	17
2819550638050082496548342591803760530474458493433527015	18
406879184019865729034777334718910859629744530492349060	19
29244826977244382413610245562242314197115034932429231	20
863804334297103749463847806972139941819289381939438	21
6667537516685544977435028474773748197524107684661	22

N = 46

1	0
2215734529913125273959	1
8881784146593291767865453361889	2
11703682551633199599333204087959969951	3
306587911895178702718145397712415138122575	4
776485478720110617606438660593282650504208281	5
40945058479430097411584276039237356119447623559	6
68414437689000104255651964065248507123911515701449	7
4647270941932315568087370040817358958619365664535786	8
150174995445410842452901446641254195169573538485246950	9
2560486891735147503871571635647982633526947006822322826	10
24706073789139808613367522443893779214184272171778685494	11
141489861969986517787781509005410558372587079003925654494	12
496305892385274577164640399406046655944451613836815751986	13
1086555398344176589394869046524831117629927411524560192750	14
1496561111410165407246779468397443173299759441299640592626	15
1293637680139193612736989695916692724073694227107706343749	16
692188978311634975663829702382908716996024328829651497859	17
222984083525308662804309362947825817941226822440840660741	18
41244243919225505901291399380271368299830163130103356475	19
4047192194206079782201461411084310899489928145676041891	20
183367721260997373294996683511013588483804618085541989	21
2891452191142160772376119702281238494346452991628059	22
6667537516685544977435028474773748197524107684661	23

TABLE 2.5 (cont.)

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N = 47

$(N(RU(k_2)))_n$	$RU(k_2)$
1	0
6647203589739375821923	1
44408920830458778155504778863641	2
81926150896366554113586678964760988995	3
2759759354358673652447281907575254641901215	4
8552990606573233584374114791639180930794948941	5
5350811079559836948584427380291443804970970604383	6
1040137885218007797746773326364061676966733954722741	7
81192868018897367993666153543983054524494384799554730	8
2992743041720775473647748587408350476980478201155765630	9
57975124598909601169984244854562752768813946220855693946	10
634812356335329433208113878736404470397938882128290159838	11
4130192320189018351415358263788714660455099507220829814206	12
16513036057742116974776483982082291994697109311680389702490	13
4143622439968612635744010337341035530156927210948560629470	14
65951391623910306233757806357767698489631236087730941840906	15
66635021227156035736269131459609950667228060555348558825733	16
42337541762855934726551895326235503231892570688545691237551	17
1655667883017604023172540285766445986777684376266922421725	18
3838366348102881358193458205308841543105644586482437509935	19
495888831316253318280591112896627693277728359013544569331	20
32167965179459365744893625857479449701743167451734556873	21
863586233645386727936912120646710086180808856965430611	22
6096278645568542158691685742876843153976539044435185	23

N = 48

1	0
19941610769218127465816	1
222 44604451418052315795806304740	2
573484965858161588522567439458818059528	3
24841193161414813900744194222014846977659730	4
94190527287125557700560706702426425169778585736	5
69877004686721089974219398191079419159062030967796	6
15789346666054711259402054853771125687674993291994520	7
1414603306533449513217968130017725962256306762098280863	8
59379096701280552407110873920622133752888411750745743600	9
130426716478704113305453850980659971977526738471486776136	10
16166012559883136195376193822010497143910570836913777412816	11
119125116913108694615586803563127978271325959733727999351100	12
540846396923384580401518307583362321047289240081449607693968	13
1548424264803496222906882463406618162263190184662796442006920	14
2831781403935265133325628193500278328251549935727681849028016	15
3320129358102624385270764046249179246342257011817928452544591	16
2481339280107298251473353308312391873124660882429327575700280	17
1162985159633640640019014544974646557122377740872970115691988	18
331819754707948815518524301441353878726674667011751209526440	19
54876739216892318273245359476541309312337663997897264931986	20
4854444321930525954994563702148010190119054713519397930856	21
199701206411339231481629174716499202386852235822117161860	22
2877283797277881665269245591855341886779323905984745528	23
6096278645568542158691685742876843153976539044435185	24

TABLE 2.5 (cont.)

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 $(N(RU(k_2)))_n$ RU(k_2)

N = 49

1	0
59824832307654382397496	1
1110223023174404356963012894951236	2
4014404530969726982052273971227203825256	3
22359482482129936789341569583059093157437746	4
1037089447884837727262197541495571270746670832296	5
911980300964284940857473483338724653224257988839316	6
239355772159542628130102721141445744404851132494758456	7
24585093997714501907525128075329559631738164727598588351	8
1173470143133400880158081584652387772097081639651314116016	9
29170983361565502951627857088212523424114713860423594606856	10
408337769491348247666205165733699913525313877926218510226576	11
3398444249384678906469451128450472382524823835102958196510716	12
17461855522845992341615077590265854146788632515808611392163536	13
56802924411615851233132994205625897768672878637012988187467976	14
11875370881806314359123212326640991421061851700814066160006896	15
160536334088221377113796521009227925037822380233089912216475791	16
139968944533397428965899590530920583500839243074113320390223256	17
77769200827946879201330484480435408837273228766310480340407476	18
26896792349213691485450622295908559955808844903933938559835016	19
5568143854972073204388302752951732469072590894031299957475826	20
647755019578151162250729115004694912673820664664512230482696	21
38113220220093421146640695075130525248122678893111659868836	22
934037164117417364194171241683197878226037166869751687256	23
6053285248188621896314383785111649088103498225146815121	24

N = 50

1	0
179474496922963147192537	1
5551115118683788903274820447438492	2
28100881676824131722561981134170699582412	3
20125260427865260093009695102560808181425722	4
11417151314550888273967802999980338204832834102842	5
11896190401003212902510380987521747771474474017370652	6
362407985352881796476326733600521897824206453308431532	7
42632404998673052412480772520553114793718590006492547927	8
23107240821459195285951879334881243137691910589385721619887	9
648968225030010989269085527976687012841418521898086224340472	10
10237727215786409281919926667433261190381545894255309978810072	11
95986226010883375348723817686071707228304070581581854688885452	12
556431205351458765885343373148439871526413673804406462501183372	13
2048907484962317509618003616539265680667652028336974719456332632	14
4874226386003890327223988699584014587195047854102508802897041272	15
7554019492454505172988695535370700363248313730007434360183832127	16
7628030738168673424741026525739095148172353971556494721337902407	17
4976994598634995964937721783739918879491698110465187578448425452	18
2059974512382643397549870567786094153161096925215459648258862812	19
524158613895205607719877258126015190745873520598556622414694042	20
779667605366091588162760767217674734666276 6626855725528038362	21
6249380046951262087353935083413738024882265202841610307476492	22
234465847813985721850329523734762926517237141308436628645212	23
309872246951349456550191853052006443995182913641449002697	24
6053285248188621896314383785111649088103498225146815121	25

It was shown in /SC76/ that

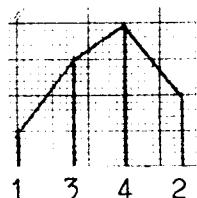
$$\sum_{k_2}^n (a_{1,k_1 k_2 k_3})_n = (N(P(k_1)))_n , \quad (2.15.3)$$

summation over k_2 for a fixed k_1 .

$(N(P(k_1)))_n$ is the NUMBER OF PERMUTATIONS OF n NATURAL NUMBERS WITH $P(k_1) = (n-k_1-1)/2$ PEAKS.

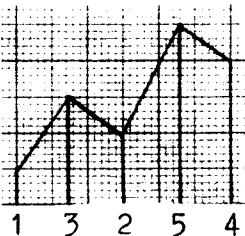
Examples of PEAKS (for details see /DA66/):

The sequence 1342 has only one PEAK;



i.e., there is only one sequence of successive increasing and decreasing numbers.

The sequence 13254 has two PEAKS:



i.e., there are two sequences of successive increasing and decreasing numbers: 132 and 254.

For $n=3$ we have

Permutations PEAKS

123	0
132	1
213	0
231	1
312	0
321	0

i.e., there are four permutations with zero PEAK and two permutations with one PEAK.

Tables for $(N(P))_n$, $n=0,1,2,3,\dots,15$, are already given in /DA66/, p.261.

We tabulate these numbers in TABLE 2.6 for $n=0,1,2,3,\dots,50$.

TABLE 2.6: Number of permutations $(N(P(k_1)))_n$ of n natural numbers with $P(k_1) = (n-k_1-1)/2$ PEAKS.
 $n=0, 1, 2, 3, \dots, 50.$

	$(N(P(k_1)))_n$	$P(k_1)$		$(N(P(k_1)))_n$	$P(k_1)$
$N = 0$		1	0		
$N = 1$		1	0		
$N = 2$		2	0		
$N = 3$		2	1		
$N = 4$		4	0		
$N = 5$		16	1		
$N = 6$		8	0		
$N = 7$		16	0		
$N = 8$		88	1		
$N = 9$		16	2		
$N = 10$				512	0
				128512	1
				1304832	2
				1841152	3
				353792	4
$N = 11$					
				1024	0
				518656	1
				8728576	2
				21253376	3
				9061376	4
				353792	5
$N = 12$					
				2048	0
				2084864	1
				56520704	2
				222398464	3
				175627264	4
				22368256	5
$N = 13$					
				4096	0
				8361984	1
				357888000	2
				2174832640	3
				2868264960	4
				795300864	5
				22368256	6
$N = 14$					
				8192	0
				33497088	1
				2230947840	2
				20261765120	3
				41731645440	4
				21016670208	5
				1903757312	6

TABLE 2.6 (cont.)

$(N(P(k_1)))_n$	$P(k_1)$	$(N(P(k_1)))_n$	$P(k_1)$
$N = 15$		$N = 20$	
16384	0	524288	0
134094848	1	137433710592	1
13754155008	2	112949304754176	2
182172651520	3	7984436548730880	3
559148810240	4	122829335169859584	4
460858269696	5	584901762421358592	5
89702612992	6	990081991141490688	6
1903757312	7	603968063567560704	7
<hr/>			
$N = 16$		$N = 21$	
32768	0	1048576	0
536608768	1	549744803840	1
84134068224	2	680032201605120	2
1594922762240	3	65569731961159680	3
7048869314560	4	1332091026832097280	4
8885192097792	5	8369943835924758528	5
3099269660672	6	19125263737773096960	6
209865342976	7	16594062955071406080	7
<hr/>			
$N = 17$		$N = 22$	
65536	0	514513339477278720	8
2146926592	1	453245464669061120	9
511780323328	2	4951498053124096	10
13684856848384	3	<hr/>	
84842998005760	4	2097152	0
155964390375424	5	2199000186880	1
87815735738368	6	4090088616099840	2
12655654469632	7	535438370914959360	3
209865342976	8	14238886515777208320	4
<hr/>			
$N = 18$		$N = 23$	
131072	0	116424418353082269696	5
8588754944	1	351453130688070942720	6
3100738912256	2	418507117183327272960	7
115620218667008	3	192176777841019453440	8
985278548541440	4	29645442651290337280	9
2550316668551168	5	1015423886506852352	10
2165206642589696	6	<hr/>	
553753414467584	7	4194304	0
29088885112832	8	8796044787712	1
<hr/>			
$N = 19$		$N = 24$	
262144	0	24582312700149760	2
34357248000	1	4353038473793372160	3
18733264797696	2	150420440721496473600	4
965271355195392	3	1582198544942090944512	5
11124607890751488	4	6201012431516898164736	6
39471306959486976	5	9859192051125874851840	7
48165109676113920	6	6388731821421641072640	8
19686087844429824	7	1553792742230904012800	9
2184860175433728	8	111275653457021763584	10
29088885112832	9	1015423886506852352	11
<hr/>			

TABLE 2.6 (cont.)

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$(N(P(k_1)))_n$	$P(k_1)$
N = 24	
8388608	0
35184271425536	1
147669797096652800	2
35266789418949672960	3
1573853022795658690560	4
21092268709406041964544	5
105800556580541665640448	6
219757197133182979276800	7
193870709194596538122240	8
69408245773147926691840	9
8663235344978094850048	10
246921480190207983616	11
<hr/>	
N = 25	
16777216	0
140737278640128	1
886757652279853056	2
284940041496433786880	3
16338065648078731345920	4
276715019854807383932928	5
1755407285349861864505344	6
4679921276516885990473728	7
5467487539701384499691520	8
2745259879825134300692480	9
537632406455257720160256	10
31915821559499276156928	11
246921480190207983616	12
<hr/>	
N = 26	
33554432	0
562949517213696	1
5323642133809201152	2
2297255485017067356160	3
168509577227723121623040	4
3581989288626948308729856	5
28449712272865369478135808	6
95943627848468518221643776	7
145213988479793780899184640	8
98645097914113762011381760	9
28299472220966475647680512	10
2916509343249013508407296	11
70251601603943959887872	12
<hr/>	
N = 27	
67108864	0
2251798907715584	1
31954800641751121920	2
18489840364946532073472	3
1728743626492555495997440	4
45848534276394672772349952	5
452025811149519397324849152	6
1904944305122746094762065920	7
3669231698969441756623405056	8
3279827854600419268320296960	9
1313104074260058798328643584	10
211493585342808702440177664	11
10576069671449583482306560	12
70251601603943959887872	13

TABLE 2.6 (cont.)

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$(N(P(k_1)))_n$	$P(k_1)$
N = 28	
134217728	0
9007197375692800	1
191782847024291905536	2
148621728533690781270016	3
17657233072224485601443840	4
581299796593602072196153344	5
7061937904515586326905487360	6
36807470238057209078740942848	7
88905502242922904756366082048	8
102288874081702802932639989760	9
55126912470524647709792534528	10
12954470493787761648536125440	11
1120952152828923980300681216	12
23119184187809597841473536	13
<hr/>	
N = 29	
268435456	0
36028793126649856	1
1150922262080143753216	2
1193384833751084963987456	3
179693387021452362421108736	4
7311084987495490092781273088	5
108749227205309443804011429888	6
694848592376649140163437395968	7
2078796153467356003638221733888	8
3023738006306208010972826697728	9
2133391941086867476009195667456	10
696795679144578813533414752256	11
93917108442490831730498338816	12
4010193615745440680463302656	13
23119184187809597841473536	14
<hr/>	
N = 30	
536870912	0
144115180022792192	1
6906470321102155415552	2
9574700804298603161976832	3
1823188336557047493418811392	4
91326887590374928361797451776	5
1654088710649251034926222934016	6
12857565113311337343479181213696	7
47146211055685496027776114753536	8
85420313967732432263115194761216	9
77172002766973164581930571661312	10
33790231828164831332875519393792	11
6622618894372234506193445322752	12
487953855010835665974965829632	13
8713962757125169296170811392	14
<hr/>	

TABLE 2.6 (cont.)

- 71 -

N = 31

$(N(P(k_1)))_n$	$P(k_1)$
1073741824	0
576460735660425216	1
41442713036473547882496	2
76770268192416379181203456	3
18452101484069342806913581056	4
1134209606152197137703364460544	5
24892452813306638127841272659968	6
233840549894018665089412689297408	7
1041495275702008988652157783769088	8
2321307023078560093623393387020288	9
2637407514518466375696739718922240	10
1505513588778714433226387610402816	11
408719714050831916491158214148096	12
46775802412164571178266269843456	13
1725280447746262076810021830656	14
8713962757125169296170811392	15

N = 32

2147483648	0
2305842974853955584	1
248672419119439779201024	2
615239656078279345694572544	3
186363501277311421169484693504	4
14016461506475891194192472309760	5
371178531509336876543845106450432	6
4189512948943818127731745936637952	7
22488363760940460437169443136602112	8
61007074321399327713598076713172992	9
85878649596348981388808994460532736	10
62506401275873810154400699838889984	11
22670821275551345294581214451073024	12
3762040751845599491938404840505344	13
238861623081046147017365734293504	14
3729407703720529571097509625856	15

N = 33

4294967296	0
9223371965987815424	1
1492101384162909439918080	2
492863140394245963959500800	3
1879016004175071195337211248640	4
172483898607088857017207815667712	5
5490845132766709986691873408811008	6
74084599281778490698040992008765440	7
476012267828973196040489657381683200	8
1557466942842093460829503181312491520	9
2682422257297868850830572875402969088	10
2444818776180810238982515735199219712	11
1151998964647199269048717874277908480	12
264032889980536202836343836691660800	13
25976052451659381870212996231331840	14
835925915762195387327217510907904	15
3729407703720529571097509625856	16

TABLE 2.6 (cont.)

- 72 -

	$(N(P(k_1)))_n$	$P(k_1)$
N = 34		
	8589934592	0
	36893488001390215168	1
	8952885006136436273971200	2
	39470830070296238581077770240	3
	18918304458253215904001582694400	4
	2114903167385267992894586857979904	5
	80666477628089894668064799668043776	6
	1295170491163790050902493340316467200	7
	9901743607993530361293551689028075520	8
	38765535142105440353237898144356761600	9
	80817826860342423169885647797240201216	10
	90864717715913871945547252149616902144	11
	54400160842635283385091822083217817600	12
	16608912636632607831807370421589770240	13
	2363478913432998673124452907089920000	14
	130653839111027779875322945274380288	15
	1798651693450888780071750349094912	16
<hr/>		
N = 35		
	17179869184	0
	147573952289028702208	1
	53718453734946660740497408	2
	316026274227547865300567326720	3
	190248756994430157481704926740480	4
	25851795620079546312335081863118848	5
	1177973459643119689189482693086150656	6
	22416723888810528602469254238092394496	7
	202839624275995557470431303868518236160	8
	943640344177998823206748341600612515840	9
	2359475218059114915036052723704635850752	10
	3231384974366384427901647472954928267264	11
	241391607678356959413407147809449181184	12
	954651001409430569756432770553473925120	13
	187166755859418215016385180163825991680	14
	15998317418717882321632598784229769216	15
	453115674910413558148408347692367872	16
	1798651693450888780071750349094912	17
<hr/>		
N = 36		
	34359738368	0
	590295809740230361088	1
	322315444776153213361455104	2
	2529821747432431322226753536000	3
	1911336305622672915045465152552960	4
	315168015122809739842545310452678656	5
	17112071529885584760148799667920961536	6
	38458299833117090801676687057373626368	7
	4099447714744130606517148554395176140800	8
	22523920120527896498602730301645578567680	9
	67006700304148509302101133387111788969984	10
	110585892437620835080144277482783180324864	11
	101538437688769432079568355518504817917952	12
	50869388807299755547314189053591761715200	13
	13252210687057991008543017569342571151360	14
	1634946692555481524390554242078308564992	15
	79399202621825590263576278958459584512	16
	970982810785059112379399707952152576	17

TABLE 2.6 (cont.)

- 73 -

N = 37

(N(P(k ₁))) _n	P(k ₁)
68719476736	0
2361183240163512287232	1
1933912148418640707770646528	2
20248565758247511327428233396224	3
19186727886902269658799227378073600	4
3833622261725529046816771284551073792	5
247448201796468430138146828112210427904	6
6546905618517241902310249385280160137216	7
81866301830389809824143884407318016688128	8
52836790890696411495880428566419918028800	9
1857054048740241445122471349644434192990208	10
3659161923065127681454979660393473162346496	11
4077615981597076690110652850757306610089984	12
2541265701180856906200049204204122325123072	13
855390819877437530182118228562602989977600	14
145083768971181345840298858731903872139264	15
1087430635191947769914364694979168698368	16
223152989053738896636387226361656246272	17
970982810785059112379399707952152576	18

N = 38

137438953472	0
9444732963127950311424	1
11603553170742009806041645056	2
162050411254729487122074527856688	3
192474735841770121927815120782622720	4
46540695521539612112247633781198946304	5
3563949003955421777151291646969273909248	6
110689246739391112760279514039175612465152	7
1617625356554395898685415405807887823405056	8
12204684216421724426400486259474758694338560	9
50365811434117847199620217406373110770302976	10
117532750933406927476879053443754302984159232	11
157246282444435765883246689365198596135190528	12
120086831412228906510929211926803104424525824	13
51074381608131694987464038898919312950558720	14
11485807166097303308346509307921747828277248	15
1240229029792350316532881552020714968580096	16
53330733013612511121767398928936299659264	17
583203324917310043943191641625494290432	18

N = 39

274877906944	0
37778931857597042524160	1
69621649590105768314510770176	2
1296786207292470383300195597156352	3
1929770921166597833378935518189846528	4
564970113597886678882878243877083414528	5
51151884834457474407148769169662206279680	6
1860126672929143348593254515801041647173632	7
31663109092985121769823906127443019907989504	8
278063816816076802400403448711460818178277370	9
1339936851662605402493254021870228852139098112	10
3677004816781169661838640978558446154714972160	11
5851394607556433825117599725151478044277342208	12
5406632951320074338788224895698068673644199936	13
2853186593778468821244142498102413537186545664	14
827215263788298960751264647943769747059900416	15
122568437175621033920543537924156543729664000	16
8121051537451801983048034121545281630633984	17
182153925387695315035143479168577682014208	18
583203324917310043943191641625494290432	19

TABLE 2.6 (cont.)

- 74 -

	$(N(P(k_1)))_n$	$P(k_1)$
N = 40		
	549755813888	0
	151115727440833530560512	1
	417731257582181483380595490816	2
	10376656794425826662524258143436800	3
	19339206370299337386054961441007468544	4
	6826734490809638081595906992070696370176	5
	731920350863145468708803359203829223522304	6
	31091975772562187912077940251227883718049792	7
	614579003824031632223068418673199357875978240	8
	6257864736367208726944194909032962801541316608	9
	35039887072898854902859657455374251110625705984	10
	112366978932689369129005955879066827051663097856	11
	210968336864975594042475848510873567626650451968	12
	233305247142752155037716693231666615481920389120	13
	150475193229194956702782973693249230199326769152	14
	55002754379010254956577893715824767277782269952	15
	10785048974277506839332597472971480463287779328	16
	1027768478401991074912990455920569401080807424	17
	39406055314539629903527588694587078439075840	18
	387635983772083031828014624002175135645696	19
N = 41		
	1099511627776	0
	604462909784774598983680	1
	2506393136775004211124203683840	2
	83027874949421989652112385989672960	3
	193734493377209426140412914928808099840	4
	82520329287194936438118587709519587966976	5
	10444860212317516066289528331623659324047360	6
	517233461834299934248384734718149528523898880	7
	11839721463146624077817180042398285534718853120	8
	139292611815296902080014471810142841261973831680	9
	902292675067486191128740557107925743266133180416	10
	3362565348769623102250476432749714620341802762240	11
	7395415400345084720297473311226848818171184414720	12
	9697071972971694251693205138149768747893527674880	13
	7547224009731626716573806222809142907244768133120	14
	3415315265649472682341105309532134085081627099136	15
	861716454536527527146509357523453241251824926720	16
	112495008042414226572195838723940861681923522560	17
	6636272493962461310899000649997155986088919040	18
	133723605294502210983703351043848240743055360	19
	387635983772083031828014624002175135645696	20

TABLE 2.6 (cont.)

- 75 -

N = 42

$(N(P(k_1)))_n$	$P(k_1)$
2199023255552	0
2417851639183078861045760	1
15 38381790240597088179983482880	2
664313229748299817368499559250001920	3
1940167881520374609052300970411729879040	4
996443455234409938893916265791956914798592	5
148703652851061073021196954274016818175672320	6
8568191475293689397830262548775854917455708160	7
226563056344331031691167243865841027366560727040	8
307000555142145701946790175720415678072729108480	9
22914876311421228050592610636197508859618354266112	10
98747421871820678276586245528151665753525929902080	11
252806976686825418568242881881392932438603244503040	12
389844661648728794572169316847823106031757725532160	13
362175727913552521020919058618371049687852431441920	14
199856776617562646433801044578738005609549284769792	15
6345151211073666274639237125118751053378318499840	16
10943551925819132373771125054249496950563846225920	17
927148403025058889247337057043537097562920058880	18
31894034187629933682944136641742553574077890560	19
283727921907431909304183316295787837183229952	20

N = 43

4398046511104	0
9671406556822475397660672	1
90230385037657510669155481681920	2
5315062258112637441040259133388881920	3
19424929778244936584130907188691048857600	4
12021347002903091628825721121527070063591424	5
2112740887027121730402468764075786118818168832	6
141403469537379801482898912454360166406385827840	7
4309476184030888184182427478402086575369397207040	8
67064187437037416181637216241054339245618600345600	9
574737406533960528560799174412414755507276563349504	10
2851150527463542067700514716035787664136607757238272	11
8449182409422053770029453593951097892720677025218560	12
15213369129840438363680869863722726820345471471452160	13
16712941762137507549210111511268478081111938826240000	14
11103701314638187459153581188558439825447658721378304	15
4355775954558235644149152112904155597519904961462272	16
965031478326118730173291843213050649700703330631680	17
111836502795686164407796683547400888361337885818880	18
5911503382630491793554450750887387630777715916800	19
107598675283001941239608109209650749883929329664	20
283727921907431909304183316295787837183229952	21

TABLE 2.6 (cont.)

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N = 44

$(N(P(k_1)))_n$	$P(k_1)$
8796093022208	0
38685626227474619544109056	1
541382697082207336913948796518400	2
42523926819532530513727500975414968320	3
194440640023741420789186521215712488325120	4
144916611647297427389769104302740336424255488	5
2996305522472603157756985772949871905489289216	6
2325837739208890475638456662192036246066718310400	7
81529868459602621756804864159959643016027952906240	8
145333012952551416421487439259541035871976334295040	9
14253763442236029616696875026858428763054930801983488	10
81071835602872141253149935021932028560438670567407616	11
276701753194244239374776087763444298493469757800448000	12
578059619005129242043594520875356113038645387654594560	13
74480215894157224029519726315761797155885708330434560	14
58929962673834710438185615919162876754989222651465728	15
281340798230638261410914146101442568221048673346256896	16
78298892765322630727730027484711379364424369517363200	17
11970038932845024088882608720505638955336466306170880	18
907479152079336658188958131319900835399135951585280	19
28165157892408048706281343590354882018235895513088	20
227681379129930886488600284336316164603920777216	21

N = 45

17592186044416	0
154742504910276710176391168	1
3248297768603919347943094087581696	2
340212528481446450195959651806383964160	3
1945979785529736911520873129693215237079040	4
1745804762168400078404850779875433974182445056	5
424265025498977259312460181263288637778850480128	6
38142258548538898308105773154034025966137660932096	7
1534986926709905015416002798082842625424437983313920	8
31267909038920299115863480117509731078872281414369280	9
349916048967331436977868436572373958684007886001012736	10
2273560613640360071259626566144112547000791502063403008	11
8896754130710665190060326917310124360599425784727207936	12
2144300264283425932534139225201541283645796252537159680	13
32171078291334364323597024749609593068423542840041144320	14
3002962043975069094647356041496390134979836749803421696	15
17226482287440213244935211036940221297664253788241788928	16
5913508920088635581718336596565477907550812709433901056	17
1159551514336014591927109378741616694582605045290762240	18
120089438613088634949736586296335506103320702206607360	19
5720332391877821336608607087394409221761587369476096	20
94513454358941105124342443281862557297280200736768	21
227681379129930886488600284336316164603920777216	22

TABLE 2.6 (cont.)

- 77 -

$(N(P(k_1)))_n$	$P(k_1)$
N = 46	
35184372088832	0
618970019641880896891518976	1
19489793110808722319280391933919232	2
2721830159762315758341594938214574981120	3
19472725931379664080316177763700794961428480	4
21019712418299871469672960791174163438724186112	5
5999067718899407233040207464201805684026109853696	6
623852617592589645227691096264969651867125790277632	7
28774032437234457226731223560112188036624013527613440	8
668337814726283322748917680696514215089329891820175360	9
8511118712293219390525556087647480099098852808795881472	10
62963439902584596197699880065195676136435185313545977856	11
281333940898565216509280284305233709409602483448302272512	12
778339156613572564910765521402634046632812410765584629760	13
1351106396311047597564055803024565223108950217746903203840	14
1475685106733371939406267789321637973414131461434367803392	15
1006115083929476923552860159836916986010302343297468727296	16
419604108572473439881082649919639860243800302998521905152	17
103198046745654910310413522357836213469647118815387975680	18
14079989659211662133406338481786353800793668450590392320	19
960790592137400305835981015448578223933910882757640192	20
27039921559304693971905495853979589408126678310322176	21
19950025215785903102716049964319565816634 757225472	22

N = 47

70368744177664	0
2475880078569106884310073344	1
116938785280563178516560917938831360	2
21775440359616069224347850001785890537472	3
194833410690027371117737099839598318038548480	4
252957039879059505207047228071346890678263087104	5
84722637999232196764001458126516375296720884465664	6
1017961111620511476233384386558174017446874269614080	7
53727201501559050908322044806623343867113142995648512	8
14201203235205465714553559097173537754848694228704296960	9
20528973266806047630578301130705044598758668872654127104	10
172390052546936079350793602375583229751915767745000505344	11
8762841581122141341788384633435576995787673831867416576000	12
2770150914404990136419632056968361203320399653850717356032	13
55321635864989306660226218997387003579291942336953204080640	14
70190732152755711219589517909710023942104360467597124173824	15
56343189454602794491891262274279747125562251593629453778944	16
28185243999692243841906157474986955786910741370813882040320	17
8537170970632094430487622998713814574528393847968484032512	18
1491982007079360678129975240491980073258570807362107473920	19
1389131324842524477895572021344762010779936229953634304	20
5993709509296408063943746894817993053627128259442376704	21
90296776277175597342965870545525768500031709763338240	22
199500252157859031027160499643195658166340757225472	23

TABLE 2.6 (cont.)

- 78 -

N = 48

$(N(P(k_1)))_n$	$P(k_1)$
140737488355326	0
990352031427964499472465920	1
791632820622102528140068417276215296	2
174208434305910337448280495572840555216896	3
1949205124514658353946344912396054616006983680	4
3042688148154935102587640746650067424224621887488	5
1195223385424896896883474113981797742218509853655040	6
165754347551255730687310199761232341039238498385657856	7
999643825999192835822636365562063558166336550549323776	8
300142225154577029563567705385457758412987278863955394560	9
4914007309283083518734725903475968868862670153602111111168	10
46711145560634231428140622864124509109723231816569019432960	11
269207493720440333930688465039466199404525498054432843104256	12
968424770818084347716841437886725207600300014608903250640896	13
221367925830677227090712981315203331445166263185610469539840	14
3241892874457990278910936515063686830574594497028265647013888	15
3038720155900586392237735203880871785342786321664955415265280	16
1803473436353359901195099340939446868086658211660112106356736	17
662635424880326514461403563650968423275007862672568977719290	18
145050989989495371430075239606817348675626770774169139281920	19
17770207620973488231755935948832320590521323780554912432128	20
1097202013314556041487258295500060266424273260795186380800	21
281284897459357097335514176243615756550971686883065856	22
190169564657928428175235445073924928592047775873499136	23

N = 49

281474976710656	0
39614081257125272659842564096	1
4209797369391029311425312979918258176	2
139369764465856944994953987524667318992896	3
19499193790953125863298828624279032622833729536	4
36590676777715292906848396411384255220719735013376	5
1684571425743 289155164358103371220885755448961007616	6
2693902379309963085587884790169080377605463819048452096	7
185409482337176910165348691172237811301288928356614537216	8
6312740461697515569184027635041579138562902010346136928256	9
116812296333710571269507435942649590108955374466301150887936	10
1253745706705864809281217348132839378092649657744913466392576	11
8167173478247304467901420662629233912260743744829479406534656	12
33307665938476489416477394956736028399112486864301246409342976	13
86747297952100088114775059635079729302961288202355282349654016	14
145800477900438556239873515127028361875845182905431099625701376	15
158428664166405772077568917688032316821422841386089000118255616	16
11050584604722975232658960433203316403126149044738367057821696	17
48625300818046086265069626850949609369576055533139078535970816	18
13091029273263406516278448784433346603050157320365024326189050	19
2051807629986344848604426466312313602882536535750828575686650	20
172568341932654909847730916643828095856317289938872587780096	21
6779920594875825855179656088221144580134825001572552933376	22
93513608341387693753065554236646869268948208302577156096	23
190169564657928428175235445073924928592047775873499136	24

TABLE 2.6 (cont.)

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$n = 50$	$(N(P(k_1)))_n$	$P(k_1)$
	562949953421312	0
	158456325028514601438252367872	1
	25258786038593913696314420232267497472	2
	11149766398352808305249334613968454955302912	3
	195050473210606918549888074310266362255734996992	4
	439868089084221639916712710081582223953550169341952	5
	237230445321577229302761252510829694098963635384614912	6
	43708383782226899776992073534426649993574617267371507712	7
	3428963362965723127886264523966029336261786480266709041152	8
	132187912668739972508971710818343192732899285914334403756032	9
	2759252733192559035004684419789538356553005298569009427382272	10
	3336064125828465141829542456158233597274342270934355418284032	11
	24494389880878240120674858279813905549188228466934214696108032	12
	1128626809755277010891001154691710409069407482076342405218435072	13
	3335187589209485710605754478100584503869313357085285891495165952	14
	6400561251856035561971453676766502166086271617020900835015524352	15
	8010983183865690262955066473679609285693589899424785797283315712	16
	6513069084362763436998328438961710974268179118188005215973670912	17
	3394843275746967810644900280984549452481581376485622123176394752	18
	1107144780747089295031973473588729176556928559212269915479212032	19
	217086213192060548604170399429450637351568107705185043440730112	20
	24011868084927574822135572062826980240738253043317022467817472	21
	1347886398960677448424649707521146025824105689705572961615872	22
	31608335579929912720865773356243628045448814004813915226112	23
	196535694915671808914892880726989984967498805398829268992	24

It has been pointed out in /SC76/ that

$$\sum_K (a_{1,k_1 k_2 k_3})_n = n! , \text{ id est, THE SUM OF ALL COEFFICIENTS IS} \\ \text{FOR A GIVEN } n \text{ EQUAL to } n! \quad (2.16.3)$$

We tabulate $n!$ for the sake of completeness in TABLE 2.7,
 $n=1,2,3,4,\dots,50.$

TABLE 2.7: $n!$ for $n=1, 2, 3, \dots, 50$.

$n!$	n
1	1
2	2
6	3
24	4
120	5
720	6
5040	7
40320	8
5628800	9
3628800	10
59916800	11
479001600	12
6227020800	13
87178291200	14
1307674368000	15
20922789888000	16
355687428096000	17
6402373705728000	18
121645100408832000	19
2432902008176640000	20
051090942171709440000	21
1124000727777607680000	22
25852016738884976640000	23
620448401733239439360000	24
15511210043330985984000000	25
403291461126605635584000000	26
10888869450418352160768000000	27
0304888344611713860501504000000	28
8841761993739701954543616000000	29
265252859812191058636308480000000	30
8222838654177922817725562880000000	31
263130836933693530167218012160000000	32
8683317618811886495518194401280000000	33
295232799039604140847618609643520000000	34
10333147966386144929666651337523200000000	35
371993326789901217467999448150835200000000	36

TABLE 2.7 (cont.)

- 82 -

n!	n
13763753091226345046315979581580902400000000	37
523022617466601111760007224100074291200000000	38
20397882081197443358640281739902897356800000000	39
815915283247897734345611269596115894272000000000	40
033452526613163807108170062053440751665152000000000	41
1405006117752879898543142606244511569936384000000000	42
60415263063373835637355132068513997507264512000000000	43
2658271574788448768043625811014615890319638528000000000	44
119622220865480194561963161495657715064383733760000000000	45
5502622159812088949850305428800254892961651752960000000000	46
25862324151116818064296435515361197996919763238912000000000	47
12413915592536072670862289047373375038521486354677760000000000	48
608281864034267560872252163321295376887552831379210240000000000	49
30414093201713378043612608166064768844377641568960512000000000000	50

Below we give some other possible representations of the numbers $(a_{1,k_1 k_2 k_3})_n$.

The relation

$k_1 + k_2 + k_3 = n+1$ permits a representation of these numbers in form of a matrix for each n .

Such a representation has been applied in /SC76/. For $n=6$ we have the following matrix

h_{31}	0	1	2	3	$n=6$
h_{21}					
0				1	1
1			135	44	179
2		135	328	16	479
3	1	44	16		61
272	416	32			$\begin{matrix} RU(h_{21}) \\ P(h_{11}-1) \end{matrix}$

Recall that $h_{11} = n - h_{21} - h_{31}$, $h_{21} = (n - k_2)/2$, $h_{31} = (n - k_3)/2$.

$RU(h_{21})$ is the number of permutations of $n=6$ natural numbers with h_{21} RUNS-UP.

$P(h_{11}-1)$ is the number of permutations of $n=6$ natural numbers with $h_{11}-1$ PEAKS.

This representation reveals still an additional SYMMETRY of the coefficients $(a_{1,k_1 k_2 k_3})_n$ namely,

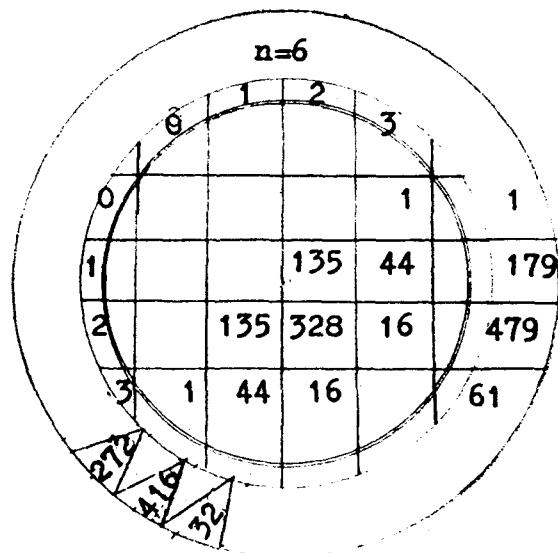
$$(a_{1,k_1 k_2 k_3})_n = (a_{1,k_1 k_3 k_2})_n$$

which follows also directly from (2.1.3) as this system remains invariant when interchanging the indices (2,3).

Another representation of these numbers is the following

n=6	0	1	2	3	
0				1	1
1			135	44	179
2		135	328	16	479
3	1	44	16		61
	272	416	328		

Still another representation of them is



DUMOND /DU79/ represented these numbers in the form

j¹	0	1	2	3
0	1	44	16	0
1	135	328	16	0
2	135	44	0	0
3	1	0	0	0

n=6

$$i=(k_1-1)/2, \quad j=k_2/2 \quad \text{for } n\text{-even} ;$$

$$i=k_1/2, \quad j=(k_2-1)/2 \quad \text{for } n\text{-odd} .$$

In conclusion, we emphasize once more that a monogenerator produces the three Jacobian elliptic functions: $\text{sn}(u,k)$, $\text{cn}(u,k)$, $\text{dn}(u,k)$ as well as the four combinatorial sets:

- the number of permutations with given runs-up,
- the number of permutations with given peaks,
- the number of permutations with given cycle-peaks, and
- the total number of permutations.

HENCE, A MONOGENERATOR ENGENDERS A SEVEN-FOLD MATHEMATICAL ENTITY. This inspires quite generally the search for monogenerators of n-fold mathematical entities. The symbolization of differential equations by means of weight diagrams of Lie algebras (see ANNEX) might be a guide in such investigations.

EXAMPLE 4: Euler's equation of motion belong also to the type of system (2.1). The analytical solution of this equation is not known.

The system describing a general gyroscope reads

$$\begin{aligned}
 \frac{dY_1(t)}{dt} &= (B-C)Y_2 Y_3 / A - mgY_6 / A + mgzY_5 / A , \\
 \frac{dY_2(t)}{dt} &= (C-A)Y_3 Y_1 / B - mgzY_4 / B + mgxY_6 / B , \\
 \frac{dY_3(t)}{dt} &= (A-B)Y_1 Y_2 / C - mgxY_5 / C + mgY_4 / C , \\
 \frac{dY_4(t)}{dt} &= -Y_2 Y_6 + Y_3 Y_5 , \\
 \frac{dY_5(t)}{dt} &= -Y_3 Y_4 + Y_1 Y_6 , \\
 \frac{dY_6(t)}{dt} &= -Y_1 Y_5 + Y_2 Y_4 .
 \end{aligned} \tag{2.1.4}$$

A, B, C are the principal moments of inertia,
m is the mass of the body,
g is the gravitational constant, and
x, y, z are the coordinates of the mass-center relative to the principal axes at the apex of the body.

Comparing (2.1.4) with (2.1) one obtains for b_{ij} and r_{ijs}

$$b_{11} = (B-C)/A, \quad b_{12} = -mgY/A, \quad b_{13} = mgz/A,$$

$$b_{21} = (C-A)/B, \quad b_{22} = -mgz/B, \quad b_{23} = mgx/B,$$

$$b_{31} = (A-B)/C, \quad b_{32} = -mgx/C, \quad b_{33} = mgY/C,$$

$$b_{41} = -1, \quad b_{42} = 1, \quad b_{51} = -1, \quad b_{52} = 1, \quad b_{61} = -1, \quad b_{62} = 1.$$

$$r_{112} = r_{113} = 1, \quad r_{126} = 1, \quad r_{135} = 1, \quad r_{211} = r_{213} = 1, \quad r_{224} = 1, \quad r_{236} = 1,$$

$$r_{311} = r_{312} = 1, \quad r_{325} = 1, \quad r_{334} = 1, \quad r_{412} = r_{416} = 1, \quad r_{423} = r_{425} = 1,$$

$$r_{513} = r_{514} = 1, \quad r_{521} = r_{526} = 1, \quad r_{611} = r_{615} = 1, \quad r_{622} = r_{624} = 1.$$

The solution of (2.1.4) reads in the form of (2.3)

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} a_{i,H,K} \cdot B^H \cdot y_1^{k_1} y_2^{k_2} y_3^{k_3} y_4^{k_4} y_5^{k_5} y_6^{k_6} \right)_n \quad (2.3.4)$$

$$i=1,2,3,4,5,6; \quad K=k_1 k_2 k_3 k_4 k_5 k_6;$$

$$H=h_{11} h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62}$$

$$B^H = b_{11}^{h_{11}} b_{12}^{h_{12}} b_{13}^{h_{13}} b_{21}^{h_{21}} b_{22}^{h_{22}} b_{23}^{h_{23}} b_{31}^{h_{31}} b_{32}^{h_{32}} b_{33}^{h_{33}} b_{41}^{h_{41}} b_{42}^{h_{42}} b_{51}^{h_{51}} b_{52}^{h_{52}} b_{61}^{h_{61}} b_{62}^{h_{62}}$$

(2.4.4)

Recurrence relation (2.5) becomes

$$\begin{aligned} (a_{i,H,K})_{n+1} &= ((k_1+1) \cdot a_{i,H_{11},k_1+1,k_2-1,k_3-1,k_4 k_5 k_6} + \\ &\quad (k_1+1) \cdot a_{i,H_{12},k_1+1,k_2 k_3 k_4 k_5 k_6-1} + (k_1+1) \cdot a_{i,H_{13},k_1+1,k_2 k_3 k_4 k_5-1,k_6} + \\ &\quad (k_2+1) \cdot a_{i,H_{21},k_1-1,k_2+1,k_3-1,k_4 k_5 k_6} + (k_2+1) \cdot a_{i,H_{22},k_1 k_2+1,k_3 k_4-1,k_5 k_6} + \\ &\quad (k_2+1) \cdot a_{i,H_{23},k_1 k_2+1,k_3 k_4 k_5 k_6-1} + (k_3+1) \cdot a_{i,H_{31},k_1-1,k_2-1,k_3+1,k_4 k_5 k_6} + \\ &\quad (k_3+1) \cdot a_{i,H_{32},k_1 k_2 k_3+1,k_4 k_5-1,k_6} + (k_3+1) \cdot a_{i,H_{33},k_1 k_2 k_3+1,k_4-1,k_5 k_6} + \\ &\quad (k_4+1) \cdot a_{i,H_{41},k_1 k_2-1,k_3 k_4+1,k_5 k_6-1} + (k_4+1) \cdot a_{i,H_{42},k_1 k_2 k_3-1,k_4+1,k_5-1,k_6} + \\ &\quad (k_5+1) \cdot a_{i,H_{51},k_1 k_2 k_3-1,k_4-1,k_5+1,k_6} + (k_5+1) \cdot a_{i,H_{52},k_1-1,k_2 k_3 k_4 k_5+1,k_6-1} + \\ &\quad (k_6+1) \cdot a_{i,H_{61},k_1-1,k_2 k_3 k_4 k_5-1,k_6+1} + (k_6+1) \cdot a_{i,H_{62},k_1 k_2-1,k_3 k_4-1,k_5 k_6+1})_n \end{aligned} \quad (2.5.4)$$

$$i=1,2,3,4,5,6;$$

$$H_{11}=h_{11}-1, h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62},$$

$$H_{12}=h_{11} h_{12}-1, h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62},$$

.....

$$H_{61} = h_{11} h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61}^{-1}, h_{62} ,$$

$$H_{62} = h_{11} h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62}^{-1} .$$

Equ.(2.1.4) remains invariant with respect to simultaneous interchange of the indices i and j of Y_i and b_{ij} :

1. $i:(1,2), (4,5)$ and

$j:(2,3)$ for $i=1,2,3$,

$(1,2)$ for $i=4,5,6$.

2. $i:(1,3), (4,6)$ and

$j:(2,3)$ for $i=1,2,3$,

$(1,2)$ for $i=4,5,6$.

These SYMMETRIES of (2.1.4) yield the following IDENTITIES

$$a_1, h_{11} h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62}, k_1 k_2 k_3 k_4 k_5 k_6 =$$

$$a_2, h_{21} h_{23} h_{22} h_{11} h_{13} h_{12} h_{31} h_{33} h_{32} h_{52} h_{51} h_{42} h_{41} h_{62} h_{61}, k_2 k_1 k_3 k_5 k_4 k_6 =$$

$$a_3, h_{31} h_{33} h_{32} h_{21} h_{23} h_{22} h_{11} h_{13} h_{12} h_{62} h_{61} h_{52} h_{51} h_{42} h_{41}, k_3 k_2 k_1 k_6 k_5 k_4 ,$$

$$a_4, h_{11} h_{12} h_{13} h_{21} h_{22} h_{23} h_{31} h_{32} h_{33} h_{41} h_{42} h_{51} h_{52} h_{61} h_{62}, k_1 k_2 k_3 k_4 k_5 k_6 =$$

$$a_5, h_{21} h_{23} h_{22} h_{11} h_{13} h_{12} h_{31} h_{33} h_{32} h_{52} h_{51} h_{42} h_{41} h_{62} h_{61}, k_2 k_1 k_3 k_5 k_4 k_6 =$$

$$a_6, h_{31} h_{33} h_{32} h_{21} h_{23} h_{22} h_{11} h_{13} h_{12} h_{62} h_{61} h_{52} h_{51} h_{42} h_{41}, k_3 k_2 k_1 k_6 k_5 k_4 .$$

(2.7.4)

This means. that one has only to compute $a_{1,H,K}$ and $a_{4,H,K}$.

If they are known then $a_{i,H,K}$ ($i=2,3,5,6$) are known too.

We give here the leading terms of $Y_1(t)$ and $Y_4(t)$ of (2.1.4) :

$$\begin{aligned} Y_1(t) = & y_1 + (t-t_0)(b_{11}y_2y_3 - b_{12}y_6 + b_{13}y_5) + \frac{(t-t_0)^2}{2!}(b_{11}b_{21}y_1y_3^2 + \\ & - b_{11}b_{22}y_3y_4 + b_{11}b_{23}y_3y_6 + b_{31}y_1y_2^2 - b_{32}y_2y_5 + b_{33}y_2y_4 - b_{12}y_5 + \\ & + b_{12}y_4 + b_{13}y_6 - b_{13}y_4) + \dots \end{aligned}$$

$$Y_4(t) = y_4 + (t-t_0)(y_3y_5 - y_2y_6) + \frac{(t-t_0)^2}{2!}(-b_{21}y_1y_3y_6 + b_{22}y_4y_6 - b_{23}y_6^2 + \\ + y_1y_2y_5 - y_2y_4^2 + b_{31}y_1y_2y_5 - b_{32}y_5^2 + b_{33}y_4y_5 - y_3y_4^2 + y_1y_3y_6) + \dots$$

Recall that $y_i = Y_i(t=t_0)$.

Before going to the next Section we summarize for the sake of convenience the benefits of the decomposed representation (2.3).

- The coefficients $a_{i,H,K}$ have only to be computed once for a given system (2.1). They are not affected if b_{ij} or the initial values y_i are altered.
- Symmetries in (2.1) yield identities of the form

$$(a_{i,H,K})_n = (a_{i_1,H_1,K_1})_n, \text{ formulae (2.7.i), } i=1,2,3,4.$$

Besides the mathematical interest of such identities, they are especially of numerical importance, as they lead to an additional reduction of computational labour, because $a_{i,H,K}$ has in this case not to be computed for all i . This also is a particularity of the decomposed representation (2.3).

- The $a_{i,H,K}$ have for certain systems a deeper mathematical meaning (combinatorial sequences /DU79/, /SC76/, Examples 2 and 3).
- The quantity B^H which reflects the physics of system (2.1) is also constant. If the b_{ij} are changed, the $(B^H)_n$ can readily be recalculated, as H is known from $a_{i,H,K}$. In practice it occurs frequently that the b_{ij} are changed (new experimental data, trial calculations, ...). For practical work it is thus very convenient to have the possibility to intervene in B only.

2.1.2 Case 2: $b(t)$ is a polynomial.

If $b_{ij}(t)$ in (2.1) are polynomials then (2.1) becomes a system of the form

where the b_{ij} are again constants.

Making (2.17) autonomous, i.e., putting $\dot{Y}_0 = t$ we obtain a system of type (2.1). Therefore, the formalism of the decomposed coefficient representation to solve (2.17) is quite similar to that to solve (2.1).

The solution functions of (2.17) can be represented as follows

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A \cdot B^H \cdot y_0^{k_0} y_1^{k_1} y_2^{k_2} \cdots y_I^{k_I} \right)_n , \quad (2.18)$$

$K = k_0 k_1 k_2 k_3 \dots k_I$: B^H is just (2.4).

A and B are again constants. The only difference of (2.18) as against to (2.3) is that in the former there is the additional factor y^k_0 .

The recurrence relation for $A = a_{i,H,K}$ is also similar to (2.5). It reads

$$(a_{i,H,K})_{n+1} = ((k_0+1) \cdot a_{i,H,k_0-1, k_1 k_2 k_3 \dots k_I} + \text{as (2.5), but with } m=0,1,2,3,\dots,I)_{n+1}. \quad (2.19)$$

The following example serves to illustrate these formulae.

EXAMPLE 5: We consider the system (2.1.4), but with $b_i = b_i(t)$:

$$dY_1(t)/dt = b_1(t)Y_2 Y_3 ,$$

$$dY_2(t)/dt = b_2(t)Y_1 Y_3 , \quad (2.17.1)$$

$$dY_3(t)/dt = b_3(t)Y_1 Y_2 .$$

Supposing that

$$b_1(t) = b_{11} + b_{12}t + b_{13}t^2 ,$$

$$b_2(t) = b_{21} + b_{22}t + b_{23}t^2 , \quad (2.20.1)$$

$$b_3(t) = b_{31} + b_{32}t + b_{33}t^2 , \text{ where } b_{ij} \text{ are constants,}$$

(2.17.1) becomes with $Y_0 = t$

$$dY_1(t)/dt = b_{11} Y_2 Y_3 + b_{12} Y_0 Y_2 Y_3 + b_{13} Y_0^2 Y_2 Y_3 ,$$

$$dY_2(t)/dt = b_{21} Y_1 Y_3 + b_{22} Y_0 Y_1 Y_3 + b_{23} Y_0^2 Y_1 Y_3 , \quad (2.17.1.1)$$

$$dY_3(t)/dt = b_{31} Y_1 Y_2 + b_{32} Y_0 Y_1 Y_2 + b_{33} Y_0^2 Y_1 Y_2 .$$

Equ.(2.18) becomes for this system

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A \cdot B^H \cdot y_0^{k_0} y_1^{k_1} y_2^{k_2} y_3^{k_3} \right)_n , \quad (2.18.1)$$

$$K = k_0 k_1 k_2 k_3 , \quad i=1,2,3.$$

Recurrence relation (2.19) reads for EXAMPLE 5

$$(a_{i,H,K})_{n+1} = ((k_0+1) \cdot a_{i,H,k_0+1,k_1 k_2 k_3} +$$

$$(k_1+1) \cdot a_{i,H_{11},k_0,k_1+1,k_2-1,k_3-1} + (k_1+1) \cdot a_{i,H_{12},k_0-1,k_1+1,k_2-1,k_3-1} +$$

$$\begin{aligned}
 & (k_1+1) \cdot a_{i,H_{13},k_o-2,k_1+1,k_2-1,k_3-1} + (k_2+1) \cdot a_{i,H_{21},k_o,k_1-1,k_2+1,k_3-1} \\
 & (k_2+1) \cdot a_{i,H_{22},k_o-1,k_1-1,k_2+1,k_3-1} + (k_2+1) \cdot a_{i,H_{23},k_o-2,k_1-1,k_2+1,k_3-1} \\
 & (k_3+1) \cdot a_{i,H_{31},k_o,k_1-1,k_2-1,k_3+1} + (k_3+1) \cdot a_{i,H_{32},k_o-1,k_1-1,k_2-1,k_3+1} \\
 & (k_3+1) \cdot a_{i,H_{33},k_o-2,k_1-1,k_2-1,k_3+1})_n
 \end{aligned}$$

With the starting values (2.6.1). (2.19.1)

System (2.17.1.1) remains invariant if one interchanges cyclicly the indices $i:(123)$.

These SYMMETRIES of (2.17.1.1) yield the IDENTITIES

$$\begin{aligned}
 & (a_{1,h_{11}h_{12}h_{13}h_{21}h_{22}h_{23}h_{31}h_{32}h_{33},k_0k_1k_2k_3})_n = \\
 & (a_{2,h_{21}h_{22}h_{23}h_{11}h_{12}h_{13}h_{31}h_{32}h_{33},k_0k_2k_1k_3})_n = \\
 & (a_{3,h_{31}h_{32}h_{33}h_{21}h_{22}h_{23}h_{11}h_{12}h_{13},k_0k_3k_2k_1})_n
 \end{aligned}$$
(2.21.1)

The benefits of the decomposed coefficient representation (2.18) are the same as those which we have summarized at the end of Subsection 2.1.1.

We turn now to discuss systems where $b_i(t)$ are arbitrary functions.

2.1.3 Case 3: $b(t)$ is an arbitrary function.

If $b_{ij}(t)$ in system (1.1) are arbitrary functions then it is appropriate to distinguish three different cases.

Case 3.1: System (1.1) can by means of TRANSFORMATION be reduced to system (2.1). The following examples serve to illustrate that.

EXAMPLE 6: Matter which is irradiated by means of a particle beam is changing its composition with time. Such a matter conversion is described by the differential equations

$$\frac{dY_i(t)}{dt} = \sum_{j=1}^J b_{ij} Y_j(t), \quad i=1,2,3,\dots,I, \quad (2.22)$$

$$b_{ij} = P_{ij} \cdot F, \text{ where}$$

Y_i indicates the quantity of the nucleus i , P_{ij} denotes the reaction probability and F designates the particle beam flux.

Supposing that P_{ij} are constants (given energy group) and

$F(t) = c e^{a(t-T)}$; a, c, T constants, and putting

$$Y_{I+1}(t) = F(t) \quad (2.23)$$

then

$$\frac{dY_{I+1}(t)}{dt} = a \cdot Y_{I+1}, \text{ and}$$

System (2.22) becomes then

$$\frac{dY_i(t)}{dt} = \sum_{j=1}^J P_{ij} Y_{I+1} Y_j \quad (2.22.1)$$

$$\frac{dY_{I+1}(t)}{dt} = a \cdot Y_{I+1}$$

This means, we have in fact a system of the type (2.1).

EXAMPLE 7: A pendulum moving in a plane is described by means of the equation

$d^2Y(t)/dt^2 = b \sin t$ or by the equivalent system

$$\begin{aligned} dY_1(t)/dt &= Y_2, \\ dY_2(t)/dt &= b \sin t. \end{aligned} \quad (2.24)$$

Putting $Y_3 = \sin t$ then $dY_3(t)/dt = \cos t = Y_4$ and

$$dY_4(t)/dt = -\sin t = -Y_3.$$

System (2.24) becomes, therefore,

$$\begin{aligned} dY_1(t)/dt &= Y_2, \\ dY_2(t)/dt &= bY_3, \\ dY_3(t)/dt &= Y_4, \\ dY_4(t)/dt &= -Y_3. \end{aligned} \quad (2.24.1)$$

In fact (2.24.1) is again a system of the type (2.1).

Case 3.2: $b_{ij}(t)$ in system (1.1) can be approximated with the help of POLYNOMIALS.

Note that for applications in natural science, such an approximation is in general sufficient, as the $b_{ij}(t)$ are usually based on experimental data which in turn are by nature never exactly determined.

Now, if $b_{ij}(t)$ in (1.1) are replaced by polynomials then system (1.1) reduces to system (2.17).

Case 3.3: Neither a transformation, as discussed in Case 3.1, nor a substitution of polynomials for $b_{ij}(t)$ is applicable. The solution of (1.1) reads in this case

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A.B(t)^H \cdot y_1^{k_1} y_2^{k_2} y_3^{k_3} \cdots y_I^{k_I} \right)_n, \text{ where } (2.25)$$
$$i=1,2,3,\dots,I, \quad k_1+k_2+\dots+k_I,$$

$H = h_{q,ls}$, $s=1,2,3,\dots,S$; $l=1,2,3,\dots,I$; $q=0,1,2,3,\dots,n-1$ for a term of order n . If b_{ls} are constants, then $q=0$ for terms of any order.

$$(B(t)^H)_n = \prod_{q=0}^{n-1} \prod_{l=1}^I \prod_{s=1}^S b_{ls}^{h_{q,ls}}, \quad (2.26)$$

here q symbolizes the number of derivations of b_{ls} , for example,

$$b_{ls}^{h_{3,ls}=5} = \left(\frac{d^3}{dt^3} b_{ls} \right)^5.$$

The recurrence relation for $A=a_{i,H,K}$ has the form

$$\begin{aligned} (a_{i,H,K})_{n+1} &= \left(\sum_{l=1}^I \sum_{s=1}^S ((k_l+1-r_{ls}) \cdot a_{i,k_1+D_1^l-r_{ls}, k_2+D_2^l-r_{ls}, \right. \\ &\quad \left. k_3+D_3^l-r_{ls}, \dots, k_I+D_I^l-r_{ls}; H_{0,ls}) + \sum_{q=1}^n \sum_{l=1}^I \sum_{s=1}^S (h_{q-1,ls}+1) \cdot a_{i,H_{q,ls},K} \right)_n \end{aligned} \quad (2.27)$$

where

$$D_i^l = 1 \text{ for } i=l, =0 \text{ else.}$$

$H_{0,ls} = H_{-1,0,ls}$, that means, the element $h_{0,ls}$ becomes $h_{0,ls}^{-1}$, and H remains unchanged otherwise.

$(H_{q,ls})_n = (H)_{n+1-q,ls+1}$, that means, the element $h_{q,ls}$ and $h_{q-1,ls}$ of the matrix H become $h_{q,ls}^{-1}$ and $h_{q-1,ls}+1$, respectively.

For example, if $S=1$, $I=2$, $n=2$ then

$$H_{211} = h_{211}^{-1}, h_{221}, h_{111}+1, h_{121}, h_{011}, h_{021}.$$

The starting values of (2.27) have the usual form

$$(a_{i,H,K})_0 = (a_{i,0,k_i=1,K=0 \text{ else}})_{n=0} = 1 \cdot i=1,2,3,\dots,I. \quad (2.28)$$

Concluding remarks to this Subsection:

- System (1.1) with $b_{ij}(t)$ arbitrary functions should, if possible, be reduced either by TRANSFORMATION of $b_{ij}(t)$ to system (2.1) or by SUBSTITUTION of polynomials for $b_{ij}(t)$ to system (2.17). Both systems (2.1) as well as (2.17) permit a decomposed representation of the solution functions with all its advantages.
- Representation (2.25) is from the numerical point of view not so advantageous as representation (2.3) and (2.17), because only A is there constant. In fact, in (2.25) $B=B(t)$ is depending on t , that is to say, $B(t)$ must be computed at each integration step.
Compared to representation (2.2), where $C_n = C_n(A, B, y_i)$ is calculated recursively as a whole at each integration step, A is in (2.25) computed separately, as it remains constant during integration and B and the initial value depending part is computed at each integration step using the labels H and K from $A=a_{i,H,K}$.

2.2 SYSTEMS CONTAINING TERMS OF THE TYPE .

$$(b_1 Y_1^{r_{11}} Y_2^{r_{12}} \dots Y_I^{r_{1I}} + b_2 Y_1^{r_{21}} Y_2^{r_{22}} \dots Y_I^{r_{2I}})^s, \quad s \text{ fraction.}$$

We will first study such systems with b_i constants, then with b_i polynomials and finally with b_i arbitrary functions.

2.2.1 Case 1: b_i are constants.

We investigate systems of the form

$$\begin{aligned} dY_1(t)/dt &= (b_{11} Y_1^{r_{111}} Y_2^{r_{112}} + b_{12} Y_1^{r_{121}} Y_2^{r_{122}})^{s_{11}} + \\ &\quad (b_{13} Y_1^{r_{131}} Y_2^{r_{132}} + b_{14} Y_1^{r_{141}} Y_2^{r_{142}})^{s_{12}} + \\ dY_2(t)/dt &= (b_{21} Y_1^{r_{211}} Y_2^{r_{212}} + b_{22} Y_1^{r_{221}} Y_2^{r_{222}})^{s_{21}} + \\ &\quad (b_{23} Y_1^{r_{231}} Y_2^{r_{232}} + b_{24} Y_1^{r_{241}} Y_2^{r_{242}})^{s_{22}} \end{aligned}$$

with s_{ij} fractions.

The general form of such systems is thus

$$dY_q(t)/dt = \sum_{r=1}^{I_q} \left(\sum_{j=J_{q,r-1}}^{J_{q,r}} b_{qj} Y_i^{r_{qji}} \right)^{s_{qr}}, \quad (2.29)$$

$q=1, 2, 3, \dots, I$; $J_{q,0}=0$; s_{qr} are fractions;

b_{qj} are constants. $J_{q,r} - J_{q,r-1}$ is the number of terms in the expression $(\dots)^{s_{qr}}$.

The power series representation of the solution functions Y_i

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,P,K} A \cdot B^H \cdot F^P \cdot y_1^{k_1} y_2^{k_2} \dots y_I^{k_I} \right)_n, \quad (2.30)$$

$i=1, 2, 3, \dots, I$.

$y_i = Y_i(t=t_0)$, $A = a_{i,H,P,K}$, $B = B(b_{ij})$, $F = F(b_{ij}, y_i)$,

$K = k_1 k_2 k_3 \dots k_I$,

$P = p_{11} p_{12} \dots p_{1I_1} p_{21} p_{22} \dots p_{2I_2} \dots p_{I1} \dots p_{II_I}$,

$H = h_{11} h_{12} \dots h_{1J_1} h_{21} h_{22} \dots h_{2J_2} \dots h_{I1} h_{I2} \dots h_{IJ_I}$,

J_{qj} is the number of subterms in $(\dots)^{s_{qj}}$,

J_q is the total number of subterms in the q -th line of (2.29).

I_q is the total number of terms $(\dots)^{s_{qj}}$ in the q -th line of (2.29)

$$B^H = b_{11}^{h_{11}} b_{12}^{h_{12}} b_{13}^{h_{13}} \dots b_{1J_1}^{h_{1J_1}} b_{21}^{h_{21}} b_{22}^{h_{22}} \dots b_{2J_2}^{h_{2J_2}} \dots b_{I1}^{h_{I1}} b_{I2}^{h_{I2}} \dots b_{IJ_I}^{h_{IJ_I}}$$

$$F^P = \prod_{q=1}^I \prod_{r=1}^{I_q} \left(\sum_{j=J_{q,r-1}}^{J_{q,r}} \prod_{s=1}^I b_{qj}^{r_{qjs}} y_s \right) \quad (2.31)$$

$$F^P = \prod_{q=1}^I \prod_{r=1}^{I_q} \left(\sum_{j=J_{q,r-1}}^{J_{q,r}} \prod_{s=1}^I b_{qj}^{r_{qjs}} y_s \right) \quad (2.32)$$

The summation in (2.30) runs over the labels H, P , and K .

They are given, if the coefficients $a_{i,H,P,K}$ are known.

Note that the representation (2.30) has, compared to the representation (2.3) the additional part

$F^P = F(b_{ij}, y_i)$, i.e., this part depends on b_{ij} as well as on

the initial values y_i . It has thus to be computed at each integration step. However, A and B remain just as in (2.3) constant in the course of the integration process.

To compute the coefficients $a_{i,H,P,K}$ we have found the recurrence relation

$$(a_{i,H,P,K})_{n+1} = \left(\sum_{l=1}^I \sum_{m=1}^{I_l} \sum_{q=1}^I \sum_{j=1}^{J_{q,r}} \sum_{s=1}^{J_{q,r}} (k_l - r_{qjl} + 1) \right).$$

$$(p_{qj} - s_{lm} D_{lm}^{qj} + 1) \cdot a_{i,H_{qj}, P_{lmqj}, k_1 - r_{qj} + D_1^1, k_2 - r_{qj2} + D_1^2, \dots,$$

$$k_r - r_{qjy} + D_1^y, \dots, k_I - r_{qJI} + D_1^I +$$

$$\sum_{q=1}^I \sum_{g=1}^{I_q} (k_q + 1) \cdot a_{i,H, p_{qg}, k_1 + D_q^1, k_2 + D_q^2, \dots, k_r + D_q^r, \dots, k_I + D_q^I)_n , \quad (2.33)$$

where

$$J_{q,0} = 0 ,$$

$$H_{qj} = h_{11} - D_{qj}^{11}, h_{12} - D_{qj}^{12}, \dots, h_{\lambda^{\mu}} - D_{qj}^{\lambda^{\mu}}, \dots, h_{IJ_I} - D_{qj}^{IJ_I} ,$$

with

$$D_{qj}^{\lambda^{\mu}} = 1 \text{ if } \lambda^{\mu} = qj, = 0 \text{ else; } D_l^r = 1 \text{ if } l = r, = 0 \text{ else.}$$

$$P_{lmqg} = p_{11} - s_{11} D_{lm}^{11} + D_{qg}^{11}, p_{12} - s_{12} D_{lm}^{12} + D_{qg}^{12}, p_{13} - s_{13} D_{lm}^{13} + D_{qg}^{13}, \dots,$$

$$p_{\lambda^{\mu}} - s_{lm} D_{lm}^{\lambda^{\mu}} + D_{qg}^{\lambda^{\mu}}, \dots, p_{II_I} - s_{II_I} D_{lm}^{II_I} + D_{qg}^{II_I},$$

$$P_{qg} = p_{11} - s_{11} D_{qg}^{11}, p_{12} - s_{12} D_{qg}^{12}, \dots, p_{\lambda^{\mu}} - s_{\lambda^{\mu}} D_{qg}^{\lambda^{\mu}}, \dots, p_{II_I} - s_{II_I} D_{qg}^{II_I}.$$

We will now illustrate this formalism by an example.

EXAMPLE 8: We consider the system

$$\frac{dY_1(t)}{dt} = (b_{11} Y_1^{r_{111}} Y_2^{r_{112}} + b_{12} Y_1^{r_{121}} Y_2^{r_{122}})^{s_{11}} , \quad (2.29.1)$$

$$\frac{dY_2(t)}{dt} = (b_{21} Y_1^{r_{211}} Y_2^{r_{212}} + b_{22} Y_1^{r_{221}} Y_2^{r_{222}})^{s_{21}} .$$

s_{11}, s_{21} fractions.

The solution representation (2.30) becomes for this system

$$Y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,P,K} A \cdot B^H \cdot (b_{11} Y_1^{r_{111}} Y_2^{r_{112}} + b_{12} Y_1^{r_{121}} Y_2^{r_{122}})^{p_{11}} \right. \\ \left. (b_{21} Y_1^{r_{211}} Y_2^{r_{212}} + b_{22} Y_1^{r_{221}} Y_2^{r_{222}})^{p_{21}} \cdot Y_1^{k_1} Y_2^{k_2} \right)_n \quad (2.30.1)$$

where

$$i=1,2; K=k_1 k_2, P=p_{11} p_{21}, H=h_{11} h_{12} h_{21} h_{22}, A=a_{i,H,P,K},$$

$$B^H = \frac{h_{11}}{b_{11}} \frac{h_{12}}{b_{12}} \frac{h_{21}}{b_{21}} \frac{h_{22}}{b_{22}}, \quad (2.31.1)$$

The recurrence relation for $a_{i,H,P,K}$ reads for this example

$$(a_{i,H,P,K})_{n+1} = ((k_1 - r_{111} + 1)(p_{11} - s_{11} + 1) \cdot a_{i,H_{11},P_{1111},k_1-r_{111}+1,k_2-r_{112}+1} +$$

$$(k_1 - r_{121} + 1)(p_{11} - s_{11} + 1) \cdot a_{i,H_{12},P_{1111},k_1-r_{121}+1,k_2-r_{122}+1} +$$

$$(k_1 - r_{211} + 1)(p_{21} + 1) \cdot a_{i,H_{21},P_{1121},k_1-r_{211}+1,k_2-r_{212}+1} +$$

$$(k_1 - r_{221} + 1)(p_{21} + 1) \cdot a_{i,H_{22},P_{1121},k_1-r_{221}+1,k_2-r_{222}+1} +$$

$$(k_2 - r_{112} + 1)(p_{11} + 1) \cdot a_{i,H_{11},P_{2111},k_1-r_{111},k_2-r_{112}+1} +$$

$$(k_2 - r_{122} + 1)(p_{11} + 1) \cdot a_{i,H_{12},P_{2111},k_1-r_{121},k_2-r_{122}+1} +$$

$$(k_2 - r_{212} + 1)(p_{21} - s_{21} + 1) \cdot a_{i,H_{21},P_{2121},k_1-r_{211},k_2-r_{212}+1} +$$

$$(k_2 - r_{222} + 1)(p_{21} - s_{21} + 1) \cdot a_{i,H_{22},P_{2121},k_1-r_{221},k_2-r_{222}+1} +$$

$$(k_1 + 1) \cdot a_{i,H,P_{11},k_1+1,k_2} + (k_2 + 1) \cdot a_{i,H,P_{21},k_1,k_2+1})_n \quad (2.33.1)$$

where

$$i=1,2;$$

$$H_{11}=h_{11}-1, h_{12} h_{21} h_{22}, H_{12}=h_{11} h_{12}-1, h_{21} h_{22},$$

$$H_{21}=h_{11} h_{12} h_{21}-1, h_{22}, H_{22}=h_{11} h_{12} h_{21} h_{22}-1;$$

$$P_{1111}=p_{11}-s_{11}+1, p_{21}, P_{1121}=p_{11}-s_{11}, p_{21}+1,$$

$$P_{2111}=p_{11}+1, p_{21}-s_{21}, P_{2121}=p_{11}, p_{21}+1-s_{21}.$$

$$P_{11} = p_{11} - s_{11}, P_{21} = p_{11}, P_{21} = s_{21} .$$

The starting values of (2.33.1) are as usual

$$(a_{i,H=0,P=0,k_i=1, K=0 \text{ else}})_{n=0} = 1.$$

We will now treat system (2.29), but with b_{ij} polynomials.

2.2.2 Case 2: $b_i(t)$ are polynomials.

Such a system can be made autonomous by putting $Y_0 = t$.

The formalism of the preceding Case 2.2.1 can then straightforwardly be adopted to the present case.

We will first for illustration purpose solve a representative system and then consider the general formalism.

EXAMPLE 9: We treat the following system

$$\begin{aligned} dY_1(t)/dt &= (b_{11}t^{r_{110}}Y_1^{r_{111}}Y_2^{r_{112}} + b_{12}t^{r_{120}}Y_1^{r_{121}}Y_2^{r_{122}})^{s_{11}} \\ dY_2(t)/dt &= (b_{21}t^{r_{210}}Y_1^{r_{211}}Y_2^{r_{212}} + b_{22}t^{r_{220}}Y_1^{r_{221}}Y_2^{r_{222}})^{s_{21}}, \end{aligned} \quad (2.34)$$

where

b_{ij} are constants and s_{11} and s_{21} are fractions.

Note that if s_{11} , s_{21} are integers then (2.34) reduces to system (2.17), Subsection 2.1.2 .

Putting $Y_0 = t$, the solution functions Y_i can, in analogy to (2.30.1) , be represented as follows

$$\begin{aligned} Y_i(t) &= \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,P,K} A \cdot B^H \cdot (b_{11}Y_0^{r_{110}}Y_1^{r_{111}}Y_2^{r_{112}} + \right. \\ &\quad b_{12}Y_0^{r_{120}}Y_1^{r_{121}}Y_2^{r_{122}})^{p_{11}} \cdot (b_{21}Y_0^{r_{210}}Y_1^{r_{211}}Y_2^{r_{212}} + \\ &\quad \left. b_{22}Y_0^{r_{220}}Y_1^{r_{221}}Y_2^{r_{222}})^{p_{21}} \cdot Y_0^{k_0} Y_1^{k_1} Y_2^{k_2} \right)_n, \end{aligned} \quad (2.35.1)$$

where $i=1,2$; $K=k_0 k_1 k_2$, $P=p_{11} p_{21}$; $H=h_{11} h_{12} h_{21} h_{22}$;

$$B^H = \frac{h_{11} h_{12} h_{21} h_{22}}{b_{11} b_{12} b_{21} b_{22}}, \quad (2.36)$$

The general representation of Y_i has thus just the form of (2.30) except that $i, s=0, 1, 2, \dots, I$ instead of $=1, 2, \dots, I$.

If $A=a_{i,H,P,K}$ is known, then the labels H, P and K are also given. Consequently, one can then compute the solution functions Y_i with the aid of (2.35.1).

By means of mathematical induction one obtains the following recurrence relation for $a_{i,H,P,K}$

$$(a_{i,H,P,K})_{n+1} =$$

$$\begin{aligned} & ((k_0 - r_{110} + 1)(p_{11} + 1) \cdot a_{i,H_{11},P_{0111}, k_0 - r_{110} + 1, k_1 - r_{111}, k_2 - r_{112}} + \\ & (k_0 - r_{120} + 1)(p_{11} + 1) \cdot a_{i,H_{12},P_{0111}, k_0 - r_{120} + 1, k_1 - r_{121}, k_2 - r_{122}} + \\ & (k_0 - r_{210} + 1)(p_{21} + 1) \cdot a_{i,H_{21},P_{0121}, k_0 - r_{210} + 1, k_1 - r_{211}, k_2 - r_{212}} + \\ & (k_0 - r_{220} + 1)(p_{21} + 1) \cdot a_{i,H_{22},P_{0121}, k_0 - r_{220} + 1, k_1 - r_{221}, k_2 - r_{222}} + \\ & k_0 \cdot a_{i,H,P,k_0 - 1, k_1 k_2} + \\ & (k_1 - r_{110} + 1)(p_{11} + 1 - s_{11}) \cdot a_{i,H_{11},P_{1111}, k_0 - r_{110}, k_1 - r_{111} + 1, k_2 - r_{112}} + \\ & (k_1 - r_{121} + 1)(p_{11} + 1 - s_{11}) \cdot a_{i,H_{12},P_{1111}, k_0 - r_{120}, k_1 - r_{121} + 1, k_2 - r_{122}} + \\ & (k_1 - r_{211} + 1)(p_{21} + 1) \cdot a_{i,H_{21},P_{1121}, k_0 - r_{210}, k_1 - r_{211} + 1, k_2 - r_{212}} + \\ & (k_1 - r_{221} + 1)(p_{21} + 1) \cdot a_{i,H_{22},P_{1121}, k_0 - r_{220}, k_1 - r_{221} + 1, k_2 - r_{222}} + \\ & k_1 \cdot a_{i,H,P_{11}, k_0 k_1 - 1, k_2} + \\ & (k_2 - r_{112} + 1)(p_{11} + 1) \cdot a_{i,H_{11},P_{2111}, k_0 - r_{110}, k_1 - r_{111}, k_2 - r_{112} + 1} + \\ & (k_2 - r_{122} + 1)(p_{11} + 1) \cdot a_{i,H_{12},P_{2111}, k_0 - r_{120}, k_1 - r_{121}, k_2 - r_{122} + 1} \end{aligned}$$

$$\begin{aligned}
 & (k_2 - r_{212} + 1)(p_{21} + s_{21}) \cdot a_{i,H_{21},P_{2121},k_0 - r_{210},k_1 - r_{211},k_2 - r_{212} + 1} \\
 & (k_2 - r_{222} + 1)(p_{21} + s_{21}) \cdot a_{i,H_{22},P_{2121},k_0 - r_{220},k_1 - r_{221},k_2 - r_{222} + 1} \\
 & k_2 \cdot a_{i,H,P_{21},k_0 k_1 k_2 - 1} n
 \end{aligned} \tag{2.37}$$

with the starting values

$$(a_{i,H=0,P=0,k_i=1,K=0} \text{ else })_{n=0} = 1 .$$

H_{ij} has the usual form

$$H_{11} = h_{11}^{-1}, h_{12} h_{21} h_{22}, H_{12} = h_{11} h_{12}^{-1}, h_{21} h_{22},$$

$$H_{21} = h_{11} h_{12} h_{21}^{-1}, h_{22}, H_{22} = h_{11} h_{12} h_{21} h_{22}^{-1}, \text{ and}$$

$$P_{lmqj} = p_{11} - s_{11} D_{lm}^{11} + D_{qj}^{11}, p_{21} - s_{21} D_{lm}^{21} + D_{qj}^{21}, \text{ with}$$

$$D_{lm}^{ij} = 1 \text{ if } l, m = i, j, = 0 \text{ else.}$$

$$P_{11} = p_{11} - s_{11}, P_{21} = p_{11}, p_{21} - s_{21} .$$

Recurrence relation (2.37) reads in a more compact form

$$(a_{i,H,P,K})_{n+1} =$$

$$\left(\sum_{l=0}^{I=2} \sum_{m=1}^{I_1=1} \sum_{q=1}^{I_q=1} \sum_{j=1}^{J_{qj}=1} (k_1 - r_{qj1} + 1)(p_{qj} - s_{lm} D_{lm}^{qj} + 1) \cdot \right. \\
 \left. (k_1 - r_{qj1} + 1)(p_{qj} - s_{lm} D_{lm}^{qj} + 1) \cdot \right. \\
 \left. (k_1 - r_{qj0} + D_1^0, k_1 - r_{qj1} + D_1^1, k_2 - r_{qj2} + D_1^2 + \right. \\
 \left. (k_0 + 1) \cdot a_{i,H,P,k_0 + 1, k_2 k_3} + \right.$$

$$\left. \sum_{q=1}^{I=2} \sum_{j=1}^{I_q=1} (k_q + 1) \cdot a_{i,H,P_q, k_0 + D_q^0, k_1 + D_q^1, k_2 + D_q^2} \right)_n \tag{2.38}$$

The symbols appearing in (2.38) have already been defined under (2.33).

Equ.(2.38) can straightforwardly be extended to any indices, id est, it can be generalized for systems of the type (2.34) of any size.

Concluding remarks:

- Representation (2.30) reveals all benefits of decomposed coefficient representation, which we have pointed out at the end of Subsection 2.1.1 .
- In (2.30) there is, however, one additional component, F^P , compared to (2.3) or (2.17). As $F=F(b_{ij}, y_i)$ depends also on the initial values y_i , it has to be computed at each integration step.
- The components B^H , F^P and $y_1^{k_1} y_2^{k_2} \dots y_I^{k_I}$ of (2.30) can numerically be evaluated once the labels H,P, and K are known. They are simultaneously generated with the coefficients $a_{i,H,P,K}$. The components F^P and $y_1^{k_1} y_2^{k_2} \dots y_I^{k_I}$ are, therefore, at each integration step computed directly, i.e., not by means of recurrence relations, as H,P and K remain of course invariant during the integration process.

We turn now to solve system (2.29) assuming that $b_{ij}(t)$ are arbitrary functions.

2.2.3 Case 3: $b_{ij}(t)$ are arbitrary functions.

As in Subsection 2.1.3 we will reduce, if possible, such a system (1.2) either by means of TRANSFORMATION to a system of the type (2.29) or with the aid of SUBSTITUTION to a system of the type (2.34).

Case 3.1: Application of TRANSFORMATION.

The following example will illustrate such a transformation.

EXAMPLE 10:

$$\begin{aligned} dY_1(t)/dt &= (b_{11}\sin(t).Y_2^4 + b_{12}Y_1^3)^{7/3}, \\ dY_2(t)/dt &= (b_{21}Y_1^3 + b_{22}e^{-at}.Y_2^5)^{5/2} \end{aligned} \quad (2.39)$$

b_{ij} and a are constants.

We put

$$\begin{aligned} Y_3(t) &= \sin(t), \quad dY_3(t)/dt = \cos(t) = Y_4, \quad dY_4(t)/dt = -\sin(t) = -Y_3, \\ Y_5(t) &= e^{-at}, \quad dY_5(t)/dt = -aY_5. \end{aligned}$$

With these transformations becomes (2.39)

$$\begin{aligned} dY_1(t)/dt &= (b_{11}Y_3Y_2^4 + b_{12}Y_1^3)^{7/3}, \\ dY_2(t)/dt &= (b_{21}Y_1^3 + b_{22}Y_5Y_2^5)^{5/2}, \\ dY_3(t)/dt &= Y_4, \\ dY_4(t)/dt &= -Y_3, \\ dY_5(t)/dt &= -aY_5. \end{aligned} \quad (2.39.1)$$

This is indeed a system of the type (2.29).

Case 3.2: Application of SUBSTITUTION.

We illustrate this again with the aid of the following example.

EXAMPLE 11.

$$\begin{aligned} \frac{dy_1(t)}{dt} &= (b_1(t) \cdot y_1^3 + b_2(t) \cdot y_2 y_1)^{5/3}, \\ \frac{dy_2(t)}{dt} &= (b_3(t) \cdot y_1^2 + b_4(t) \cdot y_1 y_2^5)^{1/3}. \end{aligned} \quad (2.40)$$

If we approximate $b_i(t)$ with the help of polynomials

$$b_i(t) = \sum_{j=0}^{J_i} b_{ij} \cdot t^j \quad \text{then (2.40) becomes in fact a system}$$

of the type (2.34).

Recall that such an approximation is in practice in general applicable.

Case 3.3: Neither a transformation , Case 3.1, nor a substitution, Case 3.2, is suitable.

The solution functions of system (1.2) reads then in form of decomposed coefficients

$$y_i(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,P,K} A \cdot B(t)^H \cdot F^P \cdot y_1^{k_1} y_2^{k_2} \dots y_I^{k_I} \right)_n \quad (2.41)$$

where $i=1,2,3,\dots,I$,

$A=A(i,H,P,K) = a_{i,H,P,K}$ is constant for a given system,

$B=B(b_{ij}(t))$ depends on t and changes thus at each integration step,

$F=F(b_{ij}(t), y_i)$ depends on t and on the initial values y_i .
It changes also at each integration step.

$B(t)^H$ and F^P are already defined in (2.26) and (2.32), respectively.

If one has determined $a_{i,H,P,K}$ then Y_i can with the aid of (2.41) be computed.

Note that no recurrence relation is involved during the integration, as $a_{i,H,P,K}$ along with the labels H, P and K which completely determine (2.41) are calculated once and for all prior to the integration of the system.

The recurrence relation for $a_{i,H,P,K}$ reads

$$(a_{i,H,P,K})_{n+1} =$$

$$\sum_{l=1}^I \sum_{m=1}^{I_1} \sum_{q=1}^I \sum_{\beta=1}^{I_q} \sum_{j=J_{q,\beta-1}+1}^{J_{q,\beta}} (k_1 - r_{q,j,l} + 1) (p_{q,\beta} - s_{lm} D_{lm}^{q,\beta} + 1).$$

$$a_{i,H_{q,j},P_{lmq,\beta},k_1-r_{q,j,1}+D_1^1,k_2-r_{q,j,2}+D_1^2,\dots,k_y-r_{q,j,y}+D_1^y,\dots,k_I-r_{q,j,I}+D_1^I} + \\ \sum_{q=1}^I \sum_{\beta=1}^{I_q} (k_q + 1) \cdot a_{i,H_{q,\beta},k_1+D_q^1,k_2+D_q^2,\dots,k_y+D_q^y,\dots,k_I+D_q^I} + \\ \sum_{q=1}^I \sum_{\beta=1}^{I_q} \sum_{j=J_{q,\beta-1}+1}^{J_{q,\beta}} (p_{q,\beta} + 1) \cdot a_{i,H_{q,j},P_{q,\beta},k_1-r_{q,j,1},k_2-r_{q,j,2},\dots,k_I-r_{q,j,I}} + \\ \sum_{v=1}^n \sum_{q=1}^I \sum_{s=1}^S (h_{v-1,q,s} + 1) \cdot a_{i,H_{v,q,s},K} \quad (2.42)$$

The symbols appearing in this formula have already been defined in the Subsections 2.1.3 and 2.2.1.

The concluding remarks which we have made at the end of Subsection 2.1.3 apply also here.

3. NUMERICAL EXPLORATION .

In this Chapter we deal with:

- The computation of the coefficients $A=a_{i,H,K}$ and $A=a_{i,H,P,K}$ '
- the error estimation of the solution functions.

3.1 THE COMPUTATION OF THE COEFFICIENTS $A=a_{i,H,K}$.

For practical applications, as we have already mentioned above, two kind of systems are of particular interest, those with b_{ij} constants and those with $b_{ij}(t)$ polynomials in t . In this Chapter we will focuse our attention on these types of systems.

3.1.1 Systems with b_{ij} constants.

To compute $a_{i,H,K}$ recurrence relation (2.5) can of course be used. However, one can also proceed according to another equivalent prescription which is illustrated in TABLE 3.1 . The process for Case 1 in this table reads symbolically

$$(a_{i,H,K})_{n+1} = OP(a_{i,H,K})_n \quad (3.1)$$

and the process for Case 2 in this table reads correspondingly

$$(a_{i,H,K} \cdot B^H)_{n+1} = OP(a_{i,H,K} \cdot B^H)_n \quad (3.2)$$

Both operations are of practical interest. The use of relation (3.1) is advantageous if one intends to change the b_{ij} in the course of the study of a system.

TABLE 3.1: Generation of coefficients $(a_{i,H,K})_{n+1}$ from $(a_{i,H,K})_n$.

Quantity	$a_{i,H,K}$	$B^H : (s_j)$	k_1	k_2	k_3	$\dots k_s \dots$	k_I	order n
Operation	$.k_s$	1)	+	+	+	+	+	
	s=1	11	r_{111}^{-1}	r_{112}	r_{113}	r_{11s}	r_{11I}	
	1	12	r_{121}^{-1}	r_{122}	r_{123}	r_{12s}	r_{12I}	
	1	13	r_{131}^{-1}	r_{132}	r_{133}	r_{13s}	r_{13I}	
							
	1	1J	r_{1J1}^{-1}	r_{1J2}	r_{1J3}	r_{1Js}	r_{1JI}	
	2	21	r_{211}	r_{212}^{-1}	r_{213}	r_{21s}	r_{21I}	
	2	22	r_{221}	r_{222}^{-1}	r_{223}	r_{22s}	r_{22I}	
	2	23	r_{231}	r_{232}^{-1}	r_{233}	r_{23s}	r_{23I}	
							
	2	2J	r_{2J1}	r_{2J2}^{-1}	r_{2J3}	r_{2Js}	r_{2JI}	$\frac{n}{+}$
							
Result of operation	s	s1	r_{s11}	r_{s12}	r_{s13}	r_{s1s}^{-1}	r_{s1I}	
	s	s2	r_{s21}	r_{s22}	r_{s23}	r_{s2s}^{-1}	r_{s2I}	
	s	s3	r_{s31}	r_{s32}	r_{s33}	r_{s3s}^{-1}	r_{s3I}	
							
	s	sJ	r_{sJ1}	r_{sJ2}	r_{sJ3}	r_{sJs}^{-1}	r_{sJI}	
							
	I	I1	r_{I11}	r_{I12}	r_{I13}	r_{I1s}	r_{I1I}^{-1}	
	I	I2	r_{I21}	r_{I22}	r_{I23}	r_{I2s}	r_{I2I}^{-1}	
	I	I3	r_{I31}	r_{I32}	r_{I33}	r_{I3s}	r_{I3I}^{-1}	
							
	I	IJ	r_{IJ1}	r_{IJ2}	r_{IJ3}	r_{IJs}	r_{IJI}^{-1}	

1) Case 1: $h_{sj} + 1$ if one wants to compute $(a_{i,H,K})_n$ only.

Case 2: $b_{sj} \cdot B^H$ if one wishes to compute $(a_{i,H,K} \cdot B^H)_n$ at once.

Process (3.1) has in any case to be used, if one wishes just to compute $a_{i,H,K}$, for example, if one intends to investigate the mathematical meaning of these numbers.

The use of process (3.2) is suitable if the only parameters which are altered in the course of a study of a system are the initial values y_i , because then the product $A \cdot B^H$ remains constant during the investigation of the system and there is no need to compute A and B^H separately merely for the purpose of integration of a system.

Generally speaking, the computation of $a_{i,H,K}$ proceeds as follows when using TABLE 3.1 :

- Take a coefficient $(a_{i,H,K})_n$ of order n ,
- apply TABLE 3.1 to it. That yields $(a_{i,H,K})_{n+1}$ of order $n+1$.
- If all coefficients of order n have passed through this process then SORT the coefficients $(a_{i,H,K})_{n+1}$ according to the indices H, K and SUM UP coefficients with IDENTICAL H, K .
- Store $(A(i,H,K))_{n+1}$ in a permanent file for ulterior use.

TABLE 3.1 becomes for the various examples treated in Subsection 2.1.1 .

TABLE 3.1 for EXAMPLE 1 (nuclear reactions).

Quantity	$a_{i,H,K}$	$B^H : (s_j)$	k_1	k_2	k_3	Order n
Operation	$.k_s$	1)	+	+	+	
	$s=1$	11	0	1	0	
	1	12	1	0	0	
	1	13	1	0	0	
	1	14	0	0	1	
of	2	21	1	0	0	+
Result	2	22	2	-1	0	
	3	31	2	0	-1	
	3	32	1	0	0	

TABLE 3.1 for EXAMPLE 2 (SU(2)-YANG-MILL's Equations) .

Quantity	$a_{i,H,K} B^H : (s_j)$	k_1	k_2	k_3	k_4	k_5	k_6	Order n
Operation	$.k_s$	1)	+	+	+	+	+	
Result of operation	s=1	11	-1	1	0	0	0	
	2	21	1	-1	2	0	0	
	2	22	1	-1	0	0	2	
	3	31	0	0	-1	1	0	
	4	41	2	0	1	-1	0	
	4	42	0	0	1	-1	2	
	5	51	0	0	0	0	-1	
	6	61	2	0	0	0	1	-1
	6	62	0	0	2	0	1	-1

TABLE 3.1 for EXAMPLE 3 (Euler's equation of motion, force free).

Quantity	$a_{i,H,K} B^H : (s_j)$	k_1	k_2	k_3	Order n
Operation	$.k_s$	1)	+	+	
Result of operation	s=1	11	-1	1	1
	2	21	1	-1	1
	3	31	1	1	-1

TABLE 3.1 For EXAMPLE 4 (Euler's equation of motion).

Quantity	$a_{i,H,K}$	$B^H : (s_j)$	k_1	k_2	k_3	k_4	k_5	k_6	Order n
Operation	$.k_s$	1)	+	+	+	+	+	+	
Result of operation	s=1	11	-1	1	1	0	0	0	
	1	12	-1	0	0	0	0	1	
	1	13	-1	0	0	0	1	0	
	2	21	1	-1	1	0	0	0	
	2	22	0	-1	0	1	0	0	
	2	23	0	-1	0	0	0	1	
	3	31	1	1	-1	0	0	0	
	3	32	0	0	-1	0	1	0	
	3	33	0	0	-1	1	0	0	
	4	41	0	1	0	-1	0	1	
	4	42	0	0	1	-1	1	0	
	5	51	0	0	1	1	-1	0	
	5	52	1	0	0	0	-1	1	
	6	61	1	0	0	0	1	-1	
	6	62	0	1	0	1	0	-1	

Recall that of all examples the starting values are

$$(a_{i,H=0,k_i=1,K=0} \text{ else })_{n=0} = 1, \quad i=1,2,3,\dots,I.$$

3.1.2 Systems with $b_{ij}(t)$ polynomials .

We can compute the coefficients $(a_{i,H,K})_n$ by means of the recurrence relation (2.19), Subsection 2.1.2 , or again by an equivalent prescription which is exhibited in TABLE 3.2 .

TABLE 3.2: Generation of coefficients $(a_{i,H,K})_{n+1}$ from $(a_{i,H,K})_n$.

Quantity	$a_{i,H,K}$	$B^H : (s_j)$	k_0	k_1	k_2	\dots	k_I	order n
Operation	$.k_s$	1)	+	+	+	\dots	+	
	s=0		-1					
	1	11	r_{110}	r_{111}^{-1}	r_{112}		r_{11I}	
	1	12	r_{120}	r_{121}^{-1}	r_{122}		r_{12I}	
Result of operation							
	1	1J	r_{1J0}	r_{1J1}^{-1}	r_{1J2}		r_{1JI}	
	2	21	r_{210}	r_{211}	r_{212}^{-1}		r_{21I}	
	2	22	r_{220}	r_{221}	r_{222}^{-1}		r_{22I}	
							
	2	2J	r_{2J0}	r_{2J1}	r_{2J2}^{-1}		r_{2JI}	
							
	I	I1	r_{I10}	r_{I11}	r_{I12}		r_{I1I}^{-1}	
	I	I2	r_{I20}	r_{I21}	r_{I22}		r_{I2I}^{-1}	
							
	I	IJ	r_{IJ0}	r_{IJ1}	r_{IJ2}		r_{IJI}^{-1}	

1) explained in TABLE 3.1 .

Below we give TABLE 3.2 for EXAMPLE 5, Euler's equation of motion with variable moments of inertia, but with no gravitational force.

TABLE 3.2 for EXAMPLE 5 (Euler's equation of motion, force free, variable moments of inertia).

Quantity	$a_{i,H,K}$	$B^H : (s_j)$	k_0	k_1	k_2	k_3	Order n
Operation	$.k_s$	1)	+	+	+	+	
Result of operation	s=0		-1	0	0	0	
	1	11	0	-1	1	1	
	1	12	1	-1	1	1	
	1	13	2	-1	1	1	
	2	21	0	1	-1	1	
	2	22	1	1	-1	1	
	2	23	2	1	-1	1	
	3	31	0	1	1	-1	
	3	32	1	1	1	-1	
	3	33	2	1	1	-1	

The starting values are as usual

$$(a_{i,H=0,k_i=1,K=0} \text{ else })_{n=0} = 1, \quad i=1,2,3,\dots,I.$$

3.2 THE COMPUTATION OF THE COEFFICIENTS $A=a_{i,H,P,K}$.

As in Section 3.1 we concentrate our study also in this Section on systems with b_{ij} constants and $b_{ij}(t)$ polynomials.

3.2.1 Systems with b_{ij} constants.

The coefficients $A=a_{i,H,P,K}$ can in principle be computed with the aid of recurrence relation (2.33). An alternative equivalent procedure to calculate them is the prescription which is exhibited in TABLE 3.3 for the following representative system

$$\frac{dy_1(t)}{dt} = (b_{11} y_1^{r_{111}} y_2^{r_{112}} + b_{12} y_1^{r_{121}} y_2^{r_{122}})^{s_{11}}, \quad (3.3)$$

$$\frac{dy_2(t)}{dt} = (b_{21} y_1^{r_{211}} y_2^{r_{212}} + b_{22} y_1^{r_{221}} y_2^{r_{222}})^{s_{21}},$$

where s_{11} , s_{21} are fractions, b_{ij} are constants.

TABLE 3.3: Generation of coefficients $(a_{i,H,P,K})_{n+1}$ from $(a_{i,H,P,K})_n$.

Quantity	$a_{i,H,P,K}$	$B^H : (s_j)$	p_{11}	p_{21}	k_1	k_2	Order n
Operation	.	1)	+	+	+	+	
Result of operation	$p_{11} \cdot r_{111}$	11	s_{11}^{-1}		$r_{111}^{-1} r_{112}$		
	$p_{11} \cdot r_{121}$	12	s_{11}^{-1}		$r_{121}^{-1} r_{122}$		
	$p_{21} \cdot r_{211}$	21	s_{11}^{-1}	-1	$r_{211}^{-1} r_{212}$		
	$p_{21} \cdot r_{221}$	22	s_{11}^{-1}	-1	$r_{221}^{-1} r_{222}$		
	k_1		s_{11}^{-1}		-1		
	$p_{11} \cdot r_{112}$	11	-1	s_{21}	$r_{111} r_{112}^{-1}$		
	$p_{11} \cdot r_{122}$	12	-1	s_{21}	$r_{121} r_{122}^{-1}$		
	$p_{21} \cdot r_{212}$	21		s_{21}^{-1}	$r_{211} r_{212}^{-1}$		
	$p_{21} \cdot r_{222}$	22		s_{21}^{-1}	$r_{221} r_{222}^{-1}$		
	k_2			s_{21}^{-1}	-1		

1) explained under TABLE 3.1.

Recall that the starting values are

$$(a_{i,H=0,P=0,k_1=1,K=0 \text{ else}})_{n=0} = 1, \quad i=1,2,3,\dots,I.$$

TABLE 3.3 can readily be generalized for systems of any size.

3.2.2 Systems with $b_{ij}(t)$ polynomials .

An alternative to recurrence relation (2.38) to compute the coefficients $a_{i,H,P,K}$ is again a prescription which we illustrate by the following representative system

$$\begin{aligned} dY_1(t)/dt &= (b_{11} Y_0^{r_{110}} Y_1^{r_{111}} Y_2^{r_{112}} + b_{12} Y_0^{r_{120}} Y_1^{r_{121}} Y_2^{r_{122}})^{s_{11}}, \quad (3.4) \\ dY_2(t)/dt &= (b_{21} Y_0^{r_{210}} Y_1^{r_{211}} Y_2^{r_{212}} + b_{22} Y_0^{r_{220}} Y_1^{r_{221}} Y_2^{r_{222}})^{s_{21}}, \end{aligned}$$

s_{11}, s_{21} fractions and b_{ij} constants.

Note that we have put $Y_0 = t$ on the right side of (3.4) .

The prescription for the generation of the coefficients $a_{i,H,P,K}$ is for this system given in the subsequent TABLE 3.4.

Concluding remarks:

- The alternative prescriptions proposed in this Section to compute the coefficients A , or $A.B^H$ are in practice handier than the general recurrence relations.
- It is particularly advisable to use these prescriptions instead of the recurrence relations if one studies merely a special system.

Hitherto we were only concerned with the computation of the coefficients $A(i,H,K)$ or $A(i,H,P,K)$, but not with the integration of the system itself.

However, we wish once more to emphasize that, in contrast with other numerical integration methods, the formalism of decomposed coefficients which we proposed in this work yields number sequences which may have a deeper mathematical meaning. Their computation is thus not only sensible for the unique purpose of the integration of a system of differential equations, but also for the generation of number sequences which can, for example, be interpreted as combinatorial sets. That has clearly been demonstrated by EXAMPLE 3, Subsection 2.1.1 .

TABLE 3.4: Generation of $(a_{i,H,P,K})_{n+1}$ from $(a_{i,H,P,K})_n$.

Quantity	$a_{i,H,P,K}$	$B^H : (s_j)$	p_{11}	p_{21}	k_o	k_1	k_2	order n
Operation	.	1)	+	+	+	+	+	
Result of operation	$p_{11} \cdot r_{110}$	11	-1		r_{110}^{-1}	r_{111}	r_{112}	
	$p_{11} \cdot r_{120}$	12	-1		r_{120}^{-1}	r_{121}	r_{122}	
	$p_{21} \cdot r_{210}$	21		-1	r_{210}^{-1}	r_{211}	r_{212}	
	$p_{21} \cdot r_{220}$	22		-1	r_{220}^{-1}	r_{221}	r_{222}	
	k_o				-1			
	$p_{11} \cdot r_{111}$	11	s_{11}^{-1}		r_{110}	r_{111}^{-1}	r_{112}	$+u$
	$p_{11} \cdot r_{121}$	12	s_{11}^{-1}		r_{120}	r_{121}^{-1}	r_{122}	
	$p_{21} \cdot r_{211}$	21	s_{11}	-1	r_{210}	r_{211}^{-1}	r_{212}	
	$p_{21} \cdot r_{221}$	22	s_{11}	-1	r_{220}	r_{221}^{-1}	r_{222}	
	k_1		s_{11}			-1		
	$p_{11} \cdot r_{112}$	11	-1	s_{21}	r_{110}	r_{111}	r_{112}^{-1}	
	$p_{11} \cdot r_{122}$	12	-1	s_{21}	r_{120}	r_{121}	r_{122}^{-1}	
	$p_{21} \cdot r_{212}$	21		s_{21}^{-1}	r_{210}	r_{211}	r_{212}^{-1}	
	$p_{21} \cdot r_{222}$	22		s_{21}^{-1}	r_{220}	r_{221}	r_{222}^{-1}	
	k_2			s_{21}			-1	

TABLE 3.4 can straightforwardly be written down for systems of any size.

For the integration of a system, besides the determination of the coefficients $A(i,H,K)$ or $A(i,H,P,K)$, the estimation of the ERROR of the computed solution functions and, what is related to it, the strategy of the INTEGRATION STEP-SIZE are indispensable.

3.3 ERROR ESTIMATION OF THE COMPUTED SOLUTION FUNCTIONS.

One distinguishes between local and global error.

The LOCAL ERROR: It is the error introduced by one integration step. The usual way to estimate it is to check the contribution of approximations of different order. That is, when using our formalism, a natural and efficient tool, as the recurrence relation for the coefficients permit approximation of the solutions to any order desired. The local error is usually used to pilot the integration-step-size.

The GLOBAL ERROR: This is the total accumulated error during the integration. To estimate it we propose here three ways:

- By means of the socalled transfer matrix,
- with the aid of comparative calculations,
- with the help of physical conservation laws.

Conservation laws: They are not available for any system to be integrated, but if they exist then they might be the simplest way to evaluate the global accumulated error. The use of such laws to control the global error is discussed in Section 3.5, Problem 3.

Comparative calculations: A way to obtain an indication of the global error is an additional integration imposing a different tolerance.

Still another possibility to verify this error is a supplementary integration using a code which is based on a different numerical method.

Transfer matrix /WA69/.p.22.

Roughly speaking, it describes the effect of a local error e_L on the total error e .

$$e = T(t) \cdot e_L, \text{ where} \quad (3.5)$$

$$T(t) = \frac{\partial Y_i(t)}{\partial Y_k(t=t_0)}, i,k=1,2,3,\dots,I. \quad (3.6)$$

$T(t)$ becomes thus for the decomposed coefficient representation (2.3)

$$T(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,K} A \cdot B^H \frac{\partial}{\partial y_k} (y_1^{k_1} y_2^{k_2} \dots y_I^{k_I}) \right)_n \quad (3.6.1)$$

and for the representation (2.30)

$$T(t) = \sum_{n=0}^{\infty} \frac{(t-t_0)^n}{n!} \left(\sum_{H,P,K} A \cdot B^H \frac{\partial}{\partial y_k} (F^P \cdot y_1^{k_1} y_2^{k_2} \dots y_I^{k_I}) \right)_n \quad (3.6.2)$$

As the labels H, P, K are automatically known from the coefficients $A(i, H, K)$ or $A(i, H, P, K)$, the TRANSFER MATRIX is given EXPLICITELY.

THE DECOMPOSED COEFFICIENT REPRESENTATION YIELDS THUS EXPLICIT EXPRESSIONS FOR THE TRANSFER MATRIX.

For N integration steps: $t_0, t_1, t_2, \dots, t_N$ the total transfer matrix $T(t_N)$ reads

$$T(t_N) = C(t_N, t_{N-1}) \dots C(t_2, t_1) C(t_1, t_0). \quad (3.7)$$

We add still the useful relation

$$C(t, t') = C(t', t)^{-1}.$$

More details on this matter are given in /WA69/, p.22 and p.133.

The integration step-size strategy will be discussed in PART II: Applications.

4. SUMMARY AND CONCLUSIONS.

The essence of this work is:

- The decomposition of the power series coefficients C_n into individual parts:

$$C_n = (A \cdot B \cdot V)_n .$$

- The derivation of recurrence relations for these coefficients, more precisely, for the coefficients A_n as the knowledge of these coefficients automatically permit computation of B and V also.

Essential characteristics of the quantities A , B and V , depending on the system considered, are summarized in TABLE 4.1 .

TABLE 4.1: Characteristics of A , B , and V for various systems of differential equations.

A	B	V	System Subsect.	Category
Constant, mathematical part	Constant, physical part	Depending on initial values	2.1.1 2.1.2	1
idem	idem	idem and on physical para- meters b_{ij}	2.2.1 2.2.2	2
idem	depending on t	depending on initial values	2.1.3	
		idem and on physical para- meters b_{ij}	2.2.3	3

As for category 1 and category 2, the benefits of the DECOMPOSED COEFFICIENT representation , compared to other numerical integration methods are :

- It permits generation of number sequences with mathematical meaning (see EXAMPLE 3 and 2, Subsection 2.1.1), that is to say, it is not merely a tool to integrate a system in contrast with other numerical integration methods.

- It renders the integration of a system transparent and economical and allows an approximation of the solution functions which is automatic and unlimited in order.

TRANSPARENT: Alteration of physical parameters of a given system affects only the quantity B or the quantities B and V , (Subsection 2.2) and changes of the initial values of the functions influence only the quantity V . This transparency, which facilitates considerably the study of a system, does not appear in other numerical integration methods.

ECONOMICAL: The two essential parts A and B remain constant in the course of the integration. The integration labour consists thus basically only in summing up the power series at each integration step. This is little labour compared to that of other numerical integration methods.

APPROXIMATION OF UNLIMITED ORDER: The recurrence relations for A_n which have been derived in this work permit an automatized and arbitrary order approximation of the solution functions.

Neither RUNGE-KUTTA methods /C083/ (review) nor MULTISTEP-methods /H174/ offer this possibility.

Whereas, the LIE-SERIES method /KN68/, /WA69/ and the INTERVAL ANALYSIS method /MO66/ present recurrence relations to compute the coefficients C_n to any order n . However, $C_n = C_n(A, B, V)$ is computed there as a whole, id est, A and B and V are there implicitly evaluated at each integration step, in contrast to the DECOMPOSED COEFFICIENT formalism (this work), where ONLY V is computed at each integration step. Apparently, the computation of V only requires less labour than the calculation of $C_n(A, B, V)$.

Briefly, we can conclude the following:

- The decomposition of the power series coefficients into its individual parts is of mathematical as well as numerical interest.
- Concerning the integration, such a decomposition is a priori particularly beneficial for systems of category 1, it might still be advantageous for systems of category 2, but it seems to be of less interest for systems of category 3.

ANNEX: SYMBOLIZATION OF DIFFERENTIAL EQUATIONS BY MEANS OF
WEIGHT DIAGRAMS OF LIE ALGEBRAS A_n .

We illustrate this symbolization for the Lie algebra A_2 .

A generalization to Lie algebras A_n , $n=3,4,\dots$, is straightforward. Weight diagrams of the Lie algebra A_2 are two dimensional. We consider here only such weight diagrams $A_2(\lambda_1, \lambda_2)$ with $\lambda_2=0$. They are all built up by the basic element $A_2(\lambda_1, \lambda_2)=(1,0)$ which corresponds to the diagram /HU72/, /NA79/

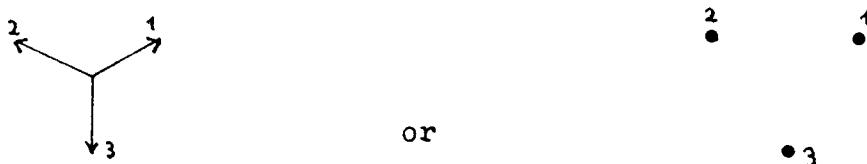


Figure A.1: The weight diagram $A_2(\lambda_1, \lambda_2) = (1,0)$.

The next higher weight diagram $A_2(2,0)$ has the form

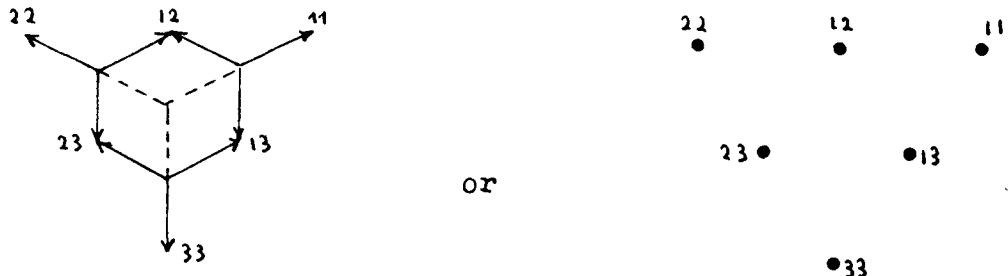


Figure A.2: The weight diagram $A_2(\lambda_1, \lambda_2) = (2,0)$.

Quite generally, each netpoint of a weight diagram $A_2(\lambda_1, 0)$ is identifiable by a combination of the numbers 1, 2, 3.

We make the following assignment to the numbers appearing in the weight diagrams $A_2(\lambda_1, 0)$:

1 indicates $Z_1(t)$, 2 indicates $Z_2(t)$, 3 indicates $Z_3(t)$.

Furthermore, we symbolize systems of differential equations of the type (1.1), i.e. of the form

$$dZ_i(t)/dt = \sum_{j=1}^m b_{ij} \prod_{s=1}^n Z_s^{r_{ijs}(t)}, \quad i=1,2,3,\dots,n, \quad (A.1)$$

where i indicates the numbers appearing in the weight diagram, for example,

$i=1,2,3$ for weight diagrams $A_2(\lambda_1, 0)$,

$i=1,2,3,4$ for weight diagrams $A_3(\lambda_1, 0, 0)$...

$i=1,2,3,\dots,n$ for weight diagrams $A_{n-1}(\lambda_1, \lambda_2=0)$, $l=4,5,\dots,n-1$.

r_{ijs} expresses the possible combinations of the numbers i which define a netpoint of a weight diagram.

The following examples illustrate this in more detail.

EXAMPLE A.1: The system of differential equations corresponding to the weight diagram $A_2(1,0)$, Fig. A.1, reads

$$\begin{aligned} dZ_1(t)/dt &= b_{11}Z_1 + b_{12}Z_2 + b_{13}Z_3 \\ dZ_2(t)/dt &= b_{21}Z_1 + b_{22}Z_2 + b_{23}Z_3 \\ dZ_3(t)/dt &= b_{31}Z_1 + b_{32}Z_2 + b_{33}Z_3 \end{aligned} \quad (A.2)$$

EXAMPLE A.2: The system of differential equations corresponding to the weight diagram $A_2(2,0)$, Fig. A.2, reads

$$\begin{aligned} dZ_1(t)/dt &= b_{11}Z_1^2 + b_{12}Z_1Z_2 + b_{13}Z_2^2 + b_{14}Z_2Z_3 + b_{15}Z_3^2 + b_{16}Z_1Z_3 \\ dZ_2(t)/dt &= b_{21}Z_1^2 + b_{22}Z_1Z_2 + b_{23}Z_2^2 + b_{24}Z_2Z_3 + b_{25}Z_3^2 + b_{26}Z_1Z_3 \\ dZ_3(t)/dt &= b_{31}Z_1^2 + b_{32}Z_1Z_2 + b_{33}Z_2^2 + b_{34}Z_2Z_3 + b_{35}Z_3^2 + b_{36}Z_1Z_3 \end{aligned} \quad (A.3)$$

Special cases of (A.3) are

$$\text{Case 1: } b_{11}=b_{12}=b_{13}=b_{15}=b_{16}=0,$$

$$b_{21}=b_{22}=b_{23}=b_{24}=b_{25}=0,$$

$b_{31}=b_{33}=b_{34}=b_{35}=b_{36}=0$, id est, (A.3) reduces to equations (2.1.3), EXAMPLE 3, Subsection 2.1.1. In other words, only the netpoints 12, 13, and 23 are active in the weight

diagram $A_2(2,0)$, Fig.A.2. If one indicates active netpoints in the weight diagram by the symbol \odot then Fig.A.2 becomes

$$\begin{array}{c} 22 \\ \bullet \end{array} \quad \begin{array}{c} 12 \\ \odot \end{array} \quad \begin{array}{c} 11 \\ \bullet \end{array}$$

$$23 \odot \quad \odot 13$$

$$\bullet 39$$

Case 2: $b_{13}=b_{14}=b_{15}=0$, $b_{23}=b_{24}=b_{25}=0$, $b_{33}=b_{34}=b_{35}=0$. In this case (A.3) reduces to equations (2.1.1), EXAMPLE 1,

Subsection 2.1.1. The only active netpoints of the weight diagram $A_2(2,0)$ are 11, 12, and 13.

EXAMPLE A.3: The weight diagram $A_2(\lambda_1=3, \lambda_2=0)$ has the form

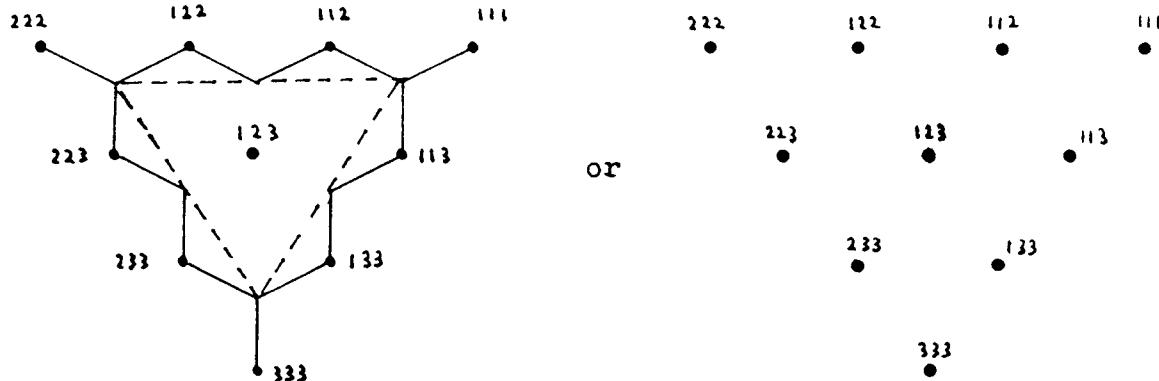


Fig.A.3: The weight diagram $A_2(3,0)$.

The corresponding system of differential equations reads

$$\begin{aligned} dZ_i(t)/dt = & b_{11}Z_1Z_2Z_3 + b_{12}Z_1^2Z_2 + b_{13}Z_1Z_2^2 + b_{14}Z_2^3 + b_{15}Z_2Z_3^2 + \\ & b_{16}Z_2^2Z_3 + b_{17}Z_3^3 + b_{18}Z_1Z_3^2 + b_{19}Z_1^2Z_3 + b_{20}Z_1^3, \quad (A.4) \\ i = & 1, 2, 3. \end{aligned}$$

Weight diagrams $A_2(0, \lambda_2)$ are built out of the elementary weight diagram $A_2(0, 1)$ which is represented in the following diagram.

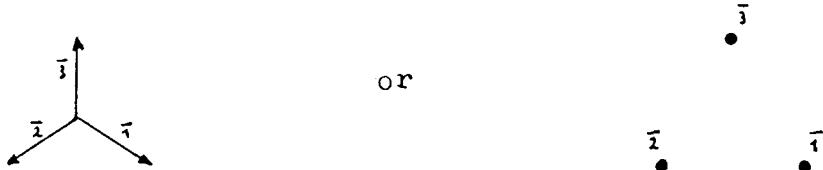


Figure A.4: The weight diagram $A_2(0,1)$.

We assigne to $\bar{1}, \bar{2}, \bar{3}$ of Fig. A.4 $\bar{z}_1, \bar{z}_2, \bar{z}_3$, where \bar{z}_i denotes the complex conjugate of z_i .

The system of differential equations which are symbolized by Fig. A.4 reads

$$d\bar{z}_i(t)/dt = b_{i1}\bar{z}_1 + b_{i2}\bar{z}_2 + b_{i3}\bar{z}_3, \quad i=1,2,3 \quad (\text{A.5})$$

Weight diagrams $A_2(\lambda_1, \lambda_2)$, $\lambda_1, \lambda_2 \neq 0$, are built out of elementary weights $A_2(1,0)$ as well as $A_2(0,1)$.

That is illustrated in the following diagram $A_2(1,1)$.

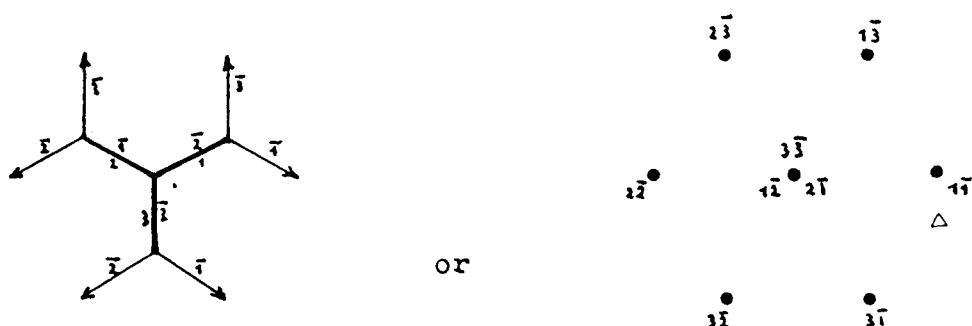


Figure A.5: The weight diagram $A_2(1,1)$.

The system of differential equations which are symbolized by Fig. A.5 reads

$$\begin{aligned} dZ_i(t)/dt = & b_{i1} Z_1 \bar{Z}_1 + b_{i2} Z_1 \bar{Z}_3 + b_{i3} Z_2 \bar{Z}_3 + b_{i4} Z_2 \bar{Z}_2 + b_{i5} Z_3 \bar{Z}_2 + b_{i6} Z_3 \bar{Z}_1 + \\ & b_{i7} Z_3 \bar{Z}_3 + b_{i8} (Z_1 \bar{Z}_2 + Z_2 \bar{Z}_1), \quad i=1,2,3; \end{aligned} \quad (A.6)$$

or the complex conjugate equations.

The following system which appears in non-linear optics/B165/, Equ.(4.12), is an example of systems of the type (A.6).

$$\begin{aligned} \frac{dA_1}{dz} = & + \frac{i\omega_1^2 C}{k_1 \cos^2 \alpha_1} A_2 A_3 A_4 e^{i \Delta k z} + \frac{i\omega_1^2}{k_1 \cos^2 \alpha_1} A_1 \sum_{j=1}^4 C_{1j} A_j A_j^*, \\ \frac{dA_2^*}{dz} = & - \frac{i\omega_2^2 C}{k_2 \cos^2 \alpha_2} A_1^* A_3 A_4 e^{i \Delta k z} - \frac{i\omega_2^2}{k_2 \cos^2 \alpha_2} A_2^* \sum_{j=1}^4 C_{2j} A_j A_j^*, \\ \frac{dA_3^*}{dz} = & - \frac{i\omega_3^2 C}{k_3 \cos^2 \alpha_3} A_1^* A_2 A_4 e^{i \Delta k z} - \frac{i\omega_3^2}{k_3 \cos^2 \alpha_3} A_3^* \sum_{j=1}^4 C_{3j} A_j A_j^*, \\ \frac{dA_4}{dz} = & + \frac{i\omega_4^2 C}{k_4 \cos^2 \alpha_4} A_1^* A_2 A_3 e^{i \Delta k z} + \frac{i\omega_4}{k_4 \cos^2 \alpha_4} A_4 \sum_{j=1}^4 C_{4j} A_j A_j^*. \end{aligned}$$

A_i are amplitudes and $A_i^* = \bar{A}_i$.

In fact, this system belongs to weight diagrams $A_3(\lambda_1, \lambda_2, \lambda_3)$ of the algebra A_3 .

These are weight diagrams in a three-dimensional space.

Each weight is characterized by a quadruplet of the numbers:
 $1, 2, 3, 4$ and $\bar{1}, \bar{2}, \bar{3}, \bar{4}$.

Concluding we can say

- the weight diagrams $A_n(\lambda_1, \lambda_2, \dots, \lambda_n)$ permit a geometric representation of systems of differential equations of the type (1.1),
- inversely, such weight diagrams inspire investigation of the corresponding systems of differential equations, in particular with a view to the mathematical interpretation of the part A (mathematical part characterizing the system) of the decomposed coefficient representation of their solution functions.
- The differential equations assigned to weight diagrams $A_n(\lambda_1, \lambda_2, \dots, \lambda_n)$ might describe physical phenomena.
Note that such diagrams are used in nuclear /SC81/ and elementary particle physics/GE64/.

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