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CLASSIFICATION OF NUCLEAR ROTATIONAL STATES USING WEIGHT DIAGRAMS OF THE LIE ALGEVRA A_2

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SEPTEMBRE 1993

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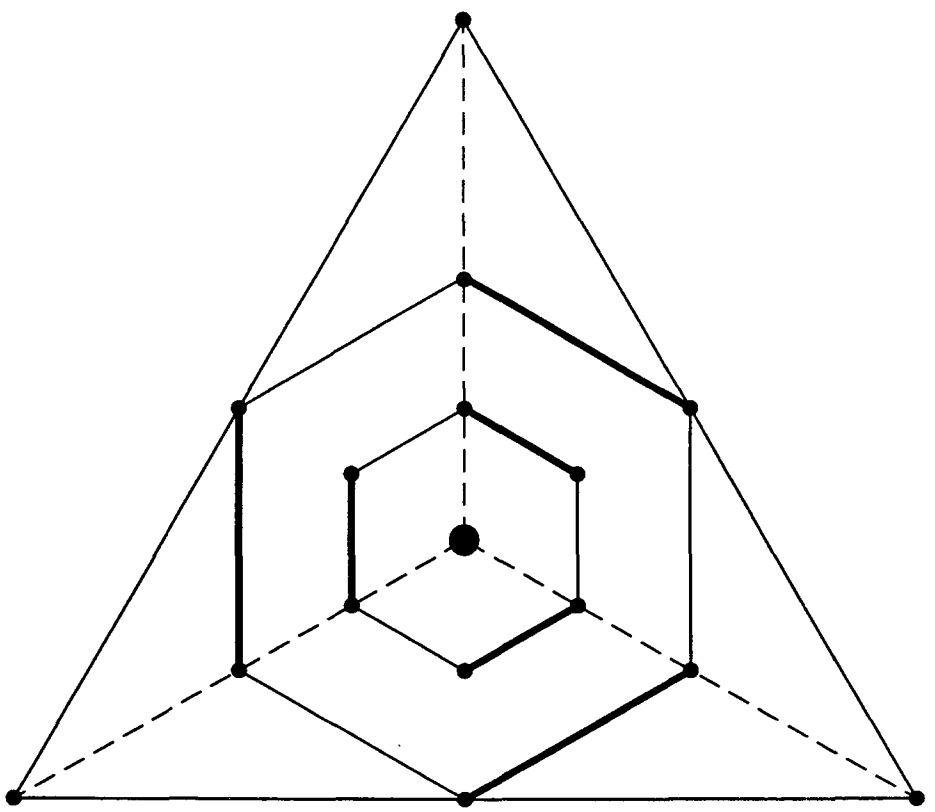
La nature fait toujours selon les conditions dont elle dispose et autant que possible les choses les plus belles et les meilleures.

(ARISTOTE, Problèmes, XVI, 2; traduction J.Barthélémy Saint-Hilaire)

... η οτι η φυσις εκ των ενδεχομενων παντα ποιει ως δυνατον αριστα και καλλιστα, το δε σχημα τουτο καλλιστον, το αυτο αυτω ομοιοτατον.

(ARISTOTELES, Problemata physica, 915a, 10, 33-35)



Logogram: Matter ↔ Mathematics (see Annex A).

Chapter 1

Introduction

The groups SU(6), SU(3) and O(6) have successfully been applied in nuclear theory by A.ARIMA, F.IACHELLO and O.SCHOLTEN [AR76], [AR78], [AR78a], [SC78].

Weight diagrams of the Lie algebra A_2 permit an elegant classification of nuclear rotational states [SC81]. The basic concept of such a classification is the assignment of nuclear rotational states to weights.

In the present work we apply in a systematic way such weight diagrams to classify nuclear rotational states.

The main objectives of such a classification are:

- An esthetical representation of nuclear rotational states in form of geometrical patterns.
- To enlarge the level energy formula given in [AR78] by terms depending on the quantum numbers M and X which directly result from the representation of nuclear rotational states by means of weight diagrams of the Lie algebra A_2. This generates additional fine structures of nuclear rotational states which in turn offers the possibility of experimentally checking the validity of the applied model.
- A systematic and complete computation of rotational state energy levels using this more general formula and comparison of the computed values with the available experimental values and the values obtained by the help of the more restrictive formula given in [AR78].
- To study the influence of the levels generated by the quantum numbers M and X on the statistical values related to nuclear levels.

The main results of this study are:

- The great manifold of nuclear rotational states can be classified by a quite reasonable number of distinct weight diagrams.
- There are only three different categories of shapes of possible weight diagrams.
- The weights and their multiplicities of these diagrams can easily be determinated by some simple laws.

- One obtains a lot of additional rotational states due to the quantum numbers M, X which specify these states in the here proposed $A_2(\lambda_1, \lambda_2)$ representation.

Chapter 2

Basic notions

2.1 The interacting boson model

The interacting boson model of collective nuclear states is based on the following assumptions:

- Two equal nucleons (protons, neutrons) pair together and appear as bosons.
- These bosons may occupy two levels, a ground-state level with angular momentum L = 0 (s-state) and an excited state level with angular momentum L = 2 (d-state).
- The five components (-2, -1, 0, 1, 2) of the d-state and the component of the s-state span a linear vector space which provides a basis for the representation of the group SU(6).
- The closed shell is considered as a vacuum state; only nucleons outside the closed shell are taken into account. As closed-shell in nuclei is currently considered a configuration which is built-up by a number of protons or a number of neutrons taking the values : 2, 8, 20, 50, 82 and 126.

One finds that the subgroups SU(5), SU(3) and O(6) of SU(6) describe three limiting cases of this model [AR76], [AR78], [AR78a]. The group SU(5) describes vibrationallike nuclei [AR78]. These are usually nuclei at the beginning of major shells. The group SU(3) describes rotationallike nuclei [AR78]. These are nuclei in the middle of major shells. Finally, the group O(6) describes nuclei towards the end of major shells [AR78].

For these three limiting cases analytical expressions can be derived for energy levels and transition probabilities. Also the intermediate cases can relatively easily be described by this model [AR78].

The results obtained by this description are in astonishing good agreement with the available experimental values.

We summarize these group-theoretical tools to describe collective nuclear states in **Table 2.1**.

Table 2.1: Groups and Lie algebras to describe collective nuclear states.

Type of nuclei	Group	Lie-algebra
rotationallike	$SU(3)$	A_2
vibrationallike	$SU(5)$	A_4
nuclei towards the end of major closed shells	$O(6)$	$D_3 = A_3$

2.2 Rotationallike nuclei

The bosonlike pairs ZN of neutrons and protons outside the next closed shell of a nucleus is in the frame work of the model which we consider a central number. In fact, it characterizes a nucleus. This can be seen in the following example.

Example 2.1 *We determine ZN for Gd-156. This nucleus is built up of 64 protons and 92 neutrons. The next closed shell (magic numbers) has 50 protons and 82 neutrons. There are, therefore, 14 protons and 10 neutrons outside this closed shell. That yields 7 proton- and 5 neutron-pairs. Thus, ZN = 7 + 5 = 12. In other words, ZN = 12 characterizes in our case Gd-156.*

Note that the ZN-value is not sufficient to identify a nucleus in an unique way. Different nuclei can yield the same ZN-value (see Table 3.1). However, it uniquely defines the representation $A_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ in SU(6), because the model considered assumes the fundamental relation

$$A_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = A_5(ZN, 0, 0, 0, 0) \tag{2.1}$$

which connects the physical quantity ZN with the Lie algebra A_5.

In the present work we investigate only rotationallike nuclei. For doing that we need according to Table 2.1 the Lie algebra A_2. We have, therefore, to express $A_5(\lambda_1, 0, 0, 0, 0)$ by means of representations $A_2(\lambda_1, \lambda_2)$. This is a well-known formula [AR78].

$$\begin{aligned}
A_5(\lambda_1,0,0,0,0) &= A_2(2\lambda_1,0) \oplus A_2(2\lambda_1-4,2) \oplus A_2(2\lambda_1-8,4) \oplus \\
&\oplus A_2(2\lambda_1-12,6) \oplus \cdots \oplus \begin{bmatrix} (0,\lambda_1) & \lambda_1 & \text{even} \\ (2,\lambda_1-1) & \lambda_1 & \text{odd} \end{bmatrix} \oplus \\
&\oplus A_2(2\lambda_1-6,0) \oplus A_2(2\lambda_1-10,2) \oplus A_2(2\lambda_1-14,4) \oplus \\
&\oplus A_2(2\lambda_1-18,6) \oplus \cdots \oplus \begin{bmatrix} (0,\lambda_1-3) & \lambda_1-3 & \text{even} \\ (2,\lambda_1-4) & \lambda_1-3 & \text{odd} \end{bmatrix} \oplus \\
&\oplus A_2(2\lambda_1-12,0) \oplus A_2(2\lambda_1-16,2) \oplus A_2(2\lambda_1-20,4) \oplus \\
&\oplus A_2(2\lambda_1-24,6) \oplus \cdots \oplus \begin{bmatrix} (0,\lambda_1-6) & \lambda_1-6 & \text{even} \\ (2,\lambda_1-7) & \lambda_1-6 & \text{odd} \end{bmatrix} \oplus
\end{aligned} \tag{2.2}$$

$$\oplus \quad A_2(2\lambda_1 - 18, 0) \oplus A_2(2\lambda_1 - 22, 2) \oplus A_2(2\lambda_1 - 26, 4) \oplus$$
$$\oplus \quad A_2(2\lambda_1 - 30, 6) \oplus \cdots \oplus \begin{bmatrix} (0, \lambda_1 - 9) & \lambda_1 - 9 & \text{even} \\ (2, \lambda_1 - 10) & \lambda_1 - 9 & \text{odd} \end{bmatrix} \oplus \cdots$$

Formulae (2.1) and (2.2) permit finding of all representations $A_2(\lambda_1, \lambda_2)$ corresponding to a given ZN-value. In other words, they yield all SU(3) representations describing a given rotationallike nucleus.

2.3 Weight diagram of the Lie-algebra A_2

A representation $A_2(\lambda_1, \lambda_2)$ is equivalent with a geometrical representation, a so-called weight diagram. Such a diagram is constructed by a series of vectors in a two-dimensional space. The number of vectors or weights W of $A_2(\lambda_1, \lambda_2)$ is given by the relation [JA62, p.255],

$$W(\lambda_1, \lambda_2) = (\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2)/2. \tag{2.3}$$

Using this formula and formula (2.2) one can calculate the number of weights of the representation $A_5(\lambda_1, 0, 0, 0, 0)$.

Quite generally, the number of weights W of $A_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ is given by the relation [JA62, p.225],

$$\begin{aligned} W(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) &= (\lambda_1 + 1)(\lambda_2 + 2)(\lambda_3 + 3)(\lambda_4 + 4)(\lambda_5 + 5) \\ &(\lambda_1 + \lambda_2 + 2)(\lambda_2 + \lambda_3 + 2)(\lambda_3 + \lambda_4 + 2)(\lambda_4 + \lambda_5 + 2)(\lambda_1 + \lambda_2 + \lambda_3 + 3) \\ &(\lambda_2 + \lambda_3 + \lambda_4 + 3)(\lambda_3 + \lambda_4 + \lambda_5 + 3)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 4) \\ &(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 4)(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + 5)/34560. \end{aligned} \tag{2.4}$$

With (2.2) it immediately follows the identity

$$W(\lambda_1, 0, 0, 0, 0) = \sum_i W(\lambda_1, \lambda_2)_i = W(ZN, 0, 0, 0, 0) \tag{2.5}$$

where i runs over all $A_2(\lambda_1, \lambda_2)$ resulting from (2.2).

We assign to a weight of $A_2(\lambda_1, \lambda_2)$ a nuclear rotational state. From this basic concept directly follows:

- The total number of rotational states of a nucleus is given by (2.5).
- The number of rotational states represented by $A_2(\lambda_1, \lambda_2)$ is given by (2.3).
- The different weight diagrams $A_2(\lambda_1, \lambda_2)$ representing the total number of rotational states of a nucleus are given by (2.2).

2.4 The construction of weight diagrams $A_2(\lambda_1, \lambda_2)$

We quote here some essential theorems which are needed to calculate the weights of $A_2(\lambda_1, \lambda_2)$.

Theorem 2.1 *If α_i is a simple root (vector) of the algebra A_2 and if Λ is a weight, then the following equation is valid:*

$$2(\Lambda, \alpha_i) = \lambda_i, \tag{2.6}$$

where i = 1, 2; λ_1, λ_2 are defined by $A_2(\lambda_1, \lambda_2)$. Λ is called a maximal weight. Proof in [JA62, p.113].

Theorem 2.2 *If Λ is a maximal weight then any other weight m has the form*

$$m = \Lambda - \sum_i k_i \alpha_i, \tag{2.7}$$

where α_i are simple roots and k_i are non-negative integers. Proof in [JA62, p.213].

Definition 2.1 *If α_1, α_2 are simple roots of the algebra A_2*

$$A_{ij} = 2(\alpha_i, \alpha_j) \tag{2.8}$$

is called a CARTAN matrix,

$$A_{ij}(A_2) = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}. \tag{2.9}$$

Theorem 2.3 *If α_i are simple roots then*

$$(\alpha_i, \alpha_j) = |\alpha_i||\alpha_j| \cos\theta_{ij} = \cos\theta_{ij}. \tag{2.10}$$

$|\alpha_i| = 1$, $i = 1, 2$. *Proof in [JA62, p.128].*

Firstly, we calculate the simple roots α_1, α_2. From (2.10) and (2.8) it follows

$$(\alpha_1, \alpha_2) = -\frac{1}{2} = \cos\theta_{12} \Rightarrow \theta_{12} = 120\,\text{deg},$$

i.e., the angle θ_{12} between the two root vectors α_1 and α_2 is 120 degrees. As the length of the roots is equal to one (see (2.10)), we obtain the coordinates of α_1 and α_2:

$$\alpha_1 = (1, 0), \alpha_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \tag{2.11}$$

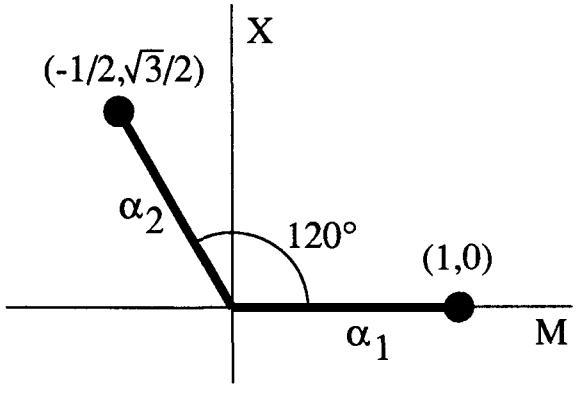


Figure 2.1: Two simple roots.

supposing that the orthogonal coordinate system of the two-dimensional root space is chosen in such a way that one axis coincides with the direction of α_1. These two simple roots are represented in Figure 2.1.

Next, we calculate the maximal weight Λ. Using (2.6) we obtain

$$\begin{aligned} 2(\Lambda_1, \Lambda_2)(\alpha_{11}, \alpha_{12}) &= \lambda_1 \\ 2(\Lambda_1, \Lambda_2)(\alpha_{21}, \alpha_{22}) &= \lambda_2. \end{aligned} \tag{2.12}$$

Inserting (2.11) yields

$$\begin{aligned} 2(\Lambda_1, \Lambda_2)(1, 0) &= \lambda_1 \\ 2(\Lambda_1, \Lambda_2)(-\frac{1}{2}, \frac{\sqrt{3}}{2}) &= \lambda_2, \end{aligned}$$

and thus

$$\begin{aligned} \Lambda_1 &= \frac{\lambda_1}{2} \\ -\Lambda_1 + \sqrt{3}\Lambda_2 &= \lambda_2 \\ \Lambda_2 &= (\lambda_2 + \frac{\lambda_1}{2})/\sqrt{3}. \end{aligned}$$

For the two <u>fundamental</u> representations $A_2(1,0)$ and $A_2(0,1)$ we obtain, therefore, the maximal weights

$$(\Lambda_1, \Lambda_2) = (\frac{1}{2}, \frac{1}{2\sqrt{3}}) \tag{2.13}$$

and

$$(\Lambda_1, \Lambda_2) = (0, \frac{1}{\sqrt{3}}), \tag{2.14}$$

respectively.

The representation $A_2(1,0)$ has according to (2.3) three weights. By means of (2.7) we obtain

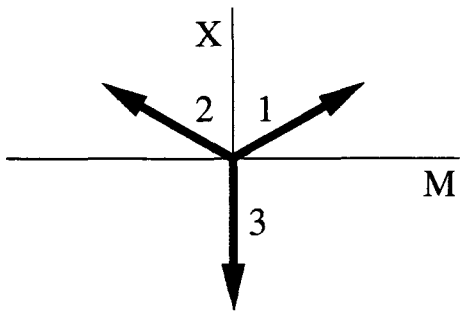


Figure 2.2: Weight diagram $A_2(1,0)$.
1, 2 and 3 denote elementary pathes.

$$m_2 = \Lambda - \alpha_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}) - (1,0) = (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$$

and

$$m_3 = m_2 - \alpha_2 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}) - (-\frac{1}{2}, \frac{\sqrt{3}}{2}) = (0, -\frac{1}{\sqrt{3}}).$$

The fundamental weight diagram $A_2(1,0)$ has, therefore, the three weights

$$m_1 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}) = \Lambda, m_2 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}}), m_3 = (0, -\frac{1}{\sqrt{3}}).$$

They are represented in Figure 2.2 in the form $m_i = (M, X)$.

Analogously, one obtains the three weights of the fundamental representation $A_2(0,1)$. They read

$$\overline{m_1} = (-\frac{1}{2}, -\frac{1}{2\sqrt{3}}), \overline{m_2} = (\frac{1}{2}, -\frac{1}{2\sqrt{3}}), \overline{m_3} = (0, \frac{1}{\sqrt{3}}).$$

The corresponding weight diagram is represented in Figure 2.3.

Note that the weight diagram $A_2(0,1)$ is generated by reflecting the weight diagram $A_2(1,0)$ on the M- and X-axis.

For the construction of weight diagrams $A_2(\lambda_1, \lambda_2)$ of higher order (higher λ_1, λ_2) one can apply a geometrical prescription or an analytical procedure.

2.4.1 Construction of weight diagrams $A_2(\lambda_1, \lambda_2)$ using a geometrical prescription

Prescription 2.1 *Centre the triangle $A_2(1,0)$, Figure 2.2, over each vertex of the triangle $A_2(1,0)$ and add the weights. Do this λ_1-times. Do the same with $A_2(0,1)$ λ_2-times.*

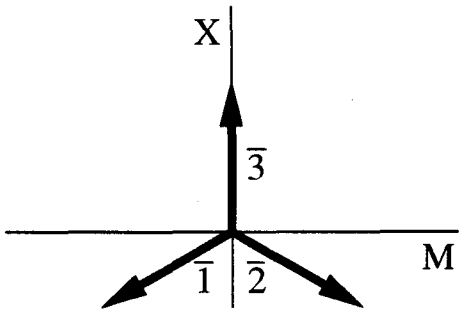
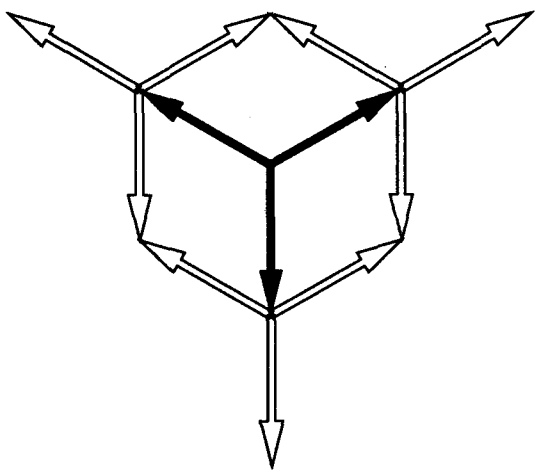


Figure 2.3: Weight diagram $A_2(0,1)$.
$\overline{1}$, $\overline{2}$ and $\overline{3}$ denote elementary pathes.

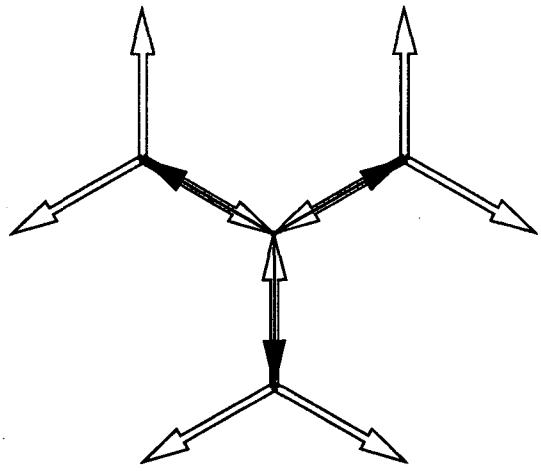
This prescription follows from a theorem which is given in [JA62, p.259].
We will illustrate the application of Prescription 2.1 in the two following examples.

Example 2.2 *Construction of the weight diagram $A_2(2,0)$ using Prescription 2.1.*



Solid arrow heads indicate weights of the $A_2(1,0)$ diagram, open arrow heads indicate weights of the $A_2(2,0)$ diagram. $W(2,0) = 6$, i.e., there are 6 weights, see (2.3).

Example 2.3 *Construction of the weight diagram $A_2(1,1)$ using Prescription 2.1.*



Solid arrow heads indicate weights of the $A_2(1,0)$ diagram, open arrow heads indicate weights of the $A_2(1,1)$ diagram. $W(1,1) = 8$, i.e., there are 8 weights (6 on the outer ring and 2 at the centre), see (2.3).

The construction of still higher weight diagrams is thus straightforward. The determination of the multiplicity of weights, however, becomes for higher weight diagrams increasingly difficult. This problem is solved in the following section.

2.4.2 Determination of the multiplicity of the weights

The analytical procedure described below is designed to calculate the M and X coordinates of each weight together with its multiplicity for a given weight diagram $A_2(\lambda_1, \lambda_2)$.

The coordinates are easily deduced recursively from the geometrical procedure given in Section 2.4.1 starting from the weight diagram $A_2(0,0)$.

To determine the multiplicity of the weights we make use of the pathes followed to reach the weights. A path is a sequence of $\lambda_1 + \lambda_2$ elementary pathes which summarizes the geometrical path followed to reach a weight from (0,0).

At each superposition of the diagram $A_2(1,0)$ or $A_2(0,1)$ one elementary path is added to the total path according to the notation given in Figures 2.2 and 2.3.

Using this concept it appears that one weight may be reached following different geometrical paths. These equivalent paths are divided into classes. Those with the same number of elementary paths 3 or $\overline{3}$ of the representations $A_2(1,0)$ and $A_2(0,1)$, respectively, belong to the same class. The number of different classes is the multiplicity of the weight.

Example 2.4 *The paths $23\overline{2}$, $13\overline{1}$ and $33\overline{3}$ yield the same weight. The paths $23\overline{2}$ and $13\overline{1}$ belong to the same class. The path $33\overline{3}$ is a different class. The multiplicity of the weight is, therefore, equal to 2.*

Only one path per class is conserved before performing the next superposition. The procedure is illustrated in the following examples where we construct the weight diagrams $A_2(1,0)$, $A_2(0,1)$, $A_2(2,0)$ and $A_2(2,1)$ starting from the weight diagram $A_2(0,0)$.

Example 2.5 *Construction of the weight diagram $A_2(1,0)$ from the weight diagram $A_2(0,0)$.*

$A_2(0,0)$		$A_2(1,0)$			
M	X	M	X	Path	Mult.
0	0	1	1	1	1
		-1	1	2	1
		0	-2	3	1

Example 2.6 *Construction of the weight diagram $A_2(0,1)$ from the weight diagram $A_2(0,0)$.*

$A_2(0,0)$		$A_2(0,1)$			
M	X	M	X	Path	Mult.
0	0	1	-1	$\overline{2}$	1
		-1	-1	$\overline{1}$	1
		0	2	$\overline{3}$	1

Example 2.7 *Construction of the weight diagram $A_2(2,0)$ from the weight diagram $A_2(1,0)$.*

$A_2(1,0)$			$A_2(2,0)$			$A_2(2,0)$			
M	X	Path	M	X	Path	M	X	Path	Mult.
1	1	1	2	2	11	-2	2	22	1
			0	2	12	-1	-1	23, 32	1
			1	-1	13	0	-4	33	1
-1	1	2	0	2	21	0	2	21	1
			-2	2	22	1	-1	13, 31	1
			-1	-1	23	2	2	11	1
0	-2	3	1	-1	31				
			-1	-1	32				
			0	-4	33				

Example 2.8 *Construction of the weight diagram $A_2(2,1)$ from the weight diagram $A_2(2,0)$.*

$A_2(2,0)$			$A_2(2,1)$			$A_2(2,1)$			
M	X	Path	M	X	Path	M	X	Path	Mult.
-2	2	22	-1	1	$22\bar{2}$	-3	1	$22\bar{1}$	1
			-3	1	$22\bar{1}$	-2	-2	$23\bar{1}$	1
			-2	4	$22\bar{3}$	-2	4	$22\bar{3}$	1
-1	1	23	0	-2	$23\bar{2}$	-1	-5	$33\bar{1}$	1
			-2	-2	$23\bar{1}$	-1	1	$(22\bar{2}, 21\bar{1}), (23\bar{3})$	2
			-1	1	$23\bar{3}$	0	-2	$(23\bar{2}, 13\bar{1}), (33\bar{3})$	2
0	-4	33	1	-5	$33\bar{2}$	0	4	$21\bar{3}$	1
			-1	-5	$33\bar{1}$	1	-5	$33\bar{2}$	1
			0	-2	$33\bar{3}$	1	1	$(21\bar{2}, 11\bar{1}), (13\bar{3})$	2
0	2	21	1	1	$21\bar{2}$	2	-2	$13\bar{2}$	1
			-1	1	$21\bar{1}$	2	4	$11\bar{3}$	1
			0	4	$21\bar{3}$	3	1	$11\bar{2}$	1
1	-1	13	2	-2	$13\bar{2}$				
			0	-2	$13\bar{1}$				
			1	1	$13\bar{3}$				
2	2	11	3	1	$11\bar{2}$				
			1	1	$11\bar{1}$				
			2	4	$11\bar{3}$				

Further details on the theory of weights of Lie algebras are for example given in [JA62], [HU72], [BO68]. Computer codes to calculate weight vectors are published in [AG69] and [BE72].

Chapter 3

Inventory of rotationallike nuclei

Prior to the classification of levels of rotationallike nuclei by means of weight diagrams $A_2(\lambda_1, \lambda_2)$ it is instructive to make an inventory of the nuclei concerned. In the Tables 3.1 and 3.2 we give these nuclei together with the characteristics which are of interest for our work.

As the number ZN is within the framework of the interacting boson model of central interest, we give in Table 3.3 the rotationallike nuclei sorted according to increasing ZN-values.

The frequencies of the occurence of the ZN-values is another interesting information. They are given in Table 3.4.

From these tables we can conclude

- $ZN < 30$, i.e., the number of rotational levels is limited (see (2.5)). That means that the number of weight diagrams $A_2(\lambda_1, \lambda_2)$ is limited (see (2.2)).
- Many nuclei have relatively few rotational levels.
- Many nuclei have according to this model the same number of rotational levels and an identical classification scheme, as the ZN-value is for many nuclei identical.

In other words, from the point of view of the classification, nuclei with identical ZN-value cannot be distinguished. That does, however, not mean that the rotational levels are the same, as we will see in Section 4.2.

Table 3.1: Rotationallike nuclei (sorted according to increasing Z).
Z = number of protons, N = number of neutrons, Z_m = number of protons of the next closed shell (magic number: 50), N_m = number of neutrons of the next closed shell (magic number: 82), $Z_p = (Z - Z_m)/2$ = number of proton pairs, $N_p = (N - N_m)/2$ = number of neutron pairs, $ZN = (Z - Z_m)/2 + (N - N_m)/2$.

	Z	N	Z_p	N_p	ZN		Z	N	Z_p	N_p	ZN		Z	N	Z_p	N_p	ZN
Nd	60	88	5	3	8	Yb	70	90	10	4	14	W	74	90	12	4	16
Nd	60	90	5	4	9	Yb	70	92	10	5	15	W	74	92	12	5	17
Nd	60	92	5	5	10	Yb	70	94	10	6	16	W	74	94	12	6	18
						Yb	70	96	10	7	17	W	74	96	12	7	19
Sm	62	90	6	4	10	Yb	70	98	10	8	18	W	74	98	12	8	20
Sm	62	92	6	5	11	Yb	70	100	10	9	19	W	74	100	12	9	21
Sm	62	94	6	6	12	Yb	70	102	10	10	20	W	74	102	12	10	22
						Yb	70	104	10	11	21	W	74	104	12	11	23
Gd	64	90	7	4	11	Yb	70	106	10	12	22	W	74	106	12	12	24
Gd	64	92	7	5	12	Yb	70	108	10	13	23	W	74	108	12	13	25
Gd	64	94	7	6	13							W	74	110	12	14	26
Gd	64	96	7	7	14	Hf	72	90	11	4	15	W	74	112	12	15	27
Gd	64	98	7	8	15	Hf	72	92	11	5	16						
						Hf	72	94	11	6	17	Os	76	94	13	6	19
Dy	66	90	8	4	12	Hf	72	96	11	7	18	Os	76	96	13	7	20
Dy	66	92	8	5	13	Hf	72	98	11	8	19	Os	76	98	13	8	21
Dy	66	94	8	6	14	Hf	72	100	11	9	20	Os	76	100	13	9	22
Dy	66	96	8	7	15	Hf	72	102	11	10	21	Os	76	102	13	10	23
Dy	66	98	8	8	16	Hf	72	104	11	11	22	Os	76	104	13	11	24
Dy	66	100	8	9	17	Hf	72	106	11	12	23	Os	76	106	13	12	25
						Hf	72	108	11	13	24	Os	76	108	13	13	26
Er	68	90	9	4	13	Hf	72	110	11	14	25	Os	76	110	13	14	27
Er	68	92	9	5	14	Hf	72	112	11	15	26	Os	76	112	13	15	28
Er	68	94	9	6	15												
Er	68	96	9	7	16							Pt	78	96	14	7	21
Er	68	98	9	8	17							Pt	78	98	14	8	22
Er	68	100	9	9	18							Pt	78	100	14	9	23
Er	68	102	9	10	19							Pt	78	102	14	10	24
Er	68	104	9	11	20							Pt	78	104	14	11	25
												Pt	78	106	14	12	26
												Pt	78	108	14	13	27
												Pt	78	110	14	14	28
												Pt	78	112	14	15	29

Table 3.2: Rotationallike nuclei (sorted according to increasing Z).
Z = number of protons, N = number of neutrons, Z_m = number of protons of the next closed shell (magic number: 82), N_m = number of neutrons of the next closed shell (magic number: 126), $Z_p = (Z - Z_m)/2$ = number of proton pairs, $N_p = (N - N_m)/2$ = number of neutron pairs, $ZN = (Z - Z_m)/2 + (N - N_m)/2$.

	Z	N	Z_p	N_p	ZN		Z	N	Z_p	N_p	ZN		Z	N	Z_p	N_p	ZN
Po	84	130	1	2	3	U	92	134	5	4	9	Cm	96	142	7	8	15
Po	84	132	1	3	4	U	92	136	5	5	10	Cm	96	144	7	9	16
Po	84	134	1	4	5	U	92	138	5	6	11	Cm	96	146	7	10	17
						U	92	140	5	7	12	Cm	96	148	7	11	18
Rn	86	130	2	2	4	U	92	142	5	8	13	Cm	96	150	7	12	19
Rn	86	132	2	3	5	U	92	144	5	9	14	Cm	96	152	7	13	20
Rn	86	134	2	4	6	U	92	146	5	10	15	Cm	96	154	7	14	21
Rn	86	136	2	5	7	U	92	148	5	11	16						
Rn	86	138	2	6	8							Cf	98	142	8	8	16
Rn	86	140	2	7	9	Pu	94	138	6	6	12	Cf	98	144	8	9	17
						Pu	94	140	6	7	13	Cf	98	146	8	10	18
Ra	88	130	3	2	5	Pu	94	142	6	8	14	Cf	98	148	8	11	19
Ra	88	132	3	3	6	Pu	94	144	6	9	15	Cf	98	150	8	12	20
Ra	88	134	3	4	7	Pu	94	146	6	10	16	Cf	98	152	8	13	21
Ra	88	136	3	5	8	Pu	94	148	6	11	17	Cf	98	154	8	14	22
Ra	88	138	3	6	9	Pu	94	150	6	12	18	Cf	98	156	8	15	23
Ra	88	140	3	7	10	Pu	94	152	6	13	19						
Ra	88	142	3	8	11							Fm	100	144	9	9	18
												Fm	100	146	9	10	19
Th	90	130	4	2	6							Fm	100	148	9	11	20
Th	90	132	4	3	7							Fm	100	150	9	12	21
Th	90	134	4	4	8							Fm	100	152	9	13	22
Th	90	136	4	5	9							Fm	100	154	9	14	23
Th	90	138	4	6	10							Fm	100	156	9	15	24
Th	90	140	4	7	11												
Th	90	142	4	8	12												
Th	90	144	4	9	13												
Th	90	146	4	10	14												

Table 3.3: Rotationallike nuclei (sorted according to increasing ZN-values).

ZN		Z	N	ZN		Z	N	ZN		Z	N
3	Po	84	130	14	Th	90	146	20	Hf	72	100
4	Po	84	132	14	U	92	144	20	W	74	98
4	Rn	86	130	14	Pu	94	142	20	Os	76	96
5	Po	84	134	15	Gd	64	98	20	Cm	96	152
5	Rn	86	132	15	Dy	64	96	20	Cf	98	150
5	Ra	88	130	15	Er	64	94	20	Fm	100	148
6	Rn	86	134	15	Yb	74	92	21	Yb	70	104
6	Ra	88	132	15	Hf	74	90	21	Hf	72	102
6	Th	90	130	15	U	92	146	21	W	74	100
7	Rn	86	136	15	Pu	94	144	21	Os	76	98
7	Ra	88	134	15	Cm	96	142	21	Pt	78	96
7	Th	90	132	16	Dy	66	98	21	Cm	96	154
8	Nd	60	88	16	Er	68	96	21	Cf	98	152
8	Rn	86	138	16	Yb	70	94	21	Fm	96	150
8	Ra	88	136	16	Hf	72	92	22	Yb	70	106
8	Th	90	134	16	W	74	90	22	Hf	72	104
9	Nd	60	90	16	U	92	148	22	W	74	102
9	Rn	86	140	16	Pu	94	146	22	Os	76	100
9	Ra	88	138	16	Cm	96	144	22	Pt	78	98
9	Th	90	136	16	Cf	98	142	22	Cf	98	154
9	U	92	134	17	Dy	66	100	22	Fm	100	152
10	Nd	60	92	17	Er	68	98	23	Yb	70	108
10	Sm	62	90	17	Yb	70	96	23	Hf	72	106
10	Ra	88	140	17	Hf	72	94	23	W	74	104
10	Th	90	138	17	W	74	92	23	Os	76	102
10	U	92	136	17	Pu	94	148	23	Pt	78	100
11	Sm	62	92	17	Cm	96	146	23	Cf	98	156
11	Gd	64	90	17	Cf	98	144	23	Fm	100	154
11	Ra	88	142	18	Er	68	100	24	Hf	72	108
11	Th	90	140	18	Yb	70	98	24	W	74	106
11	U	92	138	18	Hf	72	96	24	Os	76	104
12	Sm	62	94	18	W	74	94	24	Pt	78	102
12	Gd	64	92	18	Pu	94	150	24	Fm	100	156
12	Dy	66	90	18	Cm	96	148	25	Hf	72	110
12	Th	90	142	18	Cf	98	146	25	W	74	108
12	U	92	140	18	Fm	100	144	25	Os	76	106
12	Pu	94	138	19	Er	68	102	25	Pt	78	104
13	Gd	64	94	19	Yb	70	100	26	Hf	72	112
13	Dy	66	92	19	Hf	72	98	26	W	74	110
13	Er	68	90	19	W	74	96	26	Os	76	108
13	Th	90	144	19	Os	76	94	26	Pt	78	106
13	U	92	142	19	Pu	94	152	27	W	74	112
13	Pu	94	140	19	Cm	96	150	27	Os	76	110
14	Gd	64	96	19	Cf	98	148	27	Pt	78	108
14	Dy	66	94	19	Fm	100	146	28	Os	76	112
14	Er	68	92	20	Er	68	104	28	Pt	78	110
14	Yb	66	90	20	Yb	70	102	29	Pt	78	112

Table 3.4: Frequency F of the occurence of a given ZN-value.

ZN	F(ZN)	ZN	F(ZN)	ZN	F(ZN)
3	1	12	6	21	8
4	2	13	6	22	7
5	3	14	7	23	7
6	3	15	8	24	5
7	3	16	9	25	4
8	4	17	8	26	4
9	5	18	8	27	3
10	5	19	9	28	2
11	5	20	8	29	1

Chapter 4

Classification of nuclear rotational levels by means of weight diagrams $A_2(\lambda_1, \lambda_2)$ of the Lie algebra A_2

4.1 Weight diagrams $A_2(\lambda_1, \lambda_2)$

A nucleus with a given ZN-value has according to (2.4) W(ZN,0,0,0,0) theoretical rotational states (or weights). These states of a given nucleus are represented in a series of weight diagrams $A_2(\lambda_1, \lambda_2)_i$, where $(\lambda_1, \lambda_2)_i$ is for a given ZN determined by (2.2). A nuclear state corresponds to a weight. Taking into account the possible ZN-values (see Tables 3.1 and 3.2) one would at a first glance believe that we have to deal with a non-maniable number of different weight diagrams $A_2(\lambda_1, \lambda_2)$. It turns, however, out that this manifold can condiderably be reduced. There remain in fact only a reasonable number of weight diagrams for the classification of the rotational states of the nuclei which are tabulated in Tables 3.1 and 3.1.

That is becoming evident from Table 4.1, where we give $(\lambda_1, \lambda_2)_i$ for all ZN-values of rotational-like nuclei.

From Table 4.1 we immediately see that many of the $(\lambda_1, \lambda_2)_i$ appear there several times. In Table 4.2 we give the frequency F of all possible $(\lambda_1, \lambda_2)_i$ which are given in Table 4.1.

Another result which directly follows from Table 4.1 is the following theorem.

Theorem 4.1 *If ZNM denotes the upper limit of ZN then*

$$A_5(ZNM, 0, 0, 0, 0) = \sum_i A_2(\lambda_1, \lambda_2)_i, \tag{4.1}$$

$$A_5(ZNM - 1, 0, 0, 0, 0) = \sum_i A_2(\lambda_1, \lambda_2)_i, \text{ and} \tag{4.2}$$

$$A_5(ZNM - 2, 0, 0, 0, 0) = \sum_i A_2(\lambda_1, \lambda_2)_i, \tag{4.3}$$

comprise all possible $A_2(\lambda_1, \lambda_2)$ *resulting from*

$$A_5(ZN, 0, 0, 0, 0) = \sum_i A_2(\lambda_1, \lambda_2)_i, \tag{4.4}$$

where $ZN \leq ZNM$ *and* $\lambda_1,\ \lambda_2 = 0, 2, 4 \cdots$ *even.*

In other words, if we determine all $A_2(\lambda_1, \lambda_2)$ resulting from $A_5(29, 0, 0, 0, 0)$, $A_5(28, 0, 0, 0, 0)$ and $A_5(27, 0, 0, 0, 0)$ then we know all $A_2(\lambda_1, \lambda_2)$ which are needed for the representation of the rotational states of the nuclei given in Tables 3.1 and 3.2.

In the following we present some representative drawings of the weight diagrams $A_2(\lambda_1, \lambda_2)$ appearing in Table 4.2.

For its construction we can use the Prescription 2.1 (see Section 2.4.1) together with the analytical procedure described in Section 2.4.2.

Table 4.1: $(\lambda_1, \lambda_2)_i$ corresponding to ZN-values given in Tables 3.1 and 3.2.

$$
\begin{aligned}
(1,0,0,0,0) &= (2,0) \\
(2,0,0,0,0) &= (4,0) \oplus (0,2) \\
(3,0,0,0,0) &= (6,0) \oplus (2,2) \oplus (0,0) \\
(4,0,0,0,0) &= (8,0) \oplus (4,2) \oplus (0,4) \oplus (2,0) \\
(5,0,0,0,0) &= (10,0) \oplus (6,2) \oplus (2,4) \oplus (4,0) \oplus (0,2) \\
(6,0,0,0,0) &= (12,0) \oplus (8,2) \oplus (4,4) \oplus (0,6) \oplus (6,0) \oplus (2,2) \oplus (0,0) \\
(7,0,0,0,0) &= (14,0) \oplus (10,2) \oplus (6,4) \oplus (2,6) \oplus (8,0) \oplus (4,2) \oplus (0,4) \oplus (2,0) \\
(8,0,0,0,0) &= (16,0) \oplus (12,2) \oplus (8,4) \oplus (4,6) \oplus (0,8) \oplus (10,0) \oplus (6,2) \oplus (2,4) \oplus \\
&\quad (4,0) \oplus (0,2) \\
(9,0,0,0,0) &= (18,0) \oplus (14,2) \oplus (10,4) \oplus (6,6) \oplus (2,8) \oplus (12,0) \oplus (8,2) \oplus (4,4) \oplus \\
&\quad (0,6) \oplus (6,0) \oplus (2,2) \oplus (0,0) \\
(10,0,0,0,0) &= (20,0) \oplus (16,2) \oplus (12,4) \oplus (8,6) \oplus (4,8) \oplus (0,10) \oplus (14,0) \oplus (10,2) \oplus \\
&\quad (6,4) \oplus (2,6) \oplus (8,0) \oplus (4,2) \oplus (0,4) \oplus (2,0) \\
(11,0,0,0,0) &= (22,0) \oplus (18,2) \oplus (14,4) \oplus (10,6) \oplus (6,8) \oplus (2,10) \oplus (16,0) \oplus (12,2) \oplus \\
&\quad (8,4) \oplus (4,6) \oplus (0,8) \oplus (10,0) \oplus (6,2) \oplus (2,4) \oplus (4,0) \oplus (0,2) \\
(12,0,0,0,0) &= (24,0) \oplus (20,2) \oplus (16,4) \oplus (12,6) \oplus (8,8) \oplus (4,10) \oplus (0,12) \oplus (18,0) \oplus \\
&\quad (14,2) \oplus (10,4) \oplus (6,6) \oplus (2,8) \oplus (12,0) \oplus (8,2) \oplus (4,4) \oplus (0,6) \oplus (6,0) \\
&\quad \oplus (2,2) \\
(13,0,0,0,0) &= (26,0) \oplus (22,2) \oplus (18,4) \oplus (14,6) \oplus (10,8) \oplus (6,10) \oplus (2,12) \oplus (20,0) \\
&\quad \oplus (16,2) \oplus (12,4) \oplus (8,6) \oplus (4,8) \oplus (0,10) \oplus (14,0) \oplus (10,2) \oplus \\
&\quad (6,4) \oplus (2,6) \oplus (8,0) \oplus (4,2) \oplus (0,4) \oplus (2,0) \\
(14,0,0,0,0) &= (28,0) \oplus (24,2) \oplus (20,4) \oplus (16,6) \oplus (12,8) \oplus (8,10) \oplus (4,12) \oplus (0,14) \oplus \\
&\quad (11,0,0,0,0) \\
(15,0,0,0,0) &= (30,0) \oplus (26,2) \oplus (22,4) \oplus (18,6) \oplus (14,8) \oplus (10,10) \oplus (6,12) \oplus (2,14) \\
&\quad \oplus (12,0,0,0,0) \\
(16,0,0,0,0) &= (32,0) \oplus (28,2) \oplus (24,4) \oplus (20,6) \oplus (16,8) \oplus (12,10) \oplus (8,12) \oplus (4,14) \\
&\quad \oplus (0,16) \oplus (13,0,0,0,0) \\
(17,0,0,0,0) &= (34,0) \oplus (30,2) \oplus (26,4) \oplus (22,6) \oplus (18,8) \oplus (14,10) \oplus (10,12) \oplus (6,14) \\
&\quad \oplus (2,16) \oplus (14,0,0,0,0) \\
(18,0,0,0,0) &= (36,0) \oplus (32,2) \oplus (28,4) \oplus (24,6) \oplus (20,8) \oplus (16,10) \oplus (12,12) \oplus (8,14) \\
&\quad \oplus (4,16) \oplus (0,18) \oplus (15,0,0,0,0) \\
(19,0,0,0,0) &= (38,0) \oplus (34,2) \oplus (30,4) \oplus (26,6) \oplus (22,8) \oplus (18,10) \oplus (14,12) \oplus \\
&\quad (10,14) \oplus (6,16) \oplus (2,18) \oplus (16,0,0,0,0) \\
(20,0,0,0,0) &= (40,0) \oplus (36,2) \oplus (32,4) \oplus (28,6) \oplus (24,8) \oplus (20,10) \oplus (16,12) \oplus \\
&\quad (12,14) \oplus (8,16) \oplus (4,18) \oplus (0,20) \oplus (17,0,0,0,0) \\
(21,0,0,0,0) &= (42,0) \oplus (38,2) \oplus (34,4) \oplus (30,6) \oplus (26,8) \oplus (22,10) \oplus (18,12) \oplus \\
&\quad (14,14) \oplus (10,16) \oplus (6,18) \oplus (2,20) \oplus (18,0,0,0,0) \\
(22,0,0,0,0) &= (44,0) \oplus (40,2) \oplus (36,4) \oplus (32,6) \oplus (28,8) \oplus (24,10) \oplus (20,12) \oplus \\
&\quad (16,14) \oplus (12,16) \oplus (8,18) \oplus (4,20) \oplus (0,22) \oplus (19,0,0,0,0)
\end{aligned}
$$

Table 4.1 (*Continued*).

$$
\begin{aligned}
(23,0,0,0,0) &= (46,0)\oplus(42,2)\oplus(38,4)\oplus(34,6)\oplus(30,8)\oplus(26,10)\oplus(22,12)\oplus \\
&\quad (18,14)\oplus(14,16)\oplus(10,18)\oplus(6,20)\oplus(2,22)\oplus(20,0,0,0,0) \\
(24,0,0,0,0) &= (48,0)\oplus(44,2)\oplus(40,4)\oplus(36,6)\oplus(32,8)\oplus(28,10)\oplus(24,12)\oplus \\
&\quad (20,14)\oplus(16,16)\oplus(12,18)\oplus(8,20)\oplus(4,22)\oplus(0,24)\oplus(21,0,0,0,0) \\
(25,0,0,0,0) &= (50,0)\oplus(46,2)\oplus(42,4)\oplus(38,6)\oplus(34,8)\oplus(30,10)\oplus(26,12)\oplus \\
&\quad (22,14)\oplus(18,16)\oplus(14,18)\oplus(10,20)\oplus(6,22)\oplus(2,24)\oplus(22,0,0,0,0) \\
(26,0,0,0,0) &= (52,0)\oplus(48,2)\oplus(44,4)\oplus(40,6)\oplus(36,8)\oplus(32,10)\oplus(28,12)\oplus \\
&\quad (24,14)\oplus(20,16)\oplus(16,18)\oplus(12,20)\oplus(8,22)\oplus(4,24)\oplus(0,26)\oplus \\
&\quad (23,0,0,0,0) \\
(27,0,0,0,0) &= (54,0)\oplus(50,2)\oplus(46,4)\oplus(42,6)\oplus(38,8)\oplus(34,10)\oplus(30,12)\oplus \\
&\quad (26,14)\oplus(22,16)\oplus(18,18)\oplus(14,20)\oplus(10,22)\oplus(6,24)\oplus(2,26)\oplus \\
&\quad (24,0,0,0,0) \\
(28,0,0,0,0) &= (56,0)\oplus(52,2)\oplus(48,4)\oplus(44,6)\oplus(40,8)\oplus(36,10)\oplus(32,12)\oplus \\
&\quad (28,14)\oplus(24,16)\oplus(20,18)\oplus(16,20)\oplus(12,22)\oplus(8,24)\oplus(4,26)\oplus \\
&\quad (0,28)\oplus(25,0,0,0,0) \\
(29,0,0,0,0) &= (58,0)\oplus(54,2)\oplus(50,4)\oplus(46,6)\oplus(42,8)\oplus(38,10)\oplus(34,12)\oplus \\
&\quad (30,14)\oplus(26,16)\oplus(22,18)\oplus(18,20)\oplus(14,22)\oplus(10,24)\oplus(6,26)\oplus \\
&\quad (2,28)\oplus(26,0,0,0,0)
\end{aligned}
$$

Table 4.2: Frequency F of the occurence of a given (λ_1, λ_2).

(λ_1,λ_2)	F	(λ_1,λ_2)	F	(λ_1,λ_2)	F	(λ_1,λ_2)	F	(λ_1,λ_2)	F
(0, 0)	9	(6, 8)	7	(14, 0)	8	(22, 8)	4	(34, 2)	4
(0, 2)	10	(6,10)	6	(14, 2)	7	(22,10)	3	(34, 4)	3
(0, 4)	9	(6,12)	5	(14, 4)	7	(22,12)	3	(34, 6)	3
(0, 6)	8	(6,14)	5	(14, 6)	6	(22,14)	2	(34, 8)	2
(0, 8)	8	(6,16)	4	(14, 8)	5	(22,16)	1	(34,10)	1
(0,10)	7	(6,18)	3	(14,10)	5	(22,18)	1	(34,12)	1
(0,12)	6	(6,20)	3	(14,12)	4	(24, 0)	6	(36, 0)	4
(0,14)	6	(6,22)	2	(14,14)	3	(24, 2)	6	(36, 2)	4
(0,16)	5	(6,24)	1	(14,16)	3	(24, 4)	5	(36, 4)	3
(0,18)	4	(6,26)	1	(14,18)	2	(24, 6)	4	(36, 6)	2
(0,20)	4	(8, 0)	9	(14,20)	1	(24, 8)	4	(36, 8)	2
(0,22)	3	(8, 2)	8	(14,22)	1	(24,10)	3	(36,10)	1
(0,24)	2	(8, 4)	8	(16, 0)	8	(24,12)	2	(38, 0)	4
(0,26)	2	(8, 6)	7	(16, 2)	7	(24,14)	2	(38, 2)	3
(0,28)	1	(8, 8)	6	(16, 4)	6	(24,16)	1	(38, 4)	3
(2, 0)	10	(8,10)	6	(16, 6)	6	(26, 0)	6	(38, 6)	2
(2, 2)	9	(8,12)	5	(16, 8)	5	(26, 2)	5	(38, 8)	1
(2, 4)	9	(8,14)	4	(16,10)	4	(26, 4)	5	(38,10)	1
(2, 6)	8	(8,16)	4	(16,12)	4	(26, 6)	4	(40, 0)	4
(2, 8)	7	(8,18)	3	(16,14)	3	(26, 8)	3	(40, 2)	3
(2,10)	7	(8,20)	2	(16,16)	2	(26,10)	3	(40, 4)	2
(2,12)	6	(8,22)	2	(16,18)	2	(26,12)	2	(40, 6)	2
(2,14)	5	(8,24)	1	(16,20)	1	(26,14)	1	(40, 8)	1
(2,16)	5	(10, 0)	9	(18, 0)	7	(26,16)	1	(42, 0)	3
(2,18)	4	(10, 2)	8	(18, 2)	7	(28, 0)	6	(42, 2)	3
(2,20)	3	(10, 4)	7	(18, 4)	6	(28, 2)	5	(42, 4)	2
(2,22)	3	(10, 6)	7	(18, 6)	5	(28, 4)	4	(42, 6)	1
(2,24)	2	(10, 8)	6	(18, 8)	5	(28, 6)	4	(42, 8)	1
(2,26)	1	(10,10)	5	(18,10)	4	(28, 8)	3	(44, 0)	3
(2,28)	1	(10,12)	5	(18,12)	3	(28,10)	2	(44, 2)	2
(4, 0)	10	(10,14)	4	(18,14)	3	(28,12)	2	(44, 4)	2
(4, 2)	9	(10,16)	3	(18,16)	2	(28,14)	1	(44, 6)	1
(4, 4)	8	(10,18)	3	(18,18)	1	(30, 0)	5	(46, 0)	3
(4, 6)	8	(10,20)	2	(18,20)	1	(30, 2)	5	(46, 2)	2
(4, 8)	7	(10,22)	1	(20, 0)	7	(30, 4)	4	(46, 4)	1
(4,10)	6	(10,24)	1	(20, 2)	6	(30, 6)	3	(46, 6)	1
(4,12)	6	(12, 0)	8	(20, 4)	6	(30, 8)	3	(48, 0)	2
(4,14)	5	(12, 2)	8	(20, 6)	5	(30,10)	2	(48, 2)	2
(4,16)	4	(12, 4)	7	(20, 8)	4	(30,12)	1	(48, 4)	1
(4,18)	4	(12, 6)	6	(20,10)	4	(30,14)	1	(50, 0)	2
(4,20)	3	(12, 8)	6	(20,12)	3	(32, 0)	5	(50, 2)	1
(4,22)	2	(12,10)	5	(20,14)	2	(32, 2)	4	(50, 4)	1
(4,24)	2	(12,12)	4	(20,16)	2	(32, 4)	4	(52, 0)	2
(4,26)	1	(12,14)	4	(20,18)	1	(32, 6)	3	(52, 2)	1
(6, 0)	9	(12,16)	3	(22, 0)	7	(32, 8)	2	(54, 0)	1
(6, 2)	9	(12,18)	2	(22, 2)	6	(32,10)	2	(54, 2)	1
(6, 4)	8	(12,20)	2	(22, 4)	5	(32,12)	1	(56, 0)	1
(6, 6)	7	(12,22)	1	(22, 6)	5	(34, 0)	5	(58, 0)	1

4.1.1 Drawings of weight diagrams $A_2(\lambda_1, 0)$, λ_1-even

One can easily verify that the weight diagram $A_2(0, \lambda_1)$ is obtained by reflecting the weight diagram $A_2(\lambda_1, 0)$ on the M- and X-axis. Furthermore, it immediately follows from (2.3) that

$$W(\lambda_1, \lambda_2) = W(\lambda_2, \lambda_1), \tag{4.5}$$

i.e., the numbers of weights of $A_2(\lambda_1, \lambda_2)$ and $A_2(\lambda_2, \lambda_1)$ are identical.

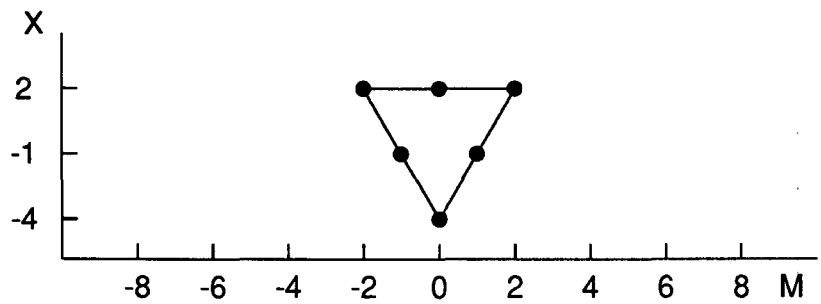


Figure 4.1: Weight diagram $A_2(2, 0)$.
Dots indicate weights. The number of weights $W(2, 0) = 6$, see (2.3).

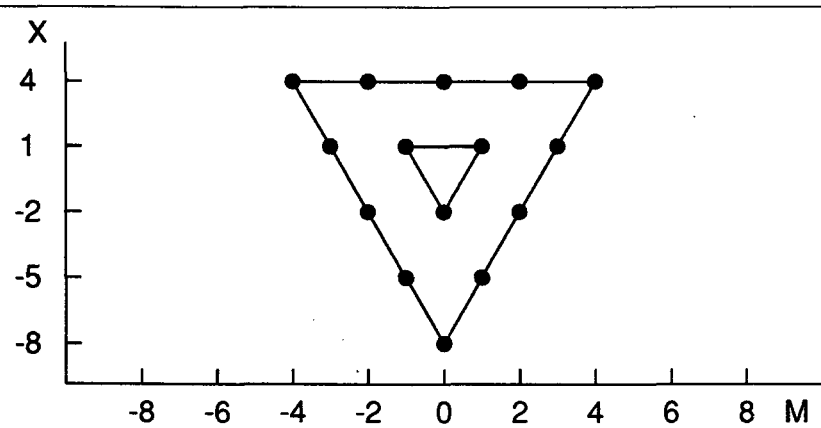


Figure 4.2: Weight diagram $A_2(4, 0)$.
Dots indicate weights. The number of weights $W(4, 0) = 15$, see (2.3).

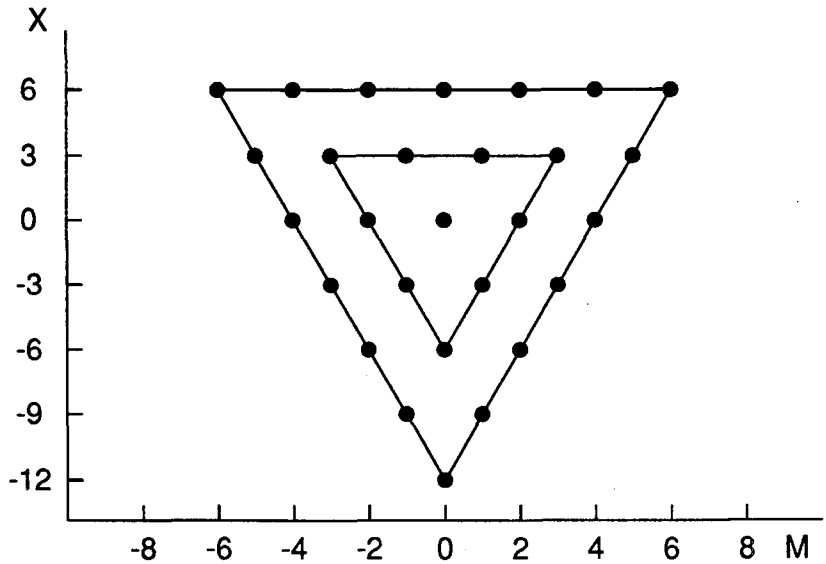


Figure 4.3: Weight diagram $A_2(6,0)$.
Dots indicate weights. The number of weights $W(6,0) = 28$, see (2.3).

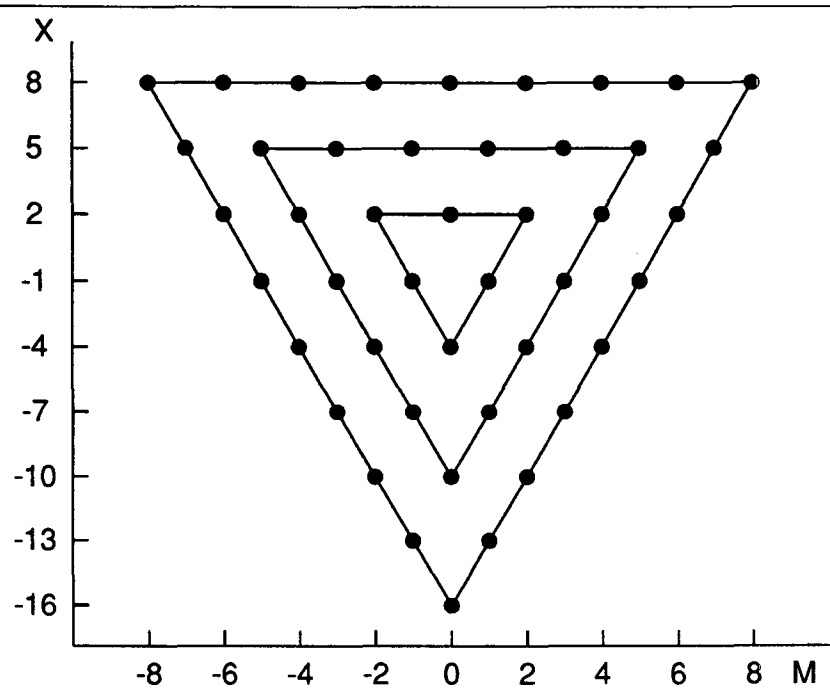


Figure 4.4: Weight diagram $A_2(8,0)$.
Dots indicate weights. The number of weights $W(8,0) = 45$, see (2.3).

In Table 4.3 we summarize the characteristics of the weights diagrams $A_2(\lambda_1, 0)$.

Table 4.3: Characteristics of weight diagrams $A_2(\lambda_1, 0)$, λ_1-even.

$(\lambda_1,0)$	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(2,0)	1	0	1	$-2 \to 2$	$-4/3 \to 2/3$	6
(4,0)	2	0	1,1	$-4 \to 4$	$-8/3 \to 4/3$	15
(6,0)	2	1	1,1	$-6 \to 6$	$-12/3 \to 6/3$	28
(8,0)	3	0	1,1,1	$-8 \to 8$	$-16/3 \to 8/3$	45
(10,0)	4	0	1,1,1,1	$-10 \to 10$	$-20/3 \to 10/3$	66
(12,0)	4	1	1,1,1,1	$-12 \to 12$	$-24/3 \to 12/3$	91
(14,0)	5	0	1,1,1,1,1	$-14 \to 14$	$-28/3 \to 14/3$	120
(16,0)	6	0	1,1,1,1,1,1	$-16 \to 16$	$-32/3 \to 16/3$	153
$(\lambda_1,0)$	r	c	1 r-times	$-\lambda_1 \to \lambda_1$	$-2\lambda_1/3 \to \lambda_1/3$	formula (2.3)

$r = (\lambda_1 + \lambda_2)/2 - [(\lambda_1 - \lambda_2)/6]$, where [ ] means that only the integer part is valid,
$c = 1$ if $\lambda_1/6$ integer, 0 otherwise.

The next weight diagram we draw are $A_2(\lambda_1, \lambda_1)$, i.e., $\lambda_1 = \lambda_2$.

4.1.2 Drawings of weight diagrams $A_2(\lambda_1, \lambda_1)$, λ_1-even

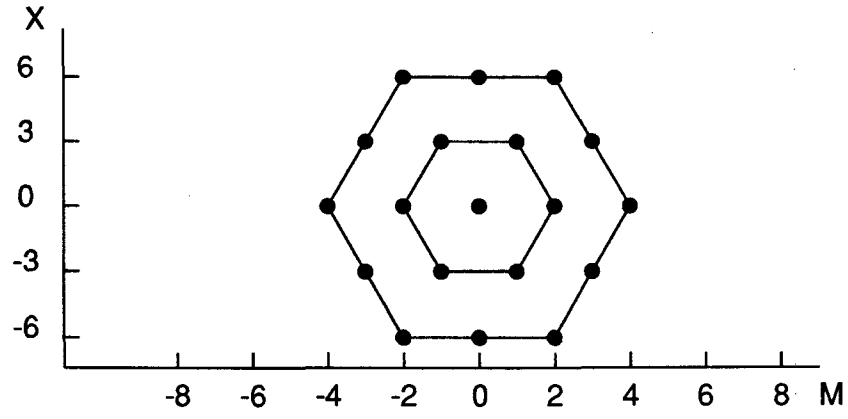


Figure 4.5: Weight diagram $A_2(2,2)$.
Dots indicate weights. The number of weights $W(2,2) = 27$, see (2.3).

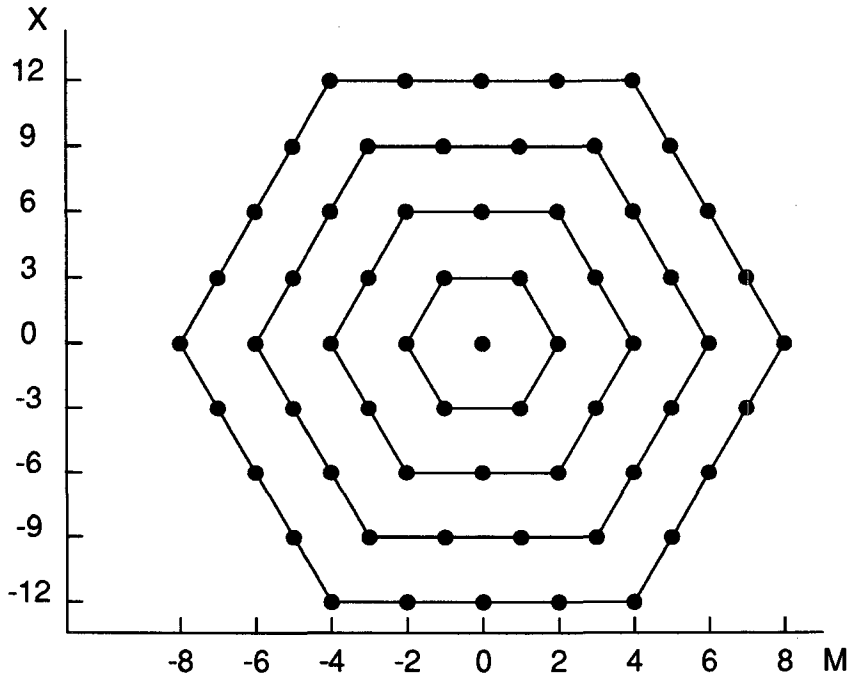


Figure 4.6: Weight diagram $A_2(4,4)$.
Dots indicate weights. The number of weights $W(4,4) = 125$, see (2.3).

We summarize the characteristics of the weight diagrams $A_2(\lambda_1, \lambda_1)$, λ_1-even, in Table 4.4.

Table 4.4: Characteristics of the weight diagrams $A_2(\lambda_1, \lambda_1)$, λ_1-even.

(λ_1, λ_1)	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(2,2)	2	3	1,2	$-4 \to 4$	$-6/3 \to 6/3$	27
(4,4)	4	5	1,2,3,4	$-8 \to 8$	$-12/3 \to 12/3$	125
(6,6)	6	7	1,2,3,4,5,6	$-12 \to 12$	$-18/3 \to 18/3$	343
(8,8)	8	9	1,2,3,4,5,6, 7,8	$-16 \to 16$	$-24/3 \to 24/3$	729
(λ_1, λ_1)	λ_1	$\lambda_1 + 1$	$1 \to \lambda_1$	$-2\lambda_1 \to 2\lambda_1$	$-\lambda_1 \to \lambda_1$	formula (2.3)

The last weight diagrams we draw are those with $\lambda_1 \neq \lambda_2$, $\lambda_2 > 0$, λ_1-even, λ_2-even.

4.1.3 Drawings of weight diagrams $A_2(\lambda_1, 2)$, λ_1-even

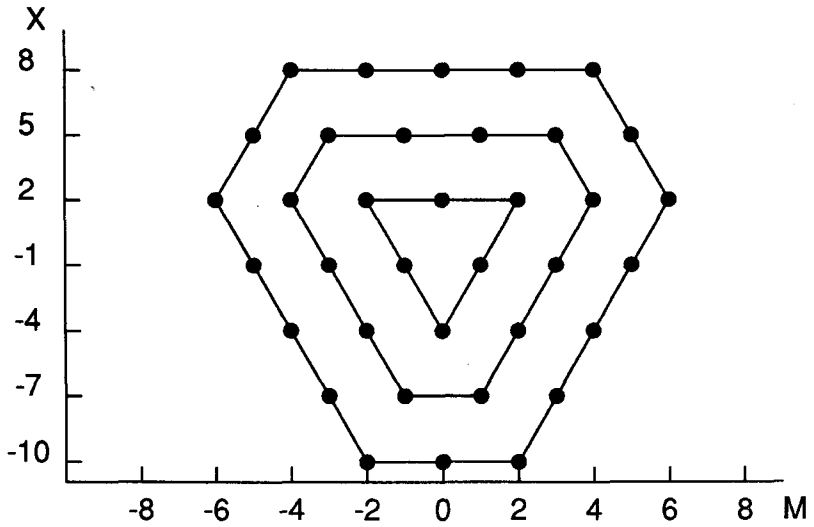


Figure 4.7: Weight diagram $A_2(4, 2)$.
Dots indicate weights. The number of weights $W(4, 2) = 60$, see (2.3).

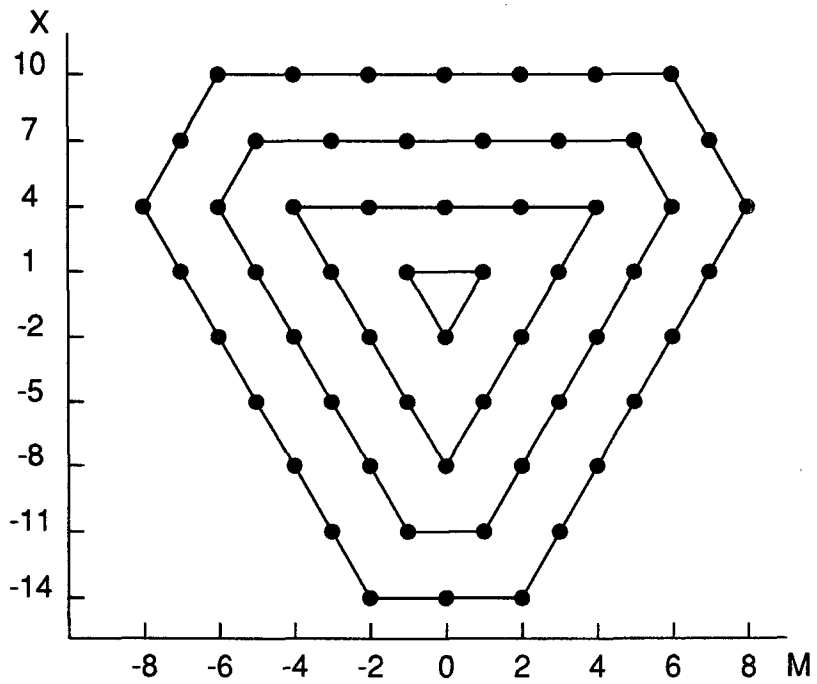


Figure 4.8: Weight diagram $A_2(6, 2)$.
Dots indicate weights. The number of weights $W(6, 2) = 105$, see (2.3).

We summarize the characteristics of the weight diagrams $A_2(\lambda_1, 2)$, λ_1-even, in Table 4.5.

Table 4.5: Characteristics of the weight diagrams $A_2(\lambda_1, 2)$, λ_1-even.

$(\lambda_1, 2)$	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(4,2)	3	0	1,2,3	$-6 \to 6$	$-10/3 \to 8/3$	60
(6,2)	4	0	1,2,3,3	$-8 \to 8$	$-14/3 \to 10/3$	105
(8,2)	4	3	1,2,3,3	$-10 \to 10$	$-18/3 \to 12/3$	162
(10,2)	5	0	1,2,3,3,3	$-12 \to 12$	$-22/3 \to 14/3$	231
(12,2)	6	0	1,2,3,3,3,3	$-14 \to 14$	$-26/3 \to 16/3$	312
(14,2)	6	3	1,2,3,3,3,3	$-16 \to 16$	$-30/3 \to 18/3$	405
$(\lambda_1, 2)$	r	c	1,2, 3 $(r-2)$-times	$-(\lambda_1 + 2) \to \lambda_1 + 2$	$-(2\lambda_1 + \lambda_2)/3 \to (\lambda_1 + 2\lambda_2)/3$	formula (2.3)

r = see Page 26,
$c = 3$ if $(\lambda_1 - 2)/3$ integer, 0 otherwise.

4.1.4 Drawings of weight diagrams $A_2(\lambda_1, 4)$, λ_1-even

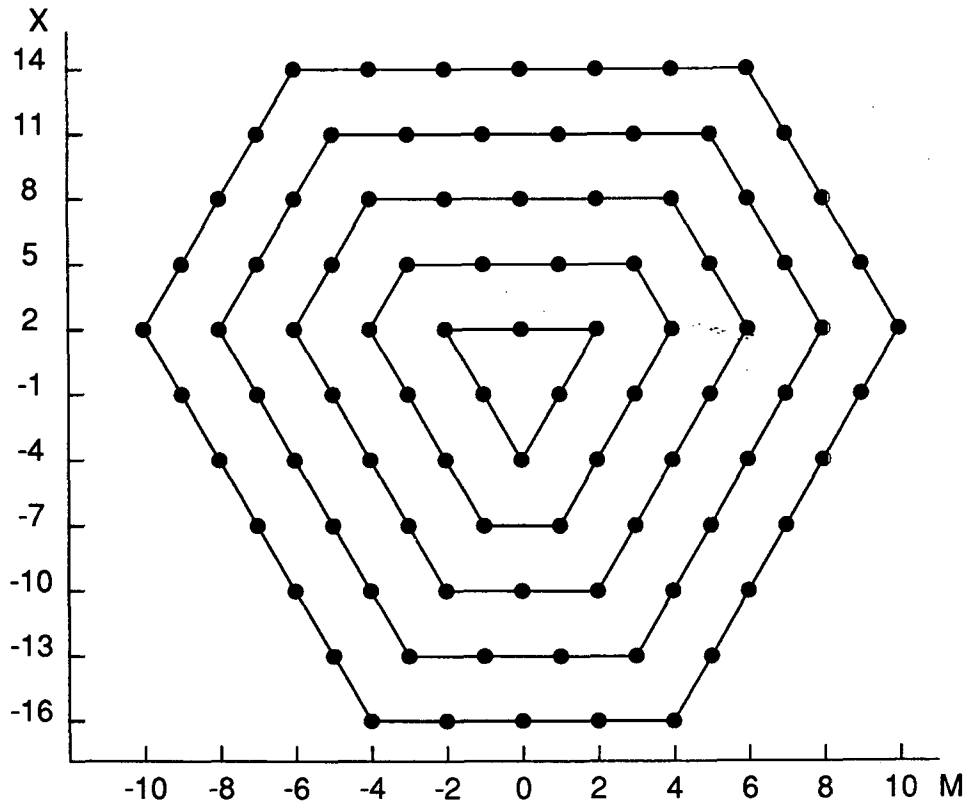


Figure 4.9: Weight diagram $A_2(6,4)$.
Dots indicate weights. The number of weights $W(6,4) = 162$, see (2.3).

The characteristics of the weight diagrams $A_2(\lambda_1, 4)$, λ_1-even, are summarized in Table 4.6.

Table 4.6: Characteristics of the weight diagrams $A_2(\lambda_1, 4)$, λ_1-even.

$(\lambda_1,4)$	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(6,4)	5	0	1,2,3,4,5	$-10 \to 10$	$-16/3 \to 14/3$	210
(8,4)	6	0	1,2,3,4,5,5	$-12 \to 12$	$-20/3 \to 16/3$	315
(10,4)	6	5	1,2,3,4,5,5	$-14 \to 14$	$-24/3 \to 18/3$	440
(12,4)	7	0	1,2,3,4,5,5,5	$-16 \to 16$	$-28/3 \to 20/3$	585
$(\lambda_1,4)$	r	c	1,2,3,4, 5 $(r-4)$-times	$-(\lambda_1+4)$ $\to \lambda_1+4$	$-(2\lambda_1+\lambda_2)/3$ $\to (\lambda_1+2\lambda_2)/3$	formula (2.3)

r = see Page 26,
$c = 5$ if $(\lambda_1 - 4)/3$ integer, 0 otherwise.

4.1.5 Drawings of weight diagrams $A_2(\lambda_1, 6)$, λ_1-even

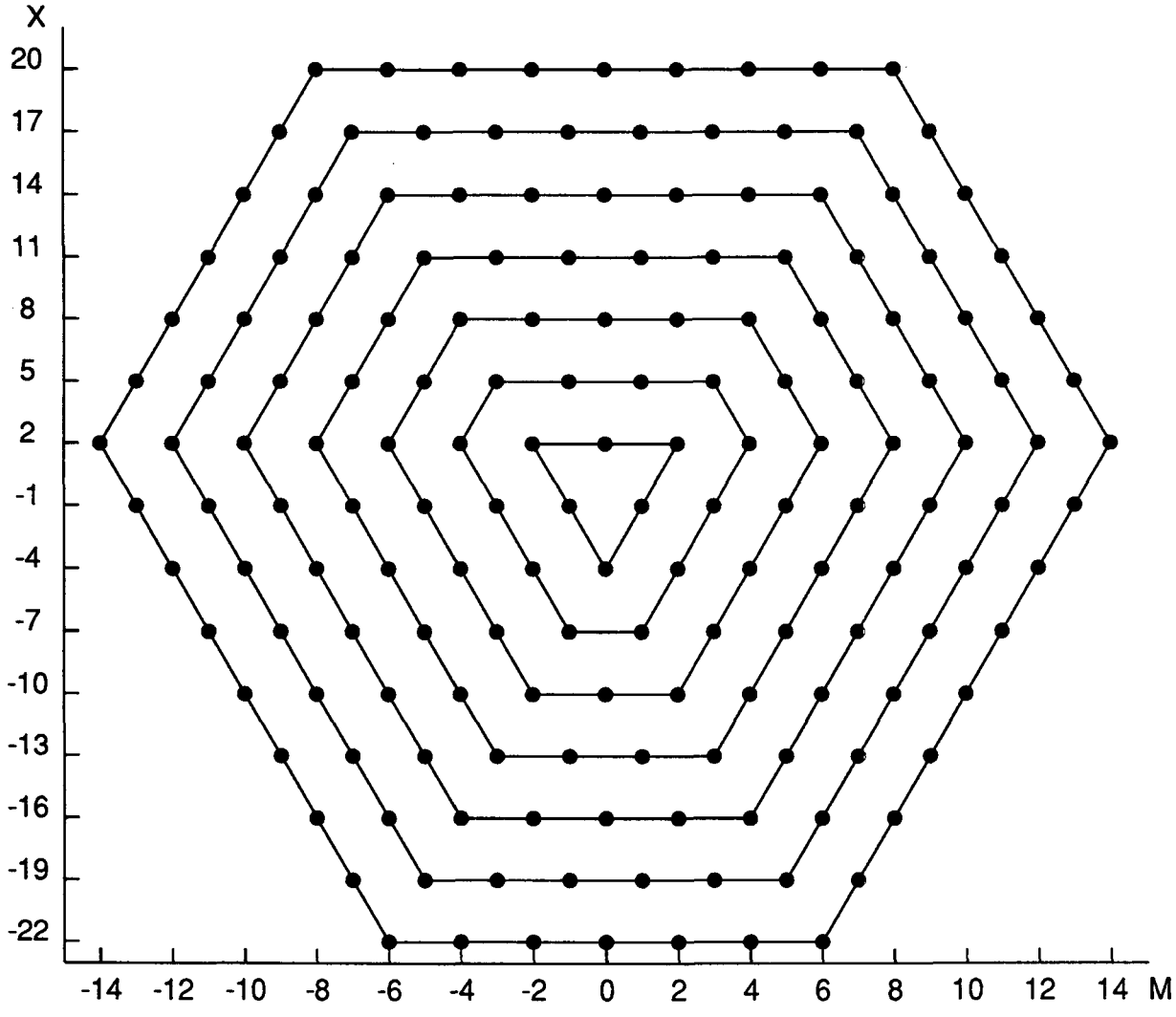


Figure 4.10: Weight diagram $A_2(8, 6)$.
Dots indicate weights. The number of weights $W(8, 6) = 504$, see (2.3).

The characteristics of the weight diagrams $A_2(\lambda_1, 6)$, λ_1-even, are summarized in Table 4.7.

Table 4.7: Characteristics of the weight diagrams $A_2(\lambda_1, 6)$, λ_1-even.

$(\lambda_1, 6)$	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(8,6)	7	0	1,2,3,4,5,6,7	-14 $\rightarrow$ 14	-22/3 $\rightarrow$ 20/3	504
(10,6)	8	0	1,2,3,4,5,6,7,7	-16 $\rightarrow$ 16	-26/3 $\rightarrow$ 22/3	693
(12,6)	8	7	1,2,3,4,5,6,7,7	-18 $\rightarrow$ 18	-30/3 $\rightarrow$ 24/3	910
(14,6)	9	0	1,2,3,4,5,6,7,7,7	-20 $\rightarrow$ 20	-34/3 $\rightarrow$ 26/3	1155
$(\lambda_1, 6)$	r	c	1,2,3,4,5,6, 7 $(r-6)$-times	$-(\lambda_1 + 6)$ $\rightarrow \lambda_1 + 6$	$-(2\lambda_1 + \lambda_2)/3$ $\rightarrow (\lambda_1 + 2\lambda_2)/3$	formula (2.3)

r = see Page 26,
c = 7 if $(\lambda_1 - 6)/3$ integer, 0 otherwise.

4.1.6 Drawings of weight diagrams $A_2(\lambda_1, 8)$, λ_1-even

The characteristics of the weight diagrams $A_2(\lambda_1, 8)$, λ_1-even, are summarized in Table 4.8.

Table 4.8: Characteristics of the weight diagrams $A_2(\lambda_1, 8)$, λ_1-even.

$(\lambda_1,8)$	Number of rings	Weights at centre	Multiplicity of rings (outer to inner)	Sequence of M-values $\Delta M = 1$	Sequence of X-values $\Delta X = 1$	Total number of weights
(10,8)	9	0	1,2,3,4,5,6,7,8,9	$-18 \to 18$	$-28/3 \to 26/3$	990
(12,8)	10	0	1,2,3,4,5,6,7,8,9,9	$-20 \to 20$	$-32/3 \to 28/3$	1287
(14,8)	10	9	1,2,3,4,5,6,7,8,9,9	$-22 \to 22$	$-36/3 \to 30/3$	1620
$(\lambda_1,8)$	r	c	1,2,3,4,5,6,7,8, 9 $(r-8)$-times	$-(\lambda_1+8)$ $\to \lambda_1+8$	$-(2\lambda_1+\lambda_2)/3$ $\to (\lambda_1+2\lambda_2)/3$	formula (2.3)

r = see Page 26,
$c = 9$ if $(\lambda_1 - 8)/3$ integer, 0 otherwise.

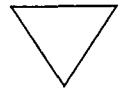
It can easily be verified that the weight diagram $A_2(\lambda_2, \lambda_1)$, $\lambda_1 \neq \lambda_2$, is generated by reflecting the weight diagram $A_2(\lambda_1, \lambda_2)$, on the M- and X-axis.

The weight diagrams enumerated in Table 4.2 in which we are interested can thus be classified into three categories:

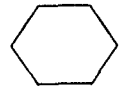
- the diagrams $A_2(\lambda_1, 0)$,
- the diagrams $A_2(\lambda_1, \lambda_2)$, $\lambda_1 = \lambda_2$,
- the diagrams $A_2(\lambda_1, \lambda_2)$, $\lambda_1 \neq \lambda_2 \neq 0$, $\lambda_1 > \lambda_2$.

Geometrically, they are characterized by the following forms:

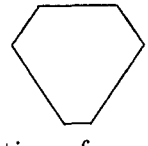
- $A_2(\lambda_1, 0)$



- $A_2(\lambda_1, \lambda_1)$



- $A_2(\lambda_1, \lambda_2)$, $\lambda_1 > \lambda_2$.



It remains still the problem fo find the characteristics of any weight diagram $A_2(\lambda_1, \lambda_2)$, λ_1-even, λ_2-even: the number of rings, the multiplicity of rings, the sequence of M-values and of X-values.

Recall that we have according to the Table 4.2 only to deal with weight diagrams $A_2(\lambda_1\text{-even}, \lambda_2\text{-even})$.

Analysing these weight diagrams we have found the following:

Theorem 4.2 *The number of rings r for $A_2(\lambda_1\text{-even}, \lambda_2\text{-even})$ is*

$$r = (\lambda_1 + \lambda_2)/2 - [(\lambda_1 - \lambda_2)/6] \tag{4.6}$$

where [ ] means that only the integer part is valid.

Theorem 4.3 *The multiplicity MW of weights on a ring (counted from outer to inner rings) is*

ring + centre	1	2	3	4	5	6	7	8	9	10	$\cdots$	$\lambda_1 + 1$
MW $\lambda_1 = \lambda_2$	1	2	3	4	5	6	7	8	9	10	$\cdots$	$\lambda_1 + 1$
$\lambda_1 \neq \lambda_2$	1	2	3	4	5	6	$\cdots$	$\lambda_2 + 1$			$\cdots$	$(r - \lambda_2)$ times

Theorem 4.4 *The M-coordinates for $A_2(\lambda_1\text{-even}, \lambda_2\text{-even})$ are given by the inequalites*

$$-(\lambda_1 + \lambda_2) \leq M \leq (\lambda_1 + \lambda_2), \textit{with } \Delta M = 1.$$

Theorem 4.5 *The X-coordinates for $A_2(\lambda_1\text{-even}, \lambda_2\text{-even})$ are given by the inequalities*

$$-(2\lambda_1 + \lambda_2)/3 \leq X \leq (\lambda_1 + 2\lambda_2)/3, \textit{with } \Delta X = 1. \tag{4.7}$$

The proof of these theorems can be made by mathematical induction.

We have still to assign the angular momentum L to the weights (M,X) of the diagrams $A_2(\lambda_1, \lambda_2)$. This is done in the following section.

4.2 L-assignment to (M,X)-coordinates of the weight diagrams $A_2(\lambda_1, \lambda_2)$, λ_1-even, λ_2-even

The L-assignment to (M,X) coordinates of the weights for the diagrams $A_2(\lambda_1, \lambda_2)$ is based on the analysis of the geometrical paths followed to reach the weight from the (0,0)-coordinates (a path is a sequence of $\lambda_1 + \lambda_2$ elementary paths, see Figures 2.2 and 2.3). The L-values can be assigned either directly (Case 1 below) or recursively (Cases 2 and 3 below) on the successive elements of the path.

The L-values should obey the following conditions:

- $|M| \leq L$;
- There exists no identical triplett L, M, X for a given $A_2(\lambda_1, \lambda_2)$ representation.

The ad-hoc procedure used for L-assignment is summarized as follows:
Case 1. If the path contains only the elementary paths 3 (λ_1-times) and $\overline{3}$ (λ_2-times):

- if λ_1 is odd, then $L = 1$,
- if λ_1 is even and
  - if λ_2 is odd, then $L = 1$,
  - if λ_2 is even, then $L = 0$.

Case 2. If the path contains λ_1-times the elementary path 3 in the first λ_1 positions, the L-recursion starts with $L = 0$ on the λ_2 remaining positions:

- if the next elementary path is $\overline{1}$ or $\overline{2}$, then $\Delta L = 1$,
- if the next elementary path is $\overline{3}$:
  - if the rank (counts of path $\overline{3}$ is odd, then $\Delta L = 1$,
  - if the rank is even, then $\Delta L = -1$.

Case 3. (General case) Starting from the second element of the path and with $L = 1$:

- if the elementary path is 1, 2, $\overline{1}$ or $\overline{2}$, $\Delta L = 1$,
- if the elementary path is 3:
  - if the rank is odd, $\Delta L = 1$,
  - if the rank is even, $\Delta L = -1$.
- if next elementary path is $\overline{3}$, $\Delta L = 0$.

Theorem 4.6 *The frequency $F(L)$ of the occurence of $L = \max|M|$ can be calculated by means of the relation*

$$F(L;\lambda_1,\lambda_2) = \sum_{i=0,1,2\cdots}^{\lambda_2/2} \Delta F(L;\lambda_1+2i,\lambda_2-2i), \tag{4.8}$$

$$\Delta F(L;\lambda_1,\lambda_2) = \begin{array}{l} 1 \text{ for } \lambda_2 \leq L \leq \lambda_1+1,\ \lambda_2 \neq 0, \\ 0 \text{ otherwise,} \end{array} \tag{4.9}$$

$$F(L;\lambda_1,0) = \begin{array}{l} 1 \text{ for L-even, } 0 \leq L \leq \lambda_1, \\ 0 \text{ otherwise,} \end{array} \tag{4.10}$$

$$F(L;\lambda_1,\lambda_2) = F(L;\lambda_2,\lambda_1). \tag{4.11}$$

The total number of quantum states belonging to a given representation $A_2(\lambda_1,\lambda_2)$ is expressed by the relation

$$W(A_2(\lambda_1,\lambda_2)) = \sum_L F(L;\lambda_1,\lambda_2).(2L+1) = (\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)/2. \tag{4.12}$$

We illustrate Theorem 4.6 by some examples.

Example 4.1 $\Delta F(L;\lambda_1,\lambda_2)$ *for* (λ_1,λ_2)*: (20,0), (18,2), (16,4), (14,6), (12,8), (10,10).*

L / λ_1,λ_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
(20, 0)	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
(18, 2)			1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
(16, 4)					1	1	1	1	1	1	1	1	1	1	1	1	1	1			
(14, 6)							1	1	1	1	1	1	1	1	1	1					
(12, 8)									1	1	1	1	1	1							
(10,10)											1	1									

According to (4.8) we obtain for $F(L;16,4)$: $F(L;16,4) = \Delta F(L;16,4)+\Delta F(L;18,2)+\Delta F(L;20,0)$. Therefore,

L	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
F(L;16,4)	1	0	2	1	3	2	3	2	3	2	3	2	3	2	3	2	3	2	2	1	1

For F(L;10,10) we get $F(L;10,10) = \Delta F(L;10,10)+\Delta F(L;12,8)+\Delta F(L;14,6)+\Delta F(L;16,4) + \Delta F(L;18,2) + \Delta F(L;20,0)$. Therefore,

L	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
F(L;10,10)	1	0	2	1	3	2	4	3	5	4	6	5	5	4	4	3	3	2	2	1	1

Example 4.2 $\Delta F(L;\lambda_1,\lambda_2)$ *for* (λ_1,λ_2)*: (12,0), (10,2), (8,4), (6,6).*

L / λ_1,λ_2	0	1	2	3	4	5	6	7	8	9	10	11	12
(12, 0)	1	0	1	0	1	0	1	0	1	0	1	0	1
(10, 2)			1	1	1	1	1	1	1	1	1	1	
(8, 4)					1	1	1	1	1	1			
(6, 6)							1	1					

Using (4.8) $F(L;8,4)$ reads: $F(L;8,4) = \Delta F(L;8,4)+\Delta F(L;10,2)+\Delta F(L;12,0)$ which give explicitely

L	0	1	2	3	4	5	6	7	8	9	10	11	12
F(L; 8, 4)	1	0	2	1	3	2	3	2	2	1	2	1	1

For $F(L;6,6)$ we obtain $F(L;6,6) = \Delta F(L;6,6) + \Delta F(L;8,4) + \Delta F(L;10,2) + \Delta F(L;12,0)$.

L	0	1	2	3	4	5	6	7	8	9	10	11	12
F(L; 6, 6)	1	0	2	1	3	2	4	3	3	2	2	1	1

For illustration we give in Table 4.9. explicit L-assignments for the $A_2(\lambda_1,\lambda_2)$-diagrams which are involved for the description of nuclei with $ZN = 7$: $A_2(14,0)$, $A_2(10,2)$, $A_2(6,4)$, $A_2(2,6)$, $A_2(8,0)$, $A_2(4,2)$, $A_2(0,4)$, $A_2(2,0)$.

Table 4.9: Values for M, X and L of the $A_2(\lambda_1, \lambda_2)$-diagrams involved in the description of nuclei with $ZN = 7$.

Values of M, X and L for the diagram $A_2(2,0)$.

M	X	L	M	X	L	M	X	L	M	X	L	M	X	L	M	X	L
-2	2	2	-1	-1	2	0	-4	0	0	2	2	1	-1	2	2	2	2

Values of M, X and L for the diagram $A_2(4,0)$.

M	X	L	M	X	L	M	X	L	M	X	L	M	X	L
-4	4	4	-2	4	4	0	-8	0	1	-5	2	2	4	4
-3	1	4	-1	-5	2	0	-2	2	1	1	4	3	1	4
-2	-2	2	-1	1	4	0	4	4	2	-2	2	4	4	4

Values of M, X and L for the diagram $A_2(8,0)$.

M	X	L	M	X	L	M	X	L	M	X	L	M	X	L	M	X	L
-8	8	8	-4	8	8	-1	-13	2	0	8	8	2	8	8	5	5	8
-7	5	8	-3	-7	4	-1	-7	4	1	-13	2	3	-7	4	6	2	6
-6	2	6	-3	-1	6	-1	-1	6	1	-7	4	3	-1	6	6	8	8
-6	8	8	-3	5	8	-1	5	8	1	-1	6	3	5	8	7	5	8
-5	-1	6	-2	-10	2	0	-16	0	1	5	8	4	-4	4	8	8	8
-5	5	8	-2	-4	4	0	-10	2	2	-10	2	4	2	6			
-4	-4	4	-2	2	6	0	-4	4	2	-4	4	4	8	8			
-4	2	6	-2	8	8	0	2	6	2	2	6	5	-1	6			

Values of M, X and L for the diagram $A_2(14,0)$.

M	X	L	M	X	L	M	X	L	M	X	L	M	X	L	M	X	L
-14	14	14	-6	-10	6	-3	5	12	0	-4	8	3	-7	8	7	-7	8
-13	11	14	-6	-4	8	-3	11	14	0	2	10	3	-1	10	7	-1	10
-12	8	12	-6	2	10	-2	-22	2	0	8	12	3	5	12	7	5	12
-12	14	14	-6	8	12	-2	-16	4	0	14	14	3	11	14	7	11	14
-11	5	12	-6	14	14	-2	-10	6	1	-25	2	4	-16	4	8	-4	8
-11	11	14	-5	-13	6	-2	-4	8	1	-19	4	4	-10	6	8	2	10
-10	2	10	-5	-7	8	-2	2	10	1	-13	6	4	-4	8	8	8	12
-10	8	12	-5	-1	10	-2	8	12	1	-7	8	4	2	10	8	14	14
-10	14	14	-5	5	12	-2	14	14	1	-1	10	4	8	12	9	-1	10
-9	-1	10	-5	11	14	-1	-25	2	1	5	12	4	14	14	9	5	12
-9	5	12	-4	-16	4	-1	-19	4	1	11	14	5	-13	6	9	11	14
-9	11	14	-4	-10	6	-1	-13	6	2	-22	2	5	-7	8	10	2	10
-8	-4	8	-4	-4	8	-1	-7	8	2	-16	4	5	-1	10	10	8	12
-8	2	10	-4	2	10	-1	-1	10	2	-10	6	5	5	12	10	14	14
-8	8	12	-4	8	12	-1	5	12	2	-4	8	5	11	14	11	5	12
-8	14	14	-4	14	14	-1	11	14	2	2	10	6	-10	6	11	11	14
-7	-7	8	-3	-19	4	0	-28	0	2	8	12	6	-4	8	12	8	12
-7	-1	10	-3	-13	6	0	-22	2	2	14	14	6	2	10	12	14	14
-7	5	12	-3	-7	8	0	-16	4	3	-19	4	6	8	12	13	11	14
-7	11	14	-3	-1	10	0	-10	6	3	-13	6	6	14	14	14	14	14

Values of M, X and L for the diagram $A_2(4,2)$.

M	X	L			M	X	L			M	X	L			M	X	L		
-6	2	6			-2	-10	2			0	2	2	5	6	3	-7	4		
-5	-1	6			-2	-4	3	4		0	8	4			3	-1	3	6	
-5	5	5			-2	2	2	5	6	1	-7	2	4		3	5	4	5	
-4	-4	4			-2	8	4			1	-1	2	3	6	4	-4	4		
-4	2	5	6		-1	-7	2	4		1	5	4	5		4	2	5	6	
-4	8	4			-1	-1	2	3	6	2	-10	2			4	8	4		
-3	-7	4			-1	5	4	5		2	-4	3	4		5	-1	6		
-3	-1	3	6		0	-10	2			2	2	2	5	6	5	5	5		
-3	5	4	5		0	-4	0	3	4	2	8	4			6	2	6		

Values of M, X and L for the diagram $A_2(6,2)$.

M	X	L			M	X	L			M	X	L			M	X	L		
-8	4	8			-3	1	4	5	8	0	10	6			4	-2	5	6	
-7	1	8			-3	7	6	7		1	-11	2	4		4	4	4	7	8
-7	7	7			-2	-14	2			1	-5	2	3	6	4	10	6		
-6	-2	6			-2	-8	3	4		1	1	4	5	8	5	-5	6		
-6	4	7	8		-2	-2	2	5	6	1	7	6	7		5	1	5	8	
-6	10	6			-2	4	4	7	8	2	-14	2			5	7	6	7	
-5	-5	6			-2	10	6			2	-8	3	4		6	-2	6		
-5	1	5	8		-1	-11	2	4		2	-2	2	5	6	6	4	7	8	
-5	7	6	7		-1	-5	2	3	6	2	4	4	7	8	6	10	6		
-4	-8	4			-1	1	4	5	8	2	10	6			7	1	8		
-4	-2	5	6		-1	7	6	7		3	-11	4			7	7	7		
-4	4	4	7	8	0	-14	2			3	-5	3	6		8	4	8		
-4	10	6			0	-8	0	3	4	3	1	4	5	8					
-3	-11	4			0	-2	2	5	6	3	7	6	7						
-3	-5	3	6		0	4	4	7	8	4	-8	4							

Values of M, X and L for the diagram $A_2(10,2)$.

M	X	L			M	X	L			M	X	L			M	X	L		
-12	8	12			-4	-10	5	6		0	2	6	9	10	5	-13	6		
-11	5	12			-4	-4	4	7	8	0	8	8	11	12	5	-7	5	8	
-11	11	11			-4	2	6	9	10	0	14	10			5	-1	6	7	10
-10	2	10			-4	8	8	11	12	1	-19	2	4		5	5	8	9	12
-10	8	11	12		-4	14	10			1	-13	2	3	6	5	11	10	11	
-10	14	10			-3	-19	4			1	-7	4	5	8	6	-10	6		
-9	-1	10			-3	-13	3	6		1	-1	6	7	10	6	-4	7	8	
-9	5	9	12		-3	-7	4	5	8	1	5	8	9	12	6	2	6	9	10
-9	11	10	11		-3	-1	6	7	10	1	11	10	11		6	8	8	11	12
-8	-4	8			-3	5	8	9	12	2	-22	2			6	14	10		
-8	2	9	10		-3	11	10	11		2	-16	3	4		7	-7	8		
-8	8	8	11	12	-2	-22	2			2	-10	2	5	6	7	-1	7	10	
-8	14	10			-2	-16	3	4		2	-4	4	7	8	7	5	8	9	12
-7	-7	8			-2	-10	2	5	6	2	2	6	9	10	7	11	10	11	
-7	-1	7	10		-2	-4	4	7	8	2	8	8	11	12	8	-4	8		
-7	5	8	9	12	-2	2	6	9	10	2	14	10			8	2	9	10	
-7	11	10	11		-2	8	8	11	12	3	-19	4			8	8	8	11	12
-6	-10	6			-2	14	10			3	-13	3	6		8	14	10		
-6	-4	7	8		-1	-19	2	4		3	-7	4	5	8	9	-1	10		
-6	2	6	9	10	-1	-13	2	3	6	3	-1	6	7	10	9	5	9	12	
-6	8	8	11	12	-1	-7	4	5	8	3	5	8	9	12	9	11	10	11	
-6	14	10			-1	-1	6	7	10	3	11	10	11		10	2	10		
-5	-13	6			-1	5	8	9	12	4	-16	4			10	8	11	12	
-5	-7	5	8		-1	11	10	11		4	-10	5	6		10	14	10		
-5	-1	6	7	10	0	-22	2			4	-4	4	7	8	11	5	12		
-5	5	8	9	12	0	-16	0	3	4	4	2	6	9	10	11	11	11		
-5	11	10	11		0	-10	2	5	6	4	8	8	11	12	12	8	12		
-4	-16	4			0	-4	4	7	8	4	14	10							

Values of M, X and L for the diagram $A_2(6,4)$.

M	X	L				M	X	L					M	X	L			
-10	2	10				-3	11	6	7				3	-7	4	5	8	
-9	-1	10				-2	-16	4					3	-1	3	6	7	10
-9	5	9				-2	-10	2	5	6			3	5	4	5	8	9
-8	-4	8				-2	-4	3	4	7	8		3	11	6	7		
-8	2	9	10			-2	2	2	5	6	9	10	4	-16	4			
-8	8	8				-2	8	4	7	8			4	-10	5	6		
-7	-7	8				-2	14	6					4	-4	4	7	8	
-7	-1	7	10			-1	-13	4	6				4	2	5	6	9	10
-7	5	8	9			-1	-7	2	4	5	8		4	8	4	7	8	
-7	11	7				-1	-1	2	3	6	7	10	4	14	6			
-6	-10	6				-1	5	4	5	8	9		5	-13	6			
-6	-4	7	8			-1	11	6	7				5	-7	5	8		
-6	2	6	9	10		0	-16	4					5	-1	6	7	10	
-6	8	7	8			0	-10	2	5	6			5	5	5	8	9	
-6	14	6				0	-4	0	3	4	7	8	5	11	6	7		
-5	-13	6				0	2	2	5	6	9	10	6	-10	6			
-5	-7	5	8			0	8	4	7	8			6	-4	7	8		
-5	-1	6	7	10		0	14	6					6	2	6	9	10	
-5	5	5	8	9		1	-13	4	6				6	8	7	8		
-5	11	6	7			1	-7	2	4	5	8		6	14	6			
-4	-16	4				1	-1	2	3	6	7	10	7	-7	8			
-4	-10	5	6			1	5	4	5	8	9		7	-1	7	10		
-4	-4	4	7	8		1	11	6	7				7	5	8	9		
-4	2	5	6	9	10	2	-16	4					7	11	7			
-4	8	4	7	8		2	-10	2	5	6			8	-4	8			
-4	14	6				2	-4	3	4	7	8		8	2	9	10		
-3	-13	4	6			2	2	2	5	6	9	10	8	8	8			
-3	-7	4	5	8		2	8	4	7	8			9	-1	10			
-3	-1	3	6	7	10	2	14	6					9	5	9			
-3	5	4	5	8	9	3	-13	4	6				10	2	10			

4.3 Classification of nuclear rotational states

Recall that we are assigning nuclear rotational states to weight vectors. The coordinates of a weight vector are interpreted as quantum numbers of a state. Any state is, therefore, within this description defined by the weight diagram $A_2(\lambda_1, \lambda_2)$, i.e. by the values λ_1, λ_2 and the coordinates of the weight vector.

We illustrate that by classifying nuclear rotational states of nuclei with $ZN = 4$. From Table 3.2 follows that Po-216 and Rn-216 have such a ZN-value. The total number of rotational states is given by (2.4), $W(4,0,0,0,0) = 126$. Using (2.2) we obtain $A_5(4,0,0,0,0) = A_2(8,0) + A_2(4,2) + A_2(0,4) + A_2(2,0)$ (see also Table 4.1).

These weight diagrams are already drawn in sections 4.1.1 and 4.1.3. The corresponding M, X, L-values are given in Table 4.9.

A nuclear rotationnal state is thus in our classification specified by the quantum numbers M, X and L and the values λ_1, λ_2 of the representation $A_2(\lambda_1, \lambda_2)$.

These quantum states are for nuclei with $ZN = 4$ explicitely given in Table 4.9. From this table directly follows that the classification of rotational levels by means of weight diagrams of $A_2(\lambda_1, \lambda_2)$ yields much more levels than the conventional classification according to the angular momentum L only. This is illustrated in Table 4.10.

Table 4.10: Number of rotational states of various nuclei.
W is the number of rotational states, with unequal tripletts (L, M, X) resulting from our $A_2(\lambda_1, \lambda_2)$-classification (see (2.4)). W_0 is the number of rotational states, with unequal L-values, in the usual classification [AR78] (see (5.2)).

ZN	Element	Z	N	W	W_0
4	Rn	84	132	126	16
6	Th	90	130	462	38
8	Nd	60	88	1287	75
10	Sm	62	90	3003	131
12	Th	90	142	6188	209
14	U	92	144	11628	316
16	Pu	94	146	20349	453
18	Er	68	100	33649	624
20	Yb	70	102	53130	836
22	Hf	72	104	80730	1090
24	W	74	106	118755	1390
26	Os	76	108	169911	1743
28	Pt	78	110	237336	2150

Chapter 5

Calculation of nuclear rotational level energies

In the preceeding Section we have classified nuclear rotational states by means of weight diagrams $A_2(\lambda_1, \lambda_2)$. There, a nuclear state is described by the coordinates of a weight vector. As these are vectors in a two dimensional space, they are defined by two coordinates. A nuclear rotational state is thus within this description characterized by the coordinates (quantum numbers) M and X and the values λ_1, λ_2 which in turn define the form of the weight diagram to which a state (weight vector) belongs.

The energy value E of a state of a nucleus can be considered as a quantity depending on numbers which define a nuclear state in the present model, i.e. on L, M, X, λ_1, λ_2 and on Z and N. Thus we have

$$E = E(Z, N; L, M, X, \lambda_1, \lambda_2). \tag{5.1}$$

A formula for E depending on Z, N; L, λ_1, λ_2 but not on M and X is already given in [AR78]. It reads

$$E(Z, N; L, \lambda_1, \lambda_2) = \alpha(Z, N).L(L+1) - \beta(Z, N).(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + 3(\lambda_1 + \lambda_2)) \tag{5.2}$$

where α and β remain constant for one and the same nucleus. They are determined from well-known energy levels. This formula yields energy levels which are degenerate in M and X, i.e. energy levels with constant L but varying M and X are not resolved.

In order to take into account all quantum states which follow from our classification using the Lie algebra A_2, we propose a more general formula for the nuclear level energy. To establish this formula we add to (5.2) terms depending on M and X, respectively. The formula for the level energies becomes then

$$\begin{aligned} &E(Z, N; L, M, X, \lambda_1, \lambda_2) = \\ &\quad \alpha(Z, N).L(L+1) - \beta(Z, N).(\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + 3(\lambda_1 + \lambda_2)) + \\ &\quad \gamma(Z, N).M(M+1) + \delta(Z, N).X(X+1) \end{aligned} \tag{5.3}$$

The constants α, β, γ and δ have to be determined from well-known level energies.

As these constants are different for different nuclei, the level energies for nuclei with the same ZN are also different in spite of having the same quantum numbers and the same values λ_1, λ_2.

From the mathematical point of view, formula (5.3) is a sort of series expansion of E(Z,N;L,M,X,λ_1,λ_2) in powers of L, M, X, λ_1, λ_2 where only up to quadratic terms are taken into account.

We now apply (5.2) and (5.3) to compute energy levels of ^{170}Er and compare them with the available experimental values.

Using (5.2) and the well-known levels $E_1(L=2) = 79keV$, $E_2(L=0) = 891keV$ one obtains

$$\alpha = E_1(L=2)/6 = 12.50, \tag{5.4}$$

$$\beta = \frac{E_2(L=0)}{6(2ZN-1)} = 4.01. \tag{5.5}$$

One cannot determine the values for γ and δ which appear in (5.3), because no levels energies depending on M, X are known. We choose, therefore, the trial values: $\gamma = 0.1$, $\delta = 0.1$.

The corresponding results are given in Table 5.1 and illustrated in Figure 5.1.

Table 5.1: Energy levels of ^{170}Er calculated with (5.3) and (5.2). $Z = 68$, $N = 102$, $ZN = 19$, $\alpha = 12.50$, $\beta = 4.01$, $\gamma = 0.1$, $\delta = 0.1$. Total number of levels computed with (5.3) = 42504. Total number of levels computed with (5.2) = 725. The higher level energies are not printed, as the listing would become too bulky.

E(5.3) keV	L	M	X	(λ_1, λ_2)	E(5.2) keV	E_{exp} [ZH87] keV
0.0	0	0	-76	(38,0)	0.0	0.0
65.47	2	0	-70	(38,0)		
65.67	2	-2	-70	(38,0)		
66.07	2	2	-70	(38,0)	75.00	78.68
70.13	2	-1	-73	(38,0)		
70.33	2	1	-73	(38,0)		
231.73	4	0	-64	(38,0)		
231.93	4	-2	-64	(38,0)		
232.33	4	2	-64	(38,0)		
232.93	4	-4	-64	(38,0)		
233.73	4	4	-64	(38,0)	250.00	260.18
236.00	4	-1	-67	(38,0)		
236.20	4	1	-67	(38,0)		
236.60	4	-3	-67	(38,0)		
237.20	4	3	-67	(38,0)		
498.80	6	0	-58	(38,0)		
499.00	6	-2	-58	(38,0)		
499.40	6	2	-58	(38,0)		
500.00	6	-4	-58	(38,0)		
500.80	6	4	-58	(38,0)		
501.80	6	-6	-58	(38,0)		
502.67	6	-1	-61	(38,0)	525.00	540.8
502.87	6	1	-61	(38,0)		
503.00	6	6	-61	(38,0)		
503.27	6	-3	-61	(38,0)		
503.87	6	3	-61	(38,0)		
504.67	6	-5	-61	(38,0)		
505.67	6	5	-61	(38,0)		
872.73	0	0	-64	(34,2)	891.00	891.0
939.80	2	0	-58	(34,2)		
940.00	2	-2	-58	(34,2)		
940.40	2	2	-58	(34,2)	966.00	934.03
943.67	2	-1	-61	(34,2)		
943.87	2	1	-61	(34,2)		
952.00	2	-1	-67	(34,2)		
952.20	2	1	-67	(34,2)		
956.47	2	0	-70	(34,2)	966.00	959.85
956.67	2	-2	-70	(34,2)		
957.07	2	2	-70	(34,2)		
1018.67	3	-1	-61	(34,2)		
1018.87	3	1	-61	(34,2)		
1019.27	3	-3	-61	(34,2)		
1019.87	3	3	-61	(34,2)	1041.00	1010.37
1022.73	3	0	-64	(34,2)		
1022.93	3	-2	-64	(34,2)		
1023.33	3	2	-64	(34,2)		

12^+	1916	1911						
						7^+	1557	1558
10^+	1374	1341	6^+	1402	1379	6^+	1413	1399
						5^+	1236	1239
			4^+	1127	1125	4^+	1103	1110
						3^+	1010	1031
			2^+	960	955	2^+	934	942
8^+	912	871	0^+	891	873		Exp	Calc
				Exp	Calc			
6^+	541	502						
4^+	260	234						
2^+	79	68						
0^+	0	0						
	Exp	Calc						

Figure 5.1: Energy levels of ^{170}Er.

Annex A

Explanation of the logogram: Matter $\leftrightarrow$ Mathematics

Hexagons:
- The points symbolize the twelve basic elements of matter: the six points of the inner hexagon symbolize the six quarks (up, down, strange, charm, bottom, top); the six points of the outer hexagon symbolize the six leptons (electron, neutrino of type electron, muon, neutrino of type muon, tau, neutrino of type tau).
- The two rings symbolize the two elementary forces: the nuclear force (inner ring), the electromagnetic and weak forces (outer ring).

Triangle:
- The three points symbolize three essential items described by mathematics:
  1. Kinds (or states) of matter.
  2. Assembly (or formation) of matter.
  3. Dynamic of matter.
- The ring symbolizes the third fundamental force, the gravitational force.

Centre: Symbolizes the origin of everything, the fundamental law.

If one uses for the mathematical description of the three items which are symbolized by the points of the triangle the Lie algebra A_5 then it follows:

1. There exist two basic kinds of matter corresponding to the representation $A_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$, λ_1: matter; λ_5: anti-matter; $\lambda_2, \lambda_3, \lambda_4$: para-matter (matter, anti-matter or a mixture of both).

2. The assembly (or formation) of general matter (matter, anti-matter) is described by the weights of the Lie-algebra A_5:

$$W_i(A_5(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)) = \prod_{r=1}^{6} r^{\alpha_{ri}} \overline{r}^{\beta_{ri}},$$

r indicates the fundamental weight r of $A_5(1,0,0,0,0)$; a quark r is assigned to each weight r.

$\overline{r}$ indicates the fundamental weight $\overline{r}$ of $A_5(0,0,0,0,1)$; an anti-quark $\overline{r}$ is assigned to each weight $\overline{r}$.

α_{ri} denotes the number of times the weight r is applied, i.e. the number of quarks of type r involved.

β_{ri} denotes the number of times the weight $\overline{r}$ is applied, i.e. the number of anti-quarks of type $\overline{r}$ involved.

$\sum_{r=1}^{6} \alpha_{ri} + \beta_{ri}$, $i = 1, 2, ...I$, is constant for a well-defined entity, for example a nucleus. I is the total number of weights (quantum states) of the nuclear entity.

Consequently, a multiple of six quarks and six anti-quarks can be involved in the formation of a nuclear quantum state. Such a state can be real or virtual. A nucleus has one real state and many virtuals states, i.e. energy levels. The real state of a nucleus is defined by the number of protons and neutrons of the nucleus. Its virtuals states (or levels) can be activated by energy input.

Example: A nucleus with m protons and n neutrons consists of $(2m+n)$ quarks up $(r = 1)$ and $(m + 2n)$ quarks down $(r = 2)$. This is its real state:

$$W_1(A_5) = 1^{2m+n}\, 2^{m+2n}\, 3^0\, 4^0\, 5^0\, 6^0.$$

Its virtual states or energy levels (weights) are

$$W_i(A_5) = 1^{\alpha_{1i}}\, 2^{\alpha_{2i}}\, 3^{\alpha_{3i}}\, 4^{\alpha_{4i}}\, 5^{\alpha_{5i}}\, 6^{\alpha_{6i}}$$

with

$$\sum_{r=1}^{6} \alpha_{ri} = 3(m+n), i = 2, 3, ..., I.$$

Deuteron has for example the real state

$$W_1(A_5) = 1^3\, 2^3\, 3^0\, 4^0\, 5^0\, 6^0$$

and the virtual states or energy levels (weights)

$$W_i(A_5) = 1^{\alpha_{1i}}\, 2^{\alpha_{2i}}\, 3^{\alpha_{3i}}\, 4^{\alpha_{4i}}\, 5^{\alpha_{5i}}\, 6^{\alpha_{6i}}$$

with

$$\sum_{r=1}^{6} \alpha_{ri} = 6, i = 2, 3, ..., I.$$

The total number I of possible states $= 46656 = 6^6$.

3. We assign to each weight r a function Y_r and to each weight $\overline{r}$ a function $\overline{Y_r}$. Physically these functions can somehow be interpreted as a measure of the communication flow (dynamic) between the quarks. A certain dynamic of matter can then be described by the system of differential equations:

$$\frac{dY_s(t)}{dt} = \sum_{i=1}^{I} b_{si} \prod_{r=1}^{6} Y_r^{\alpha_{ri}}\, \overline{Y_r}^{\beta_{ri}},$$

$$\frac{d\overline{Y_s}(t)}{dt} = \sum_{i=1}^{I} b_{si} \prod_{r=1}^{6} Y_r^{\alpha_{ri}} \, \overline{Y_r}^{\beta_{ri}},$$

$s = 1, ..., 6$. I is the total number of quantum states (weights) of a nuclear entity considered, for example a nucleus. b_{si} are coupling constants.

The solution of such systems can formally be represented as follows

$$Y_s(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (C.I.P)_{sn},$$

C are mathematical constants, I are interaction forces (coupling constants), $P = Y(t = t_0)$.

For further details of this representation of Y_s and for the numerical procedure see SCHETT et al. [SC85].

A more complete study of quantum states of nuclei using Lie algebras A_n would thus include all cases which are represented in the following table.

Table A.1: Outlook on a more comprehensive study of quantum states of nuclei using Lie algebras A_n.

Mathematical method: Lie algebra	A_2	A_3	A_4	A_5
Nuclear concept	neutron pairs or proton pairs (bosonlike)			Quark-strings (hadrons)
Description	Approximate model for rotational states of nuclei	Approximate model for states of nuclei towards the end of major closed shells	Approximate model for vibrational states of nuclei	Concept for quark-string (hadron) states of nuclei
Comment	Present report	see Table 2.1 of present report		see this Annex

Recall that the rotational states of nuclei which are presented in this report are a mathematical consequence of the use of the Lie Algebra A_2.

The quark-string states of nuclei are in a similar way a mathematical consequence of the use of the Lie algebra A_5 to describe quantum states of nuclei, i.e. the construction of neutrons and protons and in general hadrons by quarks.

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