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Donald L. Smith
Argonne National Laboratory (retired)
Coronado, CA 92118, U.S.A.

Denise Neudecker
Los Alamos National Laboratory
Los Alamos, NM 87545, U.S.A.

Roberto Capote Noy
Nuclear Data Section, International Atomic Energy Agency
A-1400 Vienna, Austria

May 2020

IAEA Nuclear Data Section
Vienna International Centre, P.O. Box 100, 1400 Vienna, Austria

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Austria

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Donald L. Smith
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Denise Neudecker
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Roberto Capote-Noy
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ABSTRACT

In two previous investigations that are documented in this IAEA report series, we examined the effects of non-Gaussian, non-symmetric probability density functions (PDFs) on the outcomes of data evaluations. Most of this earlier work involved considering just two independent input data values and their respective uncertainties. They were used to generate one evaluated data point. The input data are referred to, respectively, as the mean value and standard deviation pair (y_0,s_0) for a prior PDF $p_0(y)$ and a second mean value and standard deviation pair (y_e,s_e) for a likelihood PDF $p_e(y)$. Conceptually, these input data could be viewed as resulting from theory (subscript "0") and experiment (subscript "e"). In accordance with Bayes Theorem, the evaluated mean value and standard deviation pair (y_{sol},s_{sol}) corresponds to the posterior PDF $p(y)$ which is related to $p_0(y)$ and $p_e(y)$ by $p(y) = Cp_0(y)p_e(y)$. The prior and likelihood PDFs are both assumed to be normalized so that they integrate to unity for all $y \geq 0$. Negative values of y are viewed as non-physical so they are not permitted. The product function $p_0(y)p_e(y)$ is not normalized, so a positive multiplicative constant C is required to normalize $p(y)$. In the earlier work, both normal (Gaussian) and lognormal functions were considered for the prior PDF. The likelihood functions were all taken to be Gaussians. Gaussians are symmetric, with zero skewness, and they always possess a fixed kurtosis of 3. Lognormal functions are inherently skewed, with widely varying values of skewness and kurtosis that depend on the function parameters. In order to explore the effects of kurtosis, distinct from skewness, the present work constrains the likelihood function to be Gaussian, and it considers three distinct, inherently symmetric prior PDF types: Gaussian (kurtosis = 3), Continuous Uniform (kurtosis = 1.8), and Laplace (kurtosis = 6). A product of two Gaussians produces a Gaussian even if $y_e \neq y_0$. The product of a Gaussian PDF and a Uniform PDF, or a Laplace PDF, yields a symmetric PDF with zero skewness only when $y_e = y_0$. A pure test of the effect of kurtosis on an evaluation is provided by considering combinations of s_0 and s_e with $y_e = y_0$. The present work also investigates the extent to which $p(y)$ exhibits skewness when $y_e \neq y_0$, again by considering various values for s_0 and s_e. The Bayesian results from numerous numerical examples have been compared with corresponding least-squares solutions in order to arrive at some general conclusions regarding how the evaluated result (y_{sol},s_{sol}) depends on various combinations of the input data y_0, s_0, y_e, and s_e as well as on prior-likelihood PDF combinations: Gaussian-Gaussian, Uniform-Gaussian, and Laplace-Gaussian.

May 2020

Contents

1. Introduction

As mentioned in the Abstract, our two earlier investigations examined the effects of non-Gaussian (non-normal), asymmetric input-data probability density functions (PDFs) on data-evaluation outcomes [1,2]. The statistical methods that were used are described in the indicated references. For simplicity, portions of this earlier work involved considering just two independent input-data values and their respective uncertainties, both corresponding to a single random variable y. These data were employed to generate one evaluated data point. The input data are referred to, respectively, as a prior value for y and its standard deviation, i.e., the pair (y_0,s_0), and an additional value for y and its standard deviation, i.e., the pair (y_e,s_e). Conceptually, these two input data values could be viewed as generated by theory (subscript "0") and experiment (subscript "e"), respectively, since this is representative of scenarios encountered frequently by nuclear data evaluators. The evaluation procedure discussed in the earlier work, as well as in the present investigation, has been conducted in the random-variable space S(y) associated with these two "observable" values for y, as opposed to performing the analysis in a variable space S(x) of theoretical parameters x used to calculate values of y through the mapping of S(x) to S(y) by some theoretical model function "g", i.e., y = g(x).

In 2007, Smith suggested a data evaluation method that has come to be known by the nuclear data community as Unified Monte Carlo (UMC) [3]. More recently, variations of the original UMC technique have been suggested by various investigators, so the original approach from Smith is now denoted by UMC-G. The extension "G" appears for historical reasons. "Unified" in UMC indicates that both theoretical and experimental data are taken into consideration in a unified manner by this procedure. The term "Monte Carlo" stems from the fact that Smith suggested the use of Monte Carlo (stochastic) simulation techniques to calculate the often complicated integrals associated with realistic applications of the UMC method. The method used in the present work is most closely related to UMC-G; however, it does not employ Monte Carlo in the calculations. In very simple evaluation exercises, such as those examined in the present work, the integrals involved can be calculated readily by deterministic numerical methods rather than by resorting to Monte Carlo analyses. The method used for computation of the integrals does not alter the basic concept that underpins this evaluation method.

The approach introduced by Smith [3], and used in our earlier work [1,2], as well as in the present investigation, is based on Bayes Theorem [4,5]. The evaluated mean value and standard deviation pair (y_{sol},s_{sol}) corresponds to the posterior PDF p(y). This PDF is related to p_0(y) and p_e(y) by $p(y) = Cp_0(y)p_e(y)$. The prior and likelihood PDFs are both assumed to be normalized so that they integrate to unity over the entire range of allowed values for y. Negative values of y are considered to be non-physical, so $y \geq 0$ is assumed throughout in this work. In situations where specific PDF types might yield non-negligible probability for negative y, e.g., Gaussians with standard deviations exceeding ≈ 30%, it is essential to avoid including any negative y when calculating all integrals involved in the evaluation procedure. Finally, the product function $p_0(y)p_e(y)$ is usually not normalized, so a positive multiplicative constant C must be introduced to normalize p(y) before it can be used in the evaluation calculations.

In the earlier work, both Gaussian [6] and lognormal functions [7] were considered for the prior PDFs. All the likelihood functions were assumed to be Gaussians, in accordance with the Principle of Maximum Entropy, since usually only mean values y_e and standard deviations s_e are provided by experimenters [8,9]. Gaussians are symmetric, with zero skewness, and they always possess a fixed kurtosis of 3. Lognormal functions are inherently skewed, with widely varying values of skewness and kurtosis that depend on the lognormal function parameters.

Instances of noticeable differences between evaluated mean values and standard deviations obtained by the least-squares method [5,10] and those derived by Bayesian analyses were observed in our earlier work. Those differences for the mean values could be attributed to non-symmetric, non-Gaussian posterior PDFs *p*(y), and in particular to significant skewness associated with these PDFs. However, it was not possible from our earlier work to draw any conclusions regarding possible effects of PDF kurtosis on the evaluated mean values and standard deviations because of the simultaneous presence of skewness in the posterior PDFs.

In order to explore the effects of kurtosis, independent of skewness, the present work constrains the likelihood PDF p_e(y) to be Gaussian, as in our earlier studies [1,2], and it then considers three distinct, inherently symmetric functional forms for the prior PDF p_0(y) [11]: Gaussian (kurtosis = 3) [6], Continuous Uniform, which is referred to in this document simply as Uniform (kurtosis = 1.8) [12], and Laplace (kurtosis = 6) [13]. While many varieties of symmetric PDFs are known to statisticians [11], the choices of Uniform and Laplace PDFs employed here were made because of their simplicity as well as the fact that their constant kurtosis values bracket that of the Gaussian PDF. A product of two Gaussians always produces a Gaussian even if $y_e \neq y_0$. The product of a Gaussian PDF and a Uniform PDF, or of a Gaussian and a Laplace PDF, yields a symmetric PDF with zero skewness only when $y_e = y_0$.

A pure test of the effects of kurtosis on the outcomes of evaluations is provided by considering combinations of s_0 and s_e, always with $y_e = y_0$. The resulting posterior PDFs *p*(y) then have zero skewness. The kurtosis effects should impact only on the derived standard deviations. The present work also investigates skewness in the posterior PDFs *p*(y) when $y_e \neq y_0$, again by considering various combinations of s_0 and s_e. Bayesian evaluated results for studied examples are compared with the corresponding least-squares solutions to deduce general conclusions regarding how the evaluated results (y_{sol},s_{sol}) depend systematically on various combinations of the input data y_0, s_0, y_e, and s_e, as well as on the prior-likelihood PDF combinations: Gaussian-Gaussian, Uniform-Gaussian, and Laplace-Gaussian. The results of this work are presented in several tables and graphs that demonstrate the systematic effects associated with various choices for the input data values as well as the PDF combinations used. These numerical results compiled in this investigation also appear in their entirety in the Appendix.

2. Concepts and Formalism

The Bayesian approach to data evaluation is described conceptually in this section. First, we define in detail the nomenclature used in the present work. Random variable y, as well as the prior PDF p_0(y), likelihood PDF p_e(y), and posterior PDF *p*(y), were introduced already in Section 1. Let f(y) correspond to any well-defined, well-behaved function of the random variable y. The expectation value of f is defined in terms of the integral $\langle f \rangle = \int f(y)p(y)dy$, provided that *p*(y) is normalized. Integration extends over the entire range of values y for which the magnitude of *p*(y) contributes significantly to the integral in question, but restricted to $y \geq 0$ whenever y represents a positive physical quantity (also mentioned in Section 1). If f(y) = y, $\langle f \rangle$ corresponds to the mean value of *p*(y), and it can be written as $\langle y \rangle$ or, in the present context, as y_{sol}. If $f(y) = \langle (y-y_{sol})^2 \rangle = \langle y^2 \rangle - y_{sol}^2$, then $\langle f \rangle$ corresponds to the variance of *p*(y) which is written as var(y). The standard deviation of *p*(y) is defined as $std(y) = [var(y)]^{1/2}$, or in the present context as s_{sol}. For present purposes, the skewness and kurtosis of *p*(y) can be written as skewsol and kurtsol, respectively. The following formulas are used to determine the skewness and kurtosis of *p*(y):

$$skewsol = [\langle (y-y_{sol})^3 \rangle] / [var(y)]^{3/2} , \quad (1)$$

$$\text{kurtsol} = [\langle(y-y_{sol})^4\rangle] / [\text{var}(y)]^2 \,. \tag{2}$$

In contrast to our earlier work [1,2], the present investigation does not involve any stochastic (Monte Carlo) analyses. All the calculations have been performed deterministically. Since integrals need to be determined, and analytical formulas are often unavailable, the following numerical approach was utilized in the present work to compute these integrals.

Assume that the range of integration is (y_{low},y_{high}) and $\langle f\rangle = \int f(y)p(y)dy$ is to be approximated by a finite, discrete sum of terms. The following formulas have been used for this purpose:

$$\langle f\rangle \approx [\Sigma_{k=1,K}\, f(y_k)p(y_k)] / [\Sigma_{k=1,K}\, p(y_k)], \tag{3}$$

$$\Delta y = [(y_{high}-y_{low})/K], \tag{4}$$

$$y_k = y_{low} + (k-0.5)\Delta y \quad (k=1,K). \tag{5}$$

Note that inclusion of a term in the denominator of Eq. (3) insures that $\langle f\rangle$ will always be properly evaluated regardless of whether $p(y)$ is or is not normalized. The accuracy of the discrete-sum approximations to these integrals increases with the number of terms K. All the calculations performed for the examples considered in the present investigation were carried out using EXCEL, with $y_{low} = 0$, $y_{high} = 20$, and $K = 4{,}000$. Therefore, $\Delta y = 0.005$. Values for y_0, s_0, y_e, and s_e were selected for the studied examples so that the desired accuracy of this approximation would be adequate. Several checks to determine that this was indeed fulfilled were conducted during this investigation. This is an important consideration because some differences between corresponding least-squares and Bayesian evaluated results were found to be relatively small. To establish that these effects were real, the possibility of numerical biases due to the approximation defined by Eqs. (3), (4), and (5) needed to be excluded. In practice this numerical approach to approximating the encountered integrals proved to be very accurate.

An evaluated result produced by the least-squares method was generated for each example considered in the present work to compare with the corresponding Bayesian result. These calculations were performed using the following analytical formulas [5,10]:

$$y_{sol} = [(y_0 / s_0^2) + (y_e / s_e^2)] / [(1 / s_0^2) + (1 / s_e^2)] \,, \tag{6}$$

$$(1 / s_{sol}^2) = (1 / s_0^2) + (1 / s_e^2) \,. \tag{7}$$

Least-squares results are explicitly independent of the prior and likelihood PDFs. However, an assumption of the least-squares method is that all the data being evaluated are normal, i.e., Gaussian-distributed [5]. This issue is revisited in a later section of the present report.

For the present Bayesian evaluation approach, the PDF associated with the "experimental" data point represented by (y_e,s_e) is assumed to be Gaussian, thus $p_e(y)$ is given by:

$$p_e(y) = \exp[-0.5(y-y_e)^2/ s_e^2] / (2\pi s_e^2)^{1/2} \,. \tag{8}$$

The formulas used to characterize the three distinct, symmetric prior PDF types $p_0(y)$ considered in the present Bayesian approach are as follows:

Gaussian prior PDF [6]

$$p_0(y) = \exp[-0.5(y-y_0)^2/ s_0^2] / (2\pi s_0^2)^{½} . \quad (9)$$

Uniform prior PDF [12]

$$p_0(y) = 0 \quad \text{if } y < a \text{ or } y > b \text{ (with } 0 < a < b < \infty) , \quad (10)$$

$$p_0(y) = h \quad \text{if } a \leq y \leq b , \quad (11)$$

$$h = 1/(b-a) . \quad (12)$$

Laplace prior PDF [13]

$$p_0(y) = (1/2c) \exp[-(y_0-y)/c] \quad \text{if } y \leq y_0 , \quad (13)$$

$$p_0(y) = (1/2c) \exp[-(y-y_0)/c] \quad \text{if } y \geq y_0 , \quad (14)$$

$$c = [s_0^2/2]^{½} . \quad (15)$$

It is instructive to compare the shapes of the Gaussian, Uniform, and Laplace PDFs with a common mean value and standard deviation. This is exemplified in Fig. 1 for $<y> = 10$ and $std(y) = 1$ (10% uncertainty). The plotted curves share a common normalization.

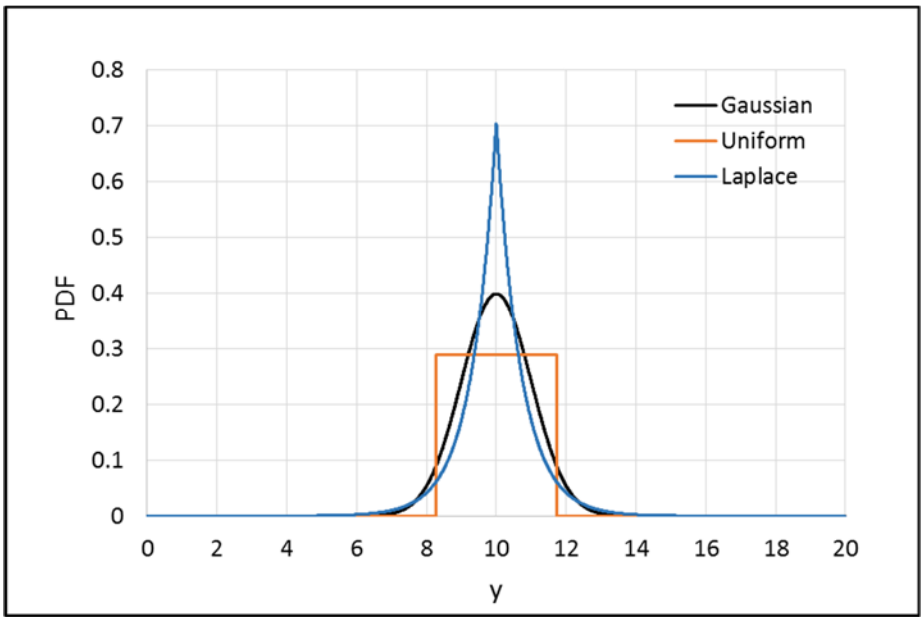


***Fig. 1**: Comparison of Gaussian, Uniform, and Laplace prior PDFs $p_0(y)$ with $<y> = 10$, $std(y) = 1$, and a common normalization.*

The use of Gaussians to represent PDFs is well justified for many situations in nuclear science [5,6]. However, Uniform PDFs are sometimes used if it is assumed that the value for a particular random variable should fall within a particular range, but otherwise there is no basis for preferring one value over another within that range [12]. The Laplace PDF is not as commonly used as the other two PDFs [13]. The present investigation is not concerned with this particular issue, but rather it has been carried out to gain an understanding of how PDF shapes influence the outcomes of evaluations performed using Bayesian methods when compared to what is obtained from using the more common least-squares approach where evaluated outcomes are not explicitly dependent on the assumed underlying PDFs.

The following three sections present and analyze results from least-squares and Bayesian evaluations performed with various selections of input data y_0, s_0, y_e, and s_e, as well as the abovementioned three prior PDF types $p_0(y)$ in combination with a Gaussian likelihood PDF $p_e(y)$. Numerical results from all the considered examples are compiled in the Appendix.

3. Gaussian – Gaussian Examples

It is straightforward to show that if $p(y) = Cp_0(y)p_e(y)$, with Gaussian $p_e(y)$ given by Eq. (8), Gaussian $p_0(y)$ given by Eq. (9), and a normalization constant C, then $p(y)$ is Gaussian with mean value y_{sol} and standard deviation s_{sol} given by Eqs. (6) and (7), respectively. Then C is:

$$C = (2\pi s_{sol}^2)^{-1/2} . \qquad [16]$$

Fig. 2 shows plots of normalized prior PDF $p_0(y)$ and likelihood $p_e(y)$, along with their un-normalized product $p_0(y)p_e(y)$, for a specific example with $y_0 = 11$, $s_0 = 1$, $y_e = 9$, and $s_e = 1$.

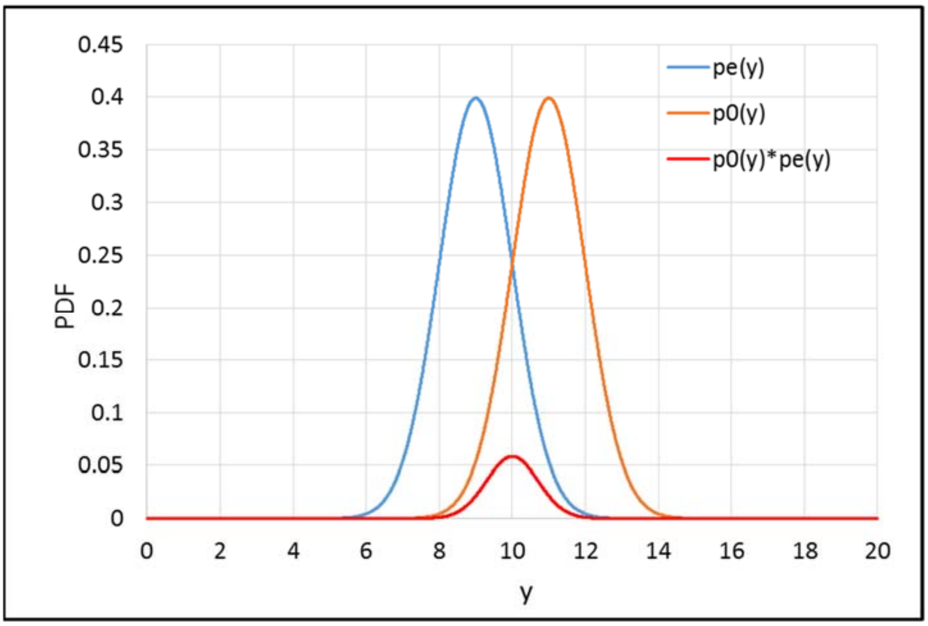


Fig. 2*: Plots of normalized Gaussians $p_0(y)$ and $p_e(y)$, along with their product $p_0(y)p_e(y)$, for $y_0 = 11$, $s_0 = 1$, $y_e = 9$, and $s_e = 1$.*

<u>Note</u>: For convenience, throughout this report the following un-subscripted notation is used in both tables and figures (and occasionally in the text) to refer to the indicated corresponding subscripted quantities that appear only in the text: p0 ≡ p_0; pe ≡ p_e; y0 ≡ y_0; ye ≡ y_e; ysol ≡ y_{sol}; s0 ≡ s_0; se ≡ s_e; and ssol ≡ s_{sol}.

If "Overlap" is defined by "Overlap = ∫ $p_0(y)p_e(y)dy$", with $p_e(y)$ and $p_0(y)$ normalized according to Eqs. (8) and (9), then "Overlap" can be calculated using Eq. (17) [1,14].

$$\text{Overlap} = \int p_0(y)p_e(y)dy = \exp[-0.5(y_0-y_e)^2/(s_0^2+s_e^2)] \,/\, [2\pi(s_0^2+s_e^2)]^{½} \,. \qquad [17]$$

"Overlap" can also be calculated using the approximation method defined by Eqs. (3), (4), and (5). Comparisons of "Overlap" values obtained by these two methods therefore provides a very sensitive test of the accuracy of the approximation method used in the present investigation.

The least-squares evaluated result for this particular example is y_{sol} = 10 and s_{sol} = 0.707106781186547. The normalized chi-square value is 2. The Bayesian solution obtained using the approximation method embodied in Eqs. (3), (4), and (5) yields the following results: y_{sol} = 10 (obviously), s_{sol} = 0.707106781186549, skewsol = 4.64417E-14, and kurtsol = 3.00000000000002. The skewness for a Gaussian should be exactly zero while the kurtosis should be exactly 3. The digits that agree for the comparable values are highlighted in yellow while those that disagree are highlighted in blue.

The value of "Overlap" obtained analytically using Eq. (17) is Overlap = 0.103776874355149. The value of "Overlap" obtained from numerical integration according to the approximation method embodied in Eqs. (3), (4), and (5) is Overlap = 0.103776874355133. Again, the digits that agree for the comparable values are highlighted in yellow while those that disagree are highlighted in blue.

Agreement between the analytical least-squares results and those obtained from the Bayesian calculations that are performed using the present approximation method is excellent to many significant figures, as is evident from examination of the highlighted digit values for the indicated comparable quantities. Similar excellent agreement is observed for the other Gaussian – Gaussian examples considered in this work, as is seen in Tables A.1 and A.2 of the Appendix. This leads to the following general conclusion:

- ***Conclusion****: If all the input data to be evaluated are assumed to be Gaussian-distributed, the Bayesian solution will be identical to that obtained by applying the least-squares method. Then, there is no benefit to employing the more computationally intensive Bayesian approach rather than using the simpler least-square approach in performing the evaluation.*

However, the Bayesian evaluation approach should be considered if some or all of the underlying PDFs of the data being evaluated are non-Gaussian. This is the case whenever non-linear functional relationships are involved, as was demonstrated in our earlier work [1,2].

4. Uniform – Gaussian Examples

In this section we explore examples that incorporate the Uniform – Gaussian PDF combination, with $y_e = y_0$. This allows examination of the effects of kurtosis free from skewness, since the prior and likelihood PDFs are both symmetric. A number of examples have been analyzed using $y_e = y_0 = 10$, $s_0 = 0.577350269$ (≈ 5.8%) and various values of s_e. The relevant results, collected from numerical information compiled in the Appendix, are assembled in Table 1.

***Table 1**: Comparisons of evaluated standard deviations s_{sol} obtained using the least-squares (LS) and Bayesian (Bay) methods for Uniform prior and Gaussian likelihood PDFs, as well as assumed equal mean values $y_e = y_0 = 10$. These results are drawn from the numerical information compiled in Tables A.3 and A.4 of the Appendix.*

s0	se	se/s0	Kurtosis	ssol(Bay)	ssol(LS)	ssol Ratio
0.57735	3	5.196152	1.815278	0.566947	0.5730789	1.0108162
0.57735	2.5	4.330127	1.822042	0.562544	0.571205	1.0153961
0.57735	2	3.464102	1.834545	0.5547	0.567763	1.0235494
0.57735	1.5	2.598076	1.861773	0.538816	0.5603645	1.0399924
0.57735	1.2	2.078461	1.897166	0.520266	0.5509363	1.0589512
0.57735	1	1.732051	1.940905	0.5	0.5395592	1.0791184
0.57735	0.75	1.299038	2.053599	0.457496	0.5114609	1.1179578
0.57735	0.62	1.073872	2.172679	0.422522	0.4832982	1.1438411
0.57735	0.5	0.866025	2.365527	0.377964	0.4398132	1.1636362
0.57735	0.4	0.69282	2.624217	0.328798	0.3818394	1.1613192
0.57735	0.3	0.519615	2.914466	0.266207	0.2984526	1.1211298
0.57735	0.2	0.34641	2.999673	0.188982	0.1999985	1.0582927
0.57735	0.1	0.173205	3	0.098533	0.1	1.0148892

The results provided in Table 1 are also plotted in Figs. 3 and 4. These figures offer two distinct yet comparable views that demonstrate the systematic behavior of ratios of the evaluated least-squares (LS) and the Bayesian (Bay) standard-deviation results, s_{sol}. These ratios are plotted versus posterior PDF kurtosis (Fig. 3) and the input data standard deviation ratio s_e/s_0 (Fig. 4).

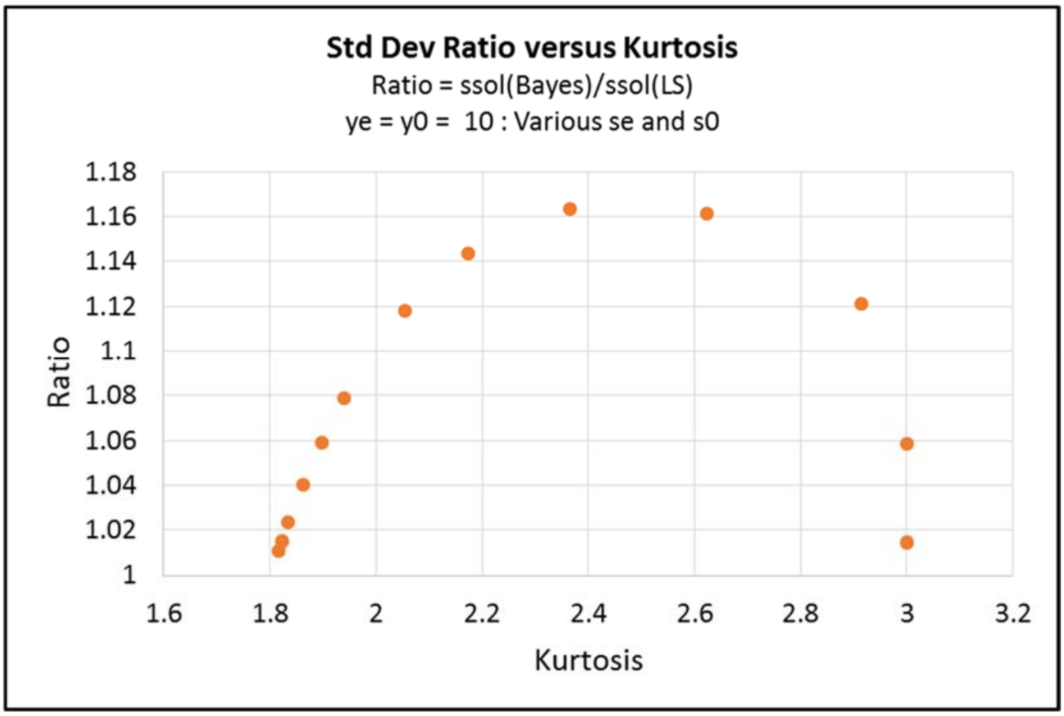


***Fig. 3**: Plot of standard-deviation ratios obtained by the least-squares and Bayesian methods versus kurtosis for several Uniform – Gaussian PDF examples.*

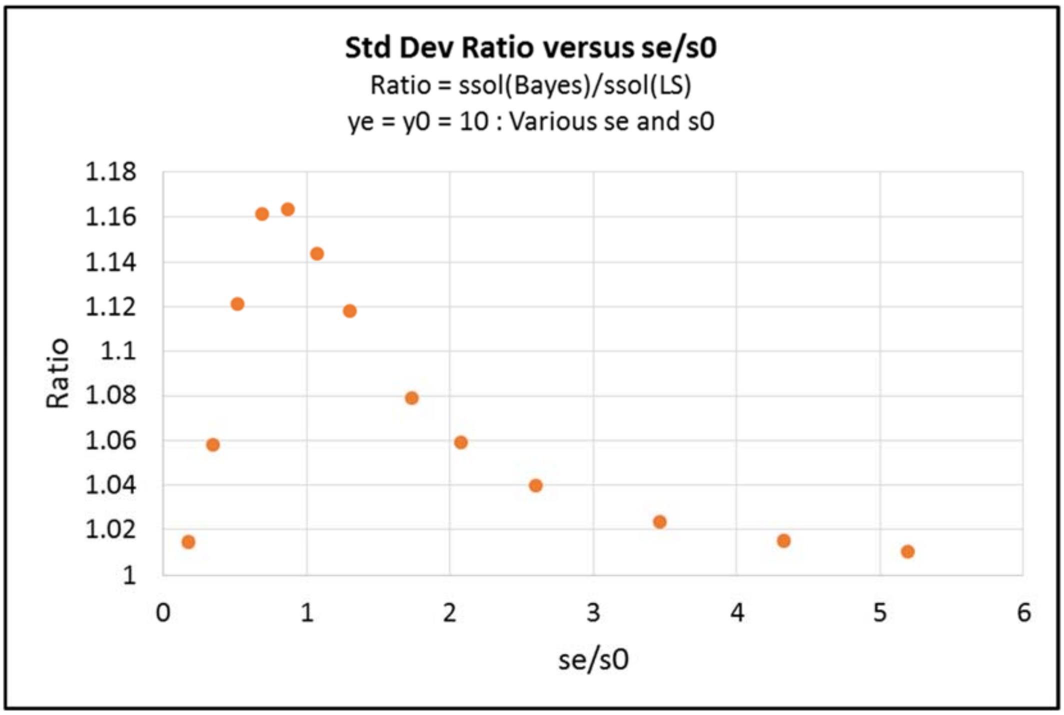


Fig. 4: *Plot of the standard-deviation ratios obtained by the least-squares and Bayesian methods versus the ratio s_e/s_0 for several Uniform – Gaussian PDF examples.*

Variations in the standard deviation ratios can be significant, i.e., they can exceed 16%. Fig. 3 shows that the largest difference between the least-squares and Bayesian results corresponds to a solution PDF *p*(y) having a kurtosis of ≈ 2.4. It is interesting to notice that this kurtosis value falls about midway between the kurtosis of 1.8 for p_0(y), a Uniform PDF, and a kurtosis of 3 for p_e(y), a Gaussian PDF. This also corresponds to a situation where the "theoretical" and "experimental" values have about the same uncertainties, and therefore contribute about equally to determination of the uncertainty in the evaluated solution. This observation is also borne out by the results shown in Fig. 4, where the maximum difference between the least-squares and Bayesian s_{sol} results correspond to $s_e \approx s_0$, i.e., to $s_e/s_0 \approx 1$. Figs. 3 and 4 also demonstrate that the difference between the least-squares and Bayesian results tends to vanish at the limits where either the prior or the "experimental" input data point dominates the evaluation due to a much lower uncertainty than the other input data point. Fig. 5 shows a plot of the prior, likelihood, and posterior PDFs for the case where $s_e \approx s_0$.

- ***Comment***: *It is evident that ssol(Bay) ≥ ssol(LS) for the considered Uniform – Gaussian cases. One might suspect that perhaps this holds true in general whenever the prior PDF kurtosis is less than that of the Gaussian PDF (and $y_e = y_0$), regardless of the nature of the prior PDF. However, such a claim cannot be substantiated by considering only one type of prior PDF (Uniform). Studies with other symmetric PDFs types having kurtosis values less than for a Gaussian would be needed to validate such a claim.*

The practical implication of the preceding observations and general comment is that in realistic nuclear data evaluations one will encounter some instances where model values have significantly larger uncertainties than the experimental uncertainties in energy ranges where experimental data dominate the evaluation, while the opposite will be true for the evaluation of data in energy ranges which are mostly dominated by theoretical models, owing to scarce and hard-to-measure experimental data. In the energy ranges transitioning between those two regimes, it is very likely to encounter cases where model values and experimental data have similar uncertainties. Studies such as the present one should be done for realistic data sets.

- ***Conclusion***: *The cases of experimental versus model uncertainties studied in the present work are potentially relevant for the evaluation of realistic nuclear data.*

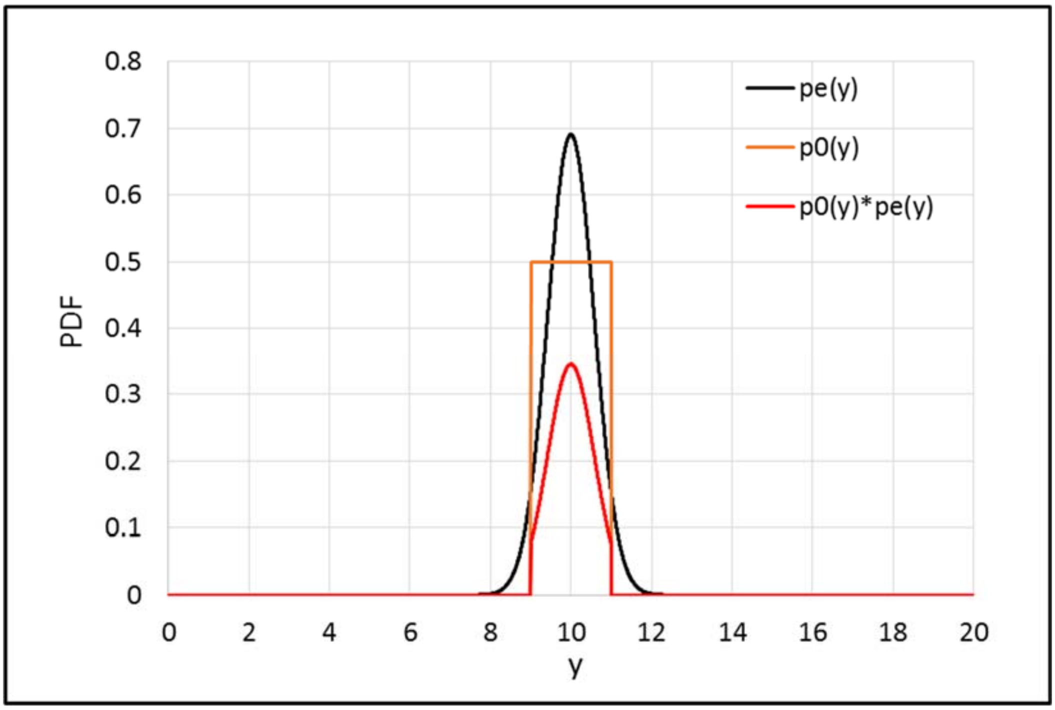


Fig. 5: *Plots of* $p_e(y)$, $p_0(y)$, *along with their product* $p_0(y)p_e(y)$, *for* $y_e = y_0 = 10$ *and* $s_e = s_0 = 0.577350269$ *(≈ 5.8%). Notice that the kurtosis of the posterior PDF is < 3 since the Gaussian "wings" are effectively truncated by the Uniform PDF.*

This analysis suggests two additional conclusions relevant to the Uniform – Gaussian case:

- ***Conclusion***: *The posterior PDF kurtosis appears to fall between the kurtosis of the prior PDF and the likelihood PDF. The largest difference between the least-squares and Bayesian evaluated standard deviations corresponds to a posterior PDF kurtosis that is about midway between the prior PDF kurtosis and the likelihood PDF kurtosis.*

- ***Conclusion***: *Differences in the posterior PDF kurtosis relative to that of a Gaussian can have a significant effect on the differences between evaluated results obtained by the least-squares and Bayesian methods, even if the input data mean values are equal. The Bayes standard deviations appear to be systematically greater than or equal to the least-squares values whenever the prior PDF is the Uniform function.*

The present investigation also considers a number of examples corresponding to $y_0 = 10$ and $s_0 = 0.577350269$ (≈ 5.8%), but with various values of s_e and $y_e \neq y_0$. A graphical rendition of the

PDFs for a typical example, i.e., $s_e = 1$ and $y_e = 11$, appears in Fig. 6. If one is accustomed to Gaussian PDF shapes, the product PDF shape (shown in red in Fig. 6) seems strange indeed.

The considered examples are grouped into five categories based on standard deviation values $s_e = 0.1, 0.2, 0.5, 1$, and 1.5, respectively. All numerical results are compiled in the Appendix.

What is shown here are systematic dependencies on the scaled displacement factor $(y_e-y_0)/s_e$ of ratios of the Bayesian and least-squares evaluated mean values y_{sol} and standard deviations s_{sol} as well as posterior PDF skewness and kurtosis values. These results are presented below in tables and figures with accompanying discussions. The displacement factor is a measure of the discrepancy between the "theoretical" value y_0 and "experimental" value y_e utilized in the evaluation procedure. In realistic evaluations, there are compelling reasons to suspect the input data – either the theoretical or experimental values – when discrepancies between these values exist that exceed more than two combined standard deviations. In such situations, careful evaluators will expend extra effort to try and trace the origins of these discrepancies. The present investigation employs hypothetical data. Doing so offers the opportunity to explore what happens in those situations where the data are chosen artificially to be very discrepant.

Extreme displacement factors (indicated in red font) for the input mean values yield large evaluated chi-squares, small "Overlaps", and various other numerical anomalies due to the discrepant nature of the input data. Furthermore, there may be numerical biases associated with incomplete integration due to including only the range of variable "y" from 0 to 20. For these reasons, one should avoid drawing any conclusions from those tabulated values shown in red font even though they may be included in the tables and plots.

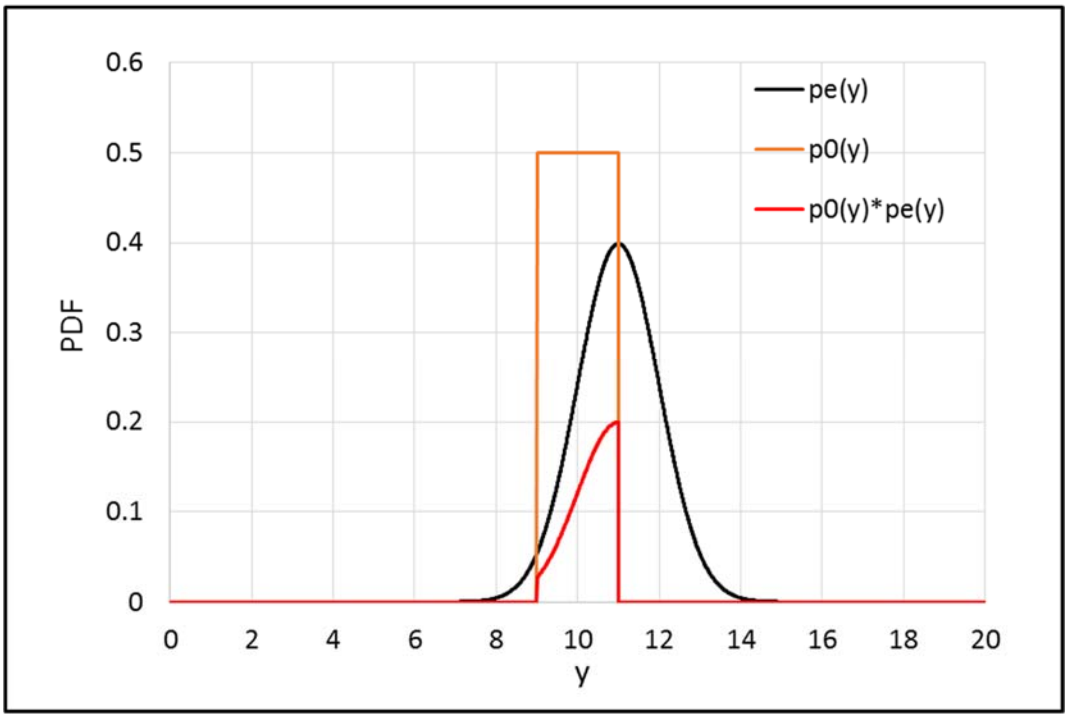


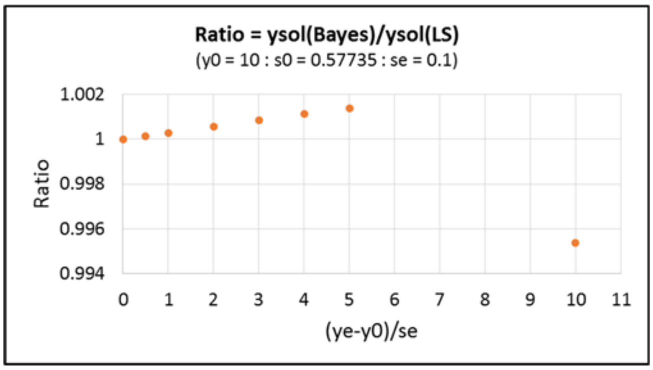
Fig. 6*: Plots of normalized $p_e(y)$ and $p_0(y)$, along with the product PDF $p_0(y)p_e(y)$, for $y_0 = 10$, $s_0 = 0.577350269$, $y_e = 11$, and $s_e = 1$. Prior PDF is Uniform and likelihood is Gaussian.*

Uniform – Gaussian Examples with $s_e = 0.1$

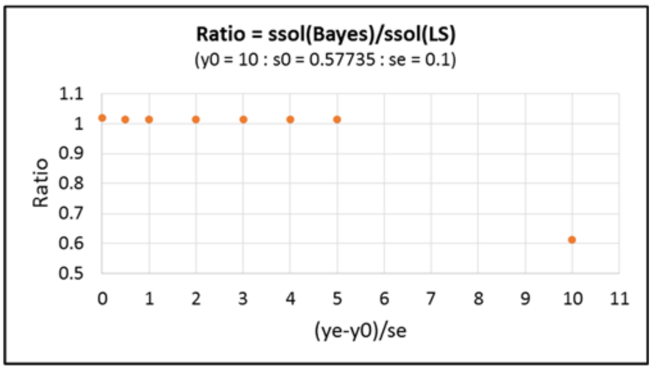
Results for y_0 = 10, s_0 = 0.57735, and s_e = 0.1 appear in Table 2 and Fig. 7 (A through D). Smooth variations with $(y_e-y_0)/s_e$ are seen in the table and plots. The mean-value ratios are close to unity for $(y_e-y_0)/s_e \leq 5$. The standard deviation ratios differ systematically from unity by ≈ 1.5 to 2% for $(y_e-y_0)/s_e \leq 5$. Skewness values are totally negligible for $(y_e-y_0)/s_e \leq 5$. The kurtosis for $(y_e-y_0)/s_e \leq 5$ is ≈ 3. It deviates considerably from 3 for $(y_e-y_0)/s_e > 5$.

***Table 2**: Tabulated Uniform – Gaussian results for y_0 = 10, s_0 = 0.57735, and s_e = 0.1.*

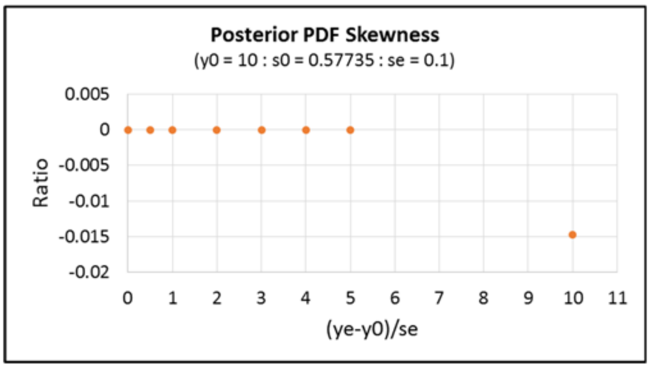
Examples with y0 = 10 : s0 = 0.57735 : se = 0.1				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	1.019803903	3.36206E-15	3
0.5	1.000144928	1.014889157	6.73977E-15	3
1	1.000288462	1.014889157	-3.38613E-15	3
2	1.000571429	1.014889157	-1.55338E-14	3
3	1.000849057	1.014889156	-1.38E-11	2.999999997
4	1.001121495	1.014889138	-6.70314E-09	2.999998801
5	1.001388875	1.014885393	-1.12601E-06	2.99983676
10	0.995381356	0.611673924	-0.014738976	3.87017511



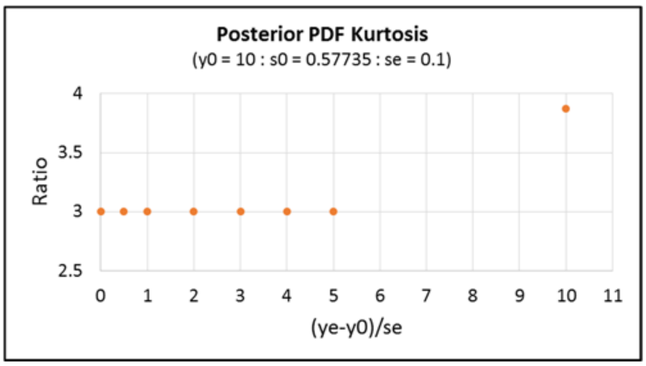
(A)



(B)



(C)



(D)

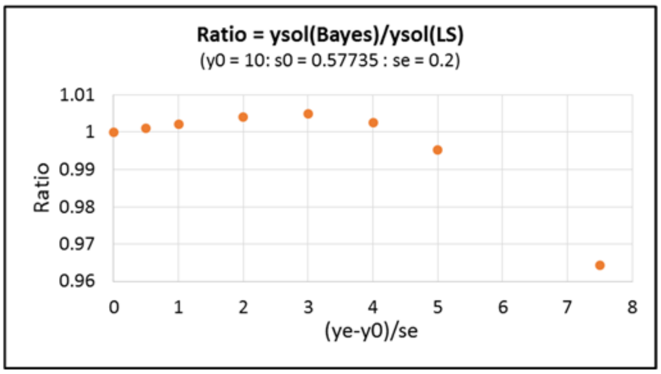
***Fig.** 7: Plots of Uniform – Gaussian results for y_0 = 10, s_0 = 0.57735, and s_e = 0.1.*

Uniform – Gaussian Examples with $s_e = 0.2$

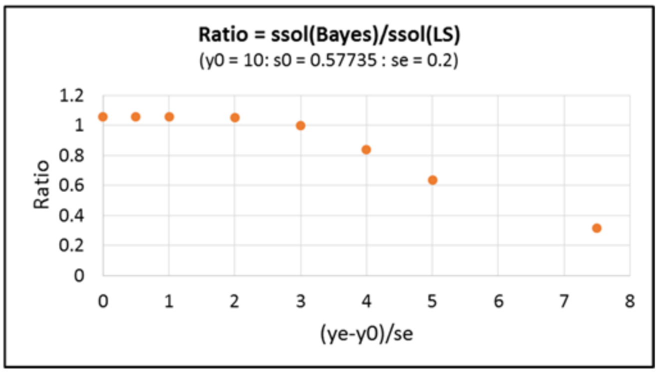
Results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.2$ are shown in Table 3 and Fig. 8 (A through D). Smooth variations with $(y_e\text{-}y_0)/s_e$ are seen in the table and plots. The mean-value ratios differ from unity by $\leq 0.5\%$ for $(y_e\text{-}y_0)/s_e \leq 5$. The standard deviation ratios differ systematically from unity by ≈ 5 to 6% for $(y_e\text{-}y_0)/s_e \leq 2$. These ratio differences from unity are ≈ 0.4 to 36% for $3 \leq (y_e\text{-}y_0)/s_e \leq 5$. The skewness is very small for $(y_e\text{-}y_0)/s_e \leq 2$ and it is very modest elsewhere. The kurtosis varies from ≈ 2.8 to 3.8 for $(y_e\text{-}y_0)/s_e \leq 5$.

***Table 3**: Tabulated Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.2$.*

Examples with y0 = 10 : s0 = 0.57735 : se = 0.2				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	1.058292662	-1.19839E-14	2.999673062
0.5	1.001061632	1.058262167	-2.72298E-05	2.998743326
1	1.002102635	1.058017279	-0.000179579	2.993031563
2	1.004052253	1.051222692	-0.003223293	2.916751448
3	1.005053007	0.996408709	-0.018053517	2.760912428
4	1.002631507	0.839783105	-0.037421378	3.001366317
5	0.995186	0.637925322	-0.041669788	3.869426535
7.5	0.964383891	0.315585798	-0.023552615	6.445149136



(A)



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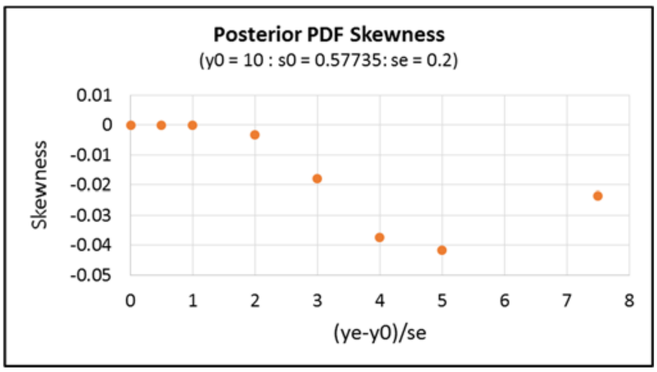

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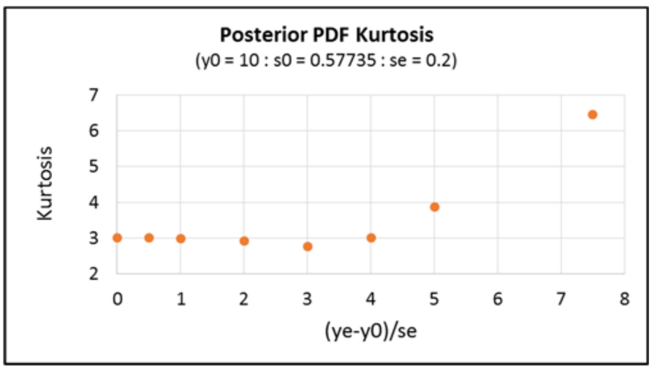


(D)

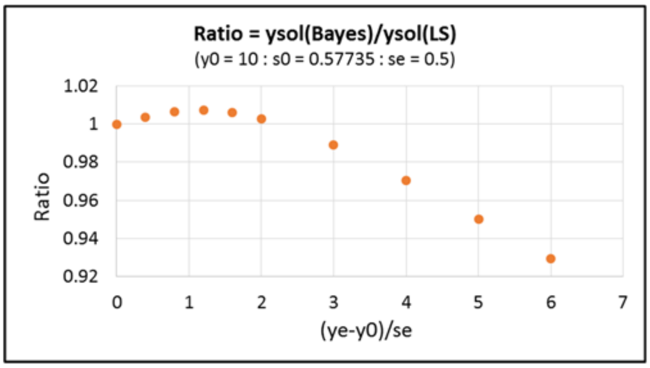
***Fig. 8**: Plots of Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.2$.*

Uniform – Gaussian Examples with $s_e = 0.5$

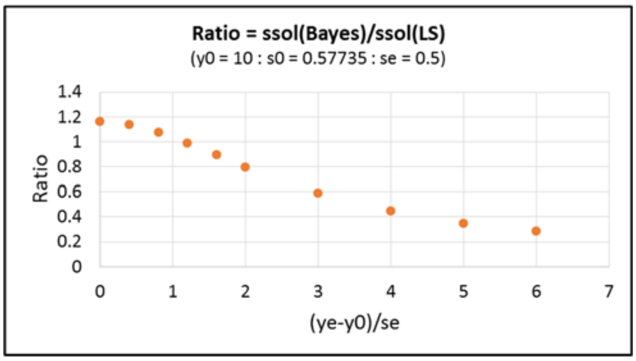
Results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.5$ are shown in Table 4 and Fig. 9 (A through D). Smooth variations with $(y_e-y_0)/s_e$ are observed in the table and plots. The mean-value ratios differ from unity by < 1.1% for $(y_e-y_0)/s_e \leq 3$. Standard deviation ratios differ from unity by ≈ 0.8 to 41% for $(y_e-y_0)/s_e \leq 3$. The skewness is noticeable for $0.8 \leq (y_e-y_0)/s_e \leq 3$. The kurtosis increases systematically from ≈ 2.4 to 5 for $(y_e-y_0)/s_e \leq 3$.

Table 4: *Tabulated Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.5$.*

Examples with y0 = 10 : s0 = 0.57735 : se = 0.5				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	1.163636242	5.30998E-14	2.365527418
0.4	1.003804067	1.141288968	-0.062887704	2.437960462
0.8	1.006440596	1.079770801	-0.113841464	2.646199402
1.2	1.007223871	0.992496037	-0.146606049	2.96641576
1.6	1.005958965	0.894297089	-0.161617826	3.367537788
2	1.002812869	0.796642156	-0.163175933	3.818156486
3	0.988973892	0.590225826	-0.138619504	4.99049609
4	0.970431869	0.447184037	-0.106771139	6.018422948
5	0.950113327	0.351374885	-0.081280228	6.803251557
6	0.929393387	0.285767465	-0.062879393	7.365674865



(A)



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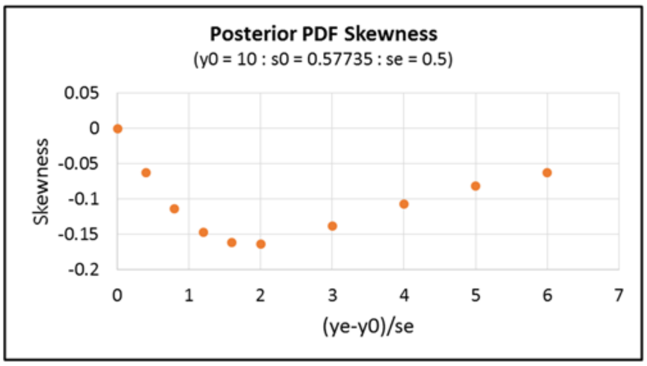

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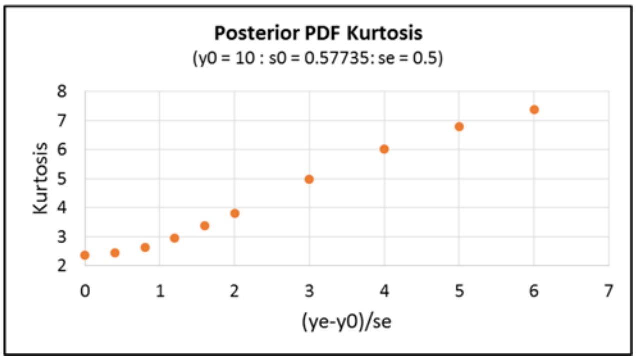


(D)

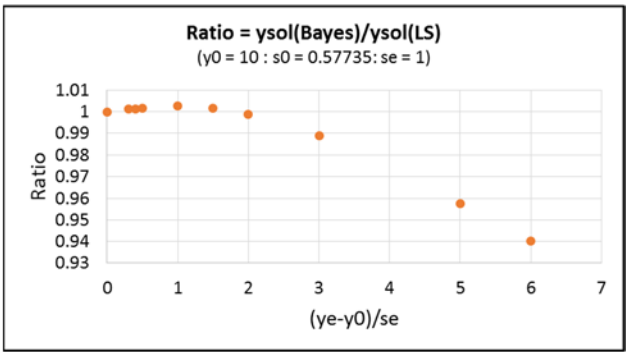
Fig. 9: *Plots of Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 0.5$.*

Uniform – Gaussian Examples with $s_e = 1$

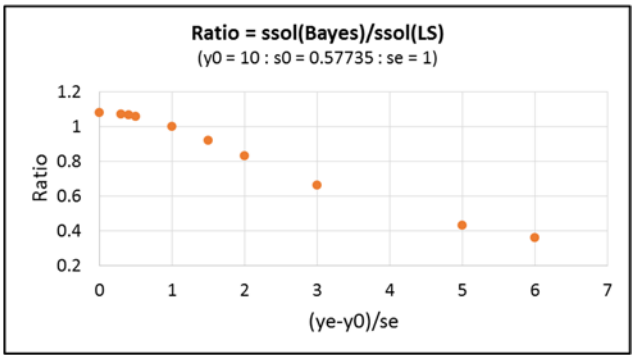
Results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1$ are shown in Table 5 and Fig. 10 (A through D). Smooth variations with $(y_e\text{-}y_0)/s_e$ are observed in the table and plots. The mean-value ratios differ from unity by < 0.3% for $(y_e\text{-}y_0)/s_e \leq 2$. Differences from unity in the standard-deviation ratios range from ≈ 0.3 to 17% for $(y_e\text{-}y_0)/s_e \leq 2$. Noticeable skewness occurs for $0.4 \leq (y_e\text{-}y_0)/s_e \leq 2$. Kurtosis increases systematically from ≈ 1.9 to 3.6 for $(y_e\text{-}y_0)/s_e \leq 2$.

Table 5: *Tabulated Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1$.*

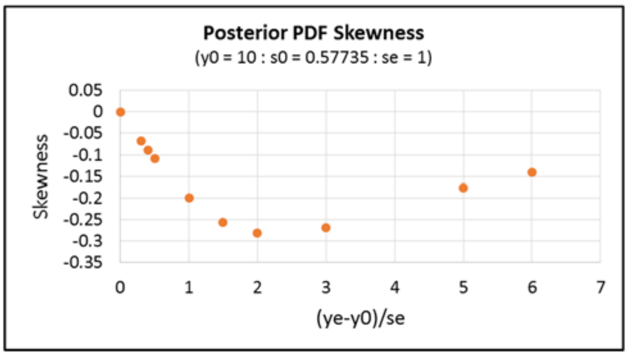
Examples with y0 = 10 : s0 = 0.57735 : se = 1				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	1.079118421	3.5649E-14	1.940905062
0.3	1.001184718	1.071690504	-0.067120884	1.983355719
0.4	1.001535011	1.065992821	-0.088654678	2.016232085
0.5	1.001849543	1.058767428	-0.109490941	2.058318989
1	1.002654555	1.002626637	-0.198656636	2.400817767
1.5	1.001779384	0.922793696	-0.256099514	2.937489323
2	0.999042663	0.832949516	-0.281664946	3.614657458
3	0.988777907	0.662060975	-0.267985038	5.102576981
5	0.957728417	0.431532614	-0.175962539	7.194155139
6	0.940303852	0.361538007	-0.140429052	7.705728835



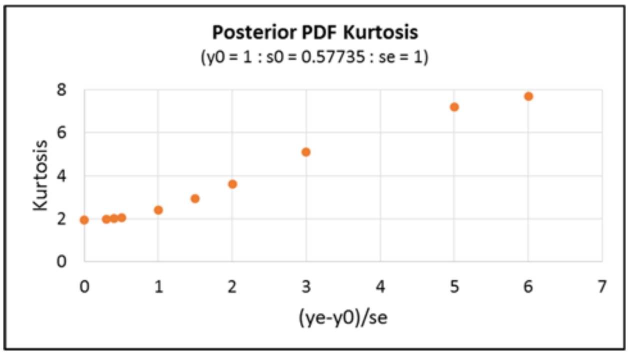
(A)



(B)



(C)



(D)

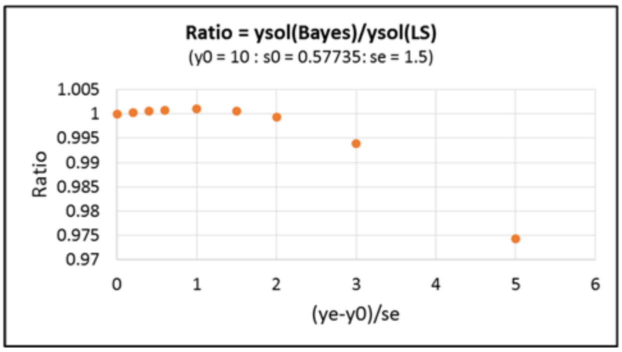
Fig. 10: *Plots of Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1$.*

<u>Uniform – Gaussian Examples with $s_e = 1.5$</u>

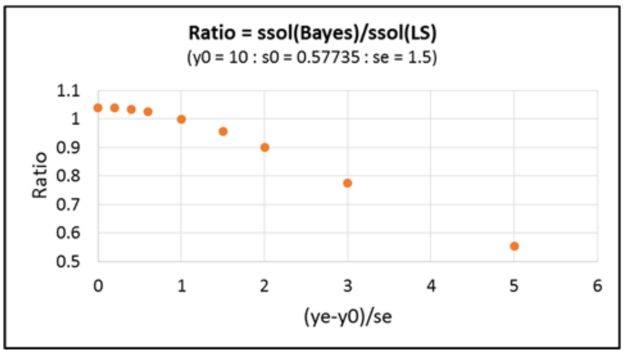
Results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1.5$ are shown in Table 6 and Fig. 11 (A through D). Smooth variations with displacement $(y_e\text{-}y_0)/s_e$ are observed in the table and plots. The mean-value ratios differ from unity by $<$ 0.1% for $(y_e\text{-}y_0)/s_e \leq 2$. The standard-deviation ratio differences from unity range from negligible to ≈ 10% for $(y_e\text{-}y_0)/s_e \leq 2$. The skewness is negligible for $(y_e\text{-}y_0)/s_e < 0.4$, but it is noticeable for $0.4 \leq (y_e\text{-}y_0)/s_e \leq 2$. The kurtosis increases systematically from ≈ 1.9 to 2.8 for $(y_e\text{-}y_0)/s_e \leq 2$.

Table 6*: Tabulated Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1.5$.*

Examples with y0 = 10 : s0 = 0.57735 : se = 1.5				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	1.039992435	3.77604E-14	1.861772951
0.2	1.000310182	1.038343307	-0.035576395	1.871974898
0.4	1.000591784	1.033430503	-0.070575795	1.902533749
0.6	1.000819824	1.025356029	-0.104445023	1.953306346
1	1.001025382	1.000434065	-0.166827669	2.114397698
1.5	1.000627614	0.95545827	-0.230921384	2.422553094
2	0.999334925	0.899622	-0.276664971	2.838261651
3	0.99391276	0.774541535	-0.315100192	3.913507399
5	0.974260171	0.553812286	-0.267883743	6.24847372



(A)



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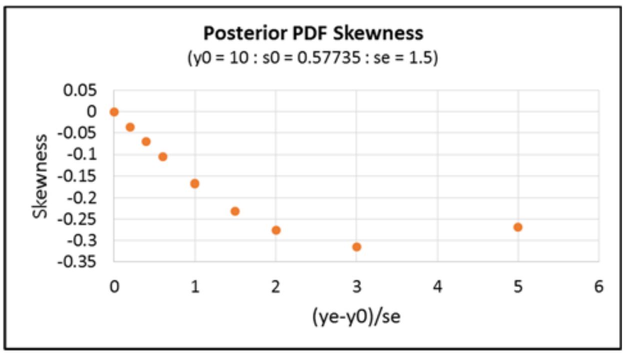

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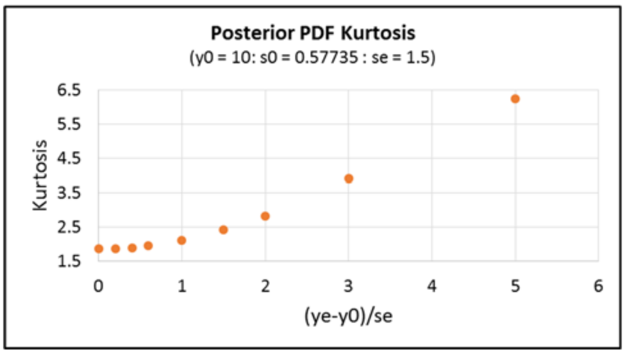


(D)

Fig. 11*: Plots of Uniform – Gaussian results for $y_0 = 10$, $s_0 = 0.57735$, and $s_e = 1.5$.*

An examination of the collected results from the preceding five hypothetical examples offers the possibility to arrive at some general conclusions about the evaluation of the two differently distributed data, one that is normal (Gaussian) and the other one that is Uniform.

- ***Conclusion***: *The derived Bayesian-to-least-squares mean-value and standard-deviation ratios, as well as skewness and kurtosis, for the collection of Uniform – Gaussian examples, are seen to vary smoothly with the scaled displacement* $(y_e\text{-}y_0)/s_e$ *for the five cases of* s_e *in these examples, as is seen in the tables and plots.*

- ***Conclusion***: *Generally, the derived mean-value ratios are reasonably close to unity in most of these examples except for extreme displacements.*

- ***Conclusion***: *Substantial differences are observed for the standard-deviation ratios. This is correlated with observed significant variations in the kurtosis of the posterior PDF rather than being traceable to skewness effects. The observed skewness values, if not always negligible, tend to be fairly modest for most of the studied examples.*

- ***Conclusion***: *Extreme values of displacement* $(y_e\text{-}y_0)/s_e$ *in the affected examples are generally seen to lead to unreasonable derived values for all four quantities. This phenomenon is traceable in some of these examples to the aforementioned incomplete integration phenomenon that occurs when the integration range is limited to 0 to 20.*

- ***Conclusion***: *For those examples that consider input data which are reasonably consistent, i.e., modest displacements* $(y_e\text{-}y_0)/s_e$*, the results of the present analysis, and the interpretations of these results, can be viewed as trustworthy.*

The analyses for all of the examples considered in the present investigation for the Uniform – Gaussian category generated chi-square values from the least-squares calculations as well as corresponding "Overlap" values from the Bayesian calculations, as described in Section 2. These results are collected in the Appendix in Tables A.3 and A.4.

Those examples where very large chi-square values (chi-square/d.f. > 6) and very small "Overlap" values (Overlap < 0.02) were generated correspond to hypothetical data that are clearly discrepant. The individual entries are identified by means of red font in these tables. As indicated in the discussion above, one should not trust the evaluated results or base any general conclusions on these discrepant cases.

5. Laplace – Gaussian Examples

As indicated in Section 4 for the Uniform – Gaussian case, examples in which $y_e = y_0$ offer the possibility of examining the effects of kurtosis, free of skewness, when the prior and likelihood PDFs are both symmetric, as is also the case for the Laplace – Gaussian combination.

A number of examples have been analyzed using $y_e = y_0 = 10$ and various values of s_0 and s_e. The relevant information, based on compiled numerical results from the Appendix, is collected in Table 7. The results in Table 7 are plotted in Figs. 12 and 13. These figures offer two distinct yet comparable views that demonstrate the systematic behavior of ratios of the evaluated least-squares (LS) and the Bayesian (Bay) standard-deviation results, s_{sol}. The ratios are plotted versus posterior PDF kurtosis (Fig. 12) and input data standard deviation ratio s_e/s_0 (Fig. 13).

***Table** 7: Comparisons of evaluated standard deviations s_{sol} obtained using the least-squares (LS) and Bayesian (Bay) methods for Laplace prior and Gaussian likelihood PDFs, and assumed equal mean values $y_e = y_0 = 10$. These results are based on the numerical data compiled in Tables A.5 and A.6 of the Appendix.*

s0	se	se/s0	Kurt	ssol(Bay)	ssol(L.S.)	ssol Ratio
1	0.05	0.05	3.05633	0.049938	0.048615	0.973523
0.5	0.05	0.1	3.112571	0.049752	0.04728	0.950325
1	0.1	0.1	3.112694	0.099504	0.094559	0.950306
0.5	0.1	0.2	3.22477	0.098058	0.089503	0.912752
1	0.2	0.2	3.22484	0.196116	0.179003	0.912742
0.5	0.15	0.3	3.335753	0.143674	0.127209	0.885398
0.5	0.2	0.4	3.445049	0.185695	0.16089	0.866422
1	0.4	0.4	3.445094	0.371391	0.321779	0.866416
0.5	0.3	0.6	3.65687	0.257248	0.217918	0.847113
1	0.6	0.6	3.656908	0.514496	0.435834	0.847109
1	0.8	0.8	3.857453	0.624695	0.527249	0.84401
0.5	0.5	1	4.044833	0.353553	0.300432	0.849749
1	1	1	4.044867	0.707107	0.600861	0.849745
1	1.5	1.5	4.451072	0.83205	0.729931	0.877267
0.5	1	2	4.770994	0.447214	0.404574	0.904654
1	2	2	4.771027	0.894427	0.809145	0.904651
0.5	1.5	3	5.205993	0.474342	0.446838	0.942018
1	3	3	5.205999	0.948683	0.893673	0.942014
0.5	2	4	5.46225	0.485071	0.466874	0.962486
0.5	3	6	5.717735	0.493197	0.483964	0.981279

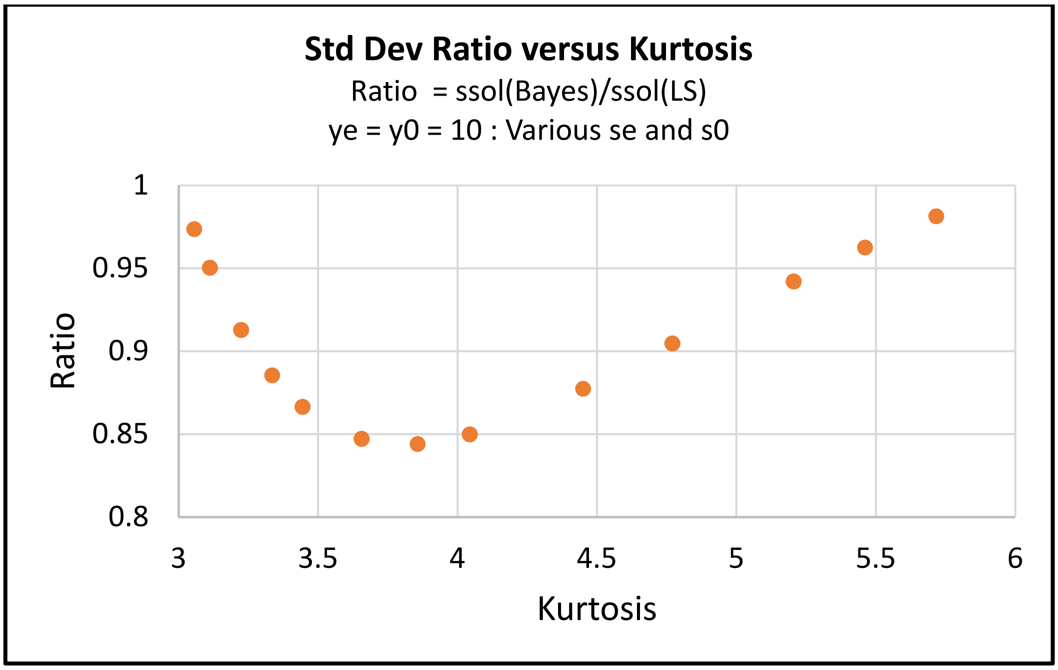


***Fig. 12**: Plot of the standard-deviation ratios obtained by the least-squares and Bayesian methods versus kurtosis for several Laplace – Gaussian examples.*

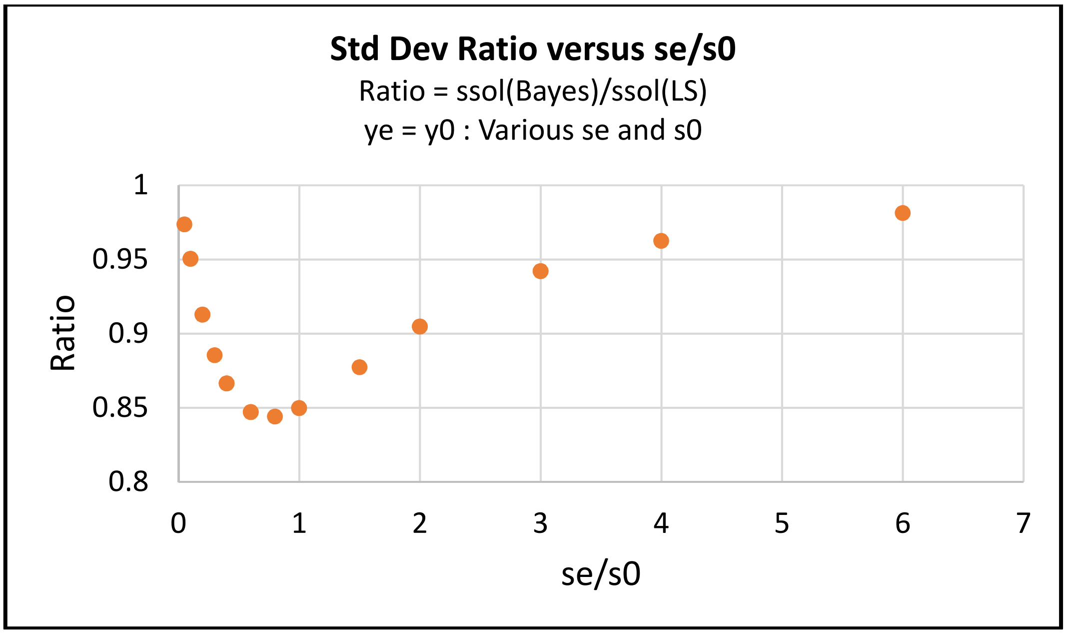


***Fig. 13**: Plot of the standard-deviation ratios obtained by the least-squares and Bayesian methods versus the ratio s_e/s_0 for several Laplace – Gaussian examples.*

It is evident from Table 7 that $y_e = y_0 = 10$, s_0 equals either 0.5 (5%) or 1 (10%), and various s_e are used for the considered examples. Variations in the standard deviation ratio values can be significant, i.e., they can exceed 16%.

Fig. 12 shows that the largest difference between the least-squares and Bayesian results corresponds to a solution PDF p(y) having a kurtosis of ≈ 3.8. This value is roughly in the middle between the kurtosis of 6 for p_0(y), a Laplace PDF, and kurtosis of 3 for p_e(y), a Gaussian PDF. Again, this is what one might expect to observe if the prior and "experimental" data carry roughly equal weights in determining the evaluated outcome.

This observation is also borne out by the information plotted in Fig. 13, where the maximum difference between the least-squares and Bayesian results correspond to values of s_e differing only modestly from s_0, i.e., to $s_e/s_0 \approx 0.8$. Again, this corresponds to a situation where the prior and "experimental" data values are roughly equally weighted in the evaluation process.

Both figures also show that the differences between the least-squares and Bayesian results vanish at both extremes, i.e., when either the prior or the "experimental" input data point dominates the evaluation due to a much lower uncertainty than the other input data point.

- ***Comment**: It is evident that ssol(Bay) ≤ ssol(LS) for the considered Laplace – Gaussian cases. One might suspect that perhaps this holds true in general whenever the prior PDF kurtosis is greater than that of the Gaussian PDF (and $y_e = y_0$), regardless of the nature of the prior PDF. However, such a claim cannot be substantiated by considering only one type of prior PDF (Laplace). Studies with other symmetric PDFs types having kurtosis values greater than for a Gaussian would be needed to validate such a claim.*

This is the exact opposite direction from the Uniform – Gaussian situation. It can be attributed to the fact that the Uniform and Laplace prior PDFs differ notably with respect to their kurtosis values relative to a Gaussian. The Uniform PDF has a kurtosis that is smaller than a Gaussian while the Laplace PDF has a kurtosis that is much larger than a Gaussian. As mentioned earlier, further investigation is needed to establish whether this is just one example of a more fundamental principle of PDF moments.

Fig. 14 is a plot of the prior, likelihood, and posterior PDFs for $s_0 = 0.5$ and $s_e = 1$.

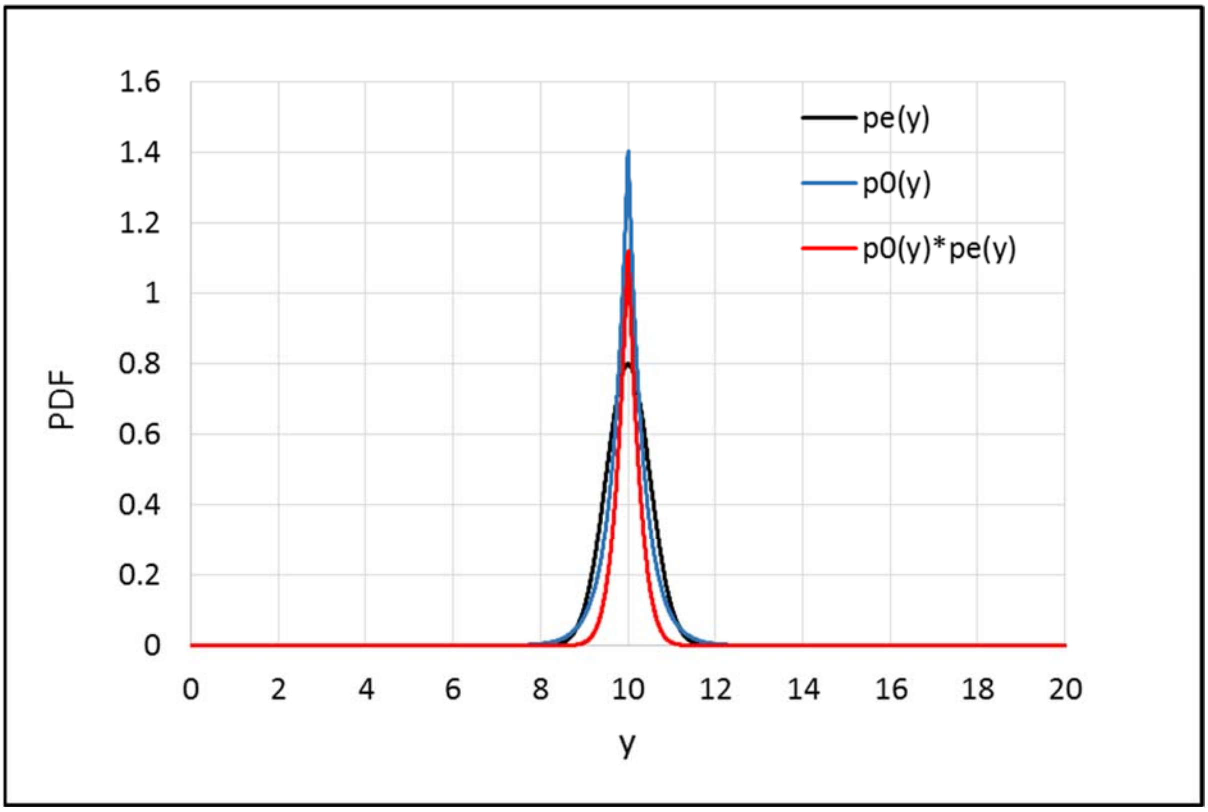


Fig. 14: *Plots of $p_e(y)$, $p_0(y)$, and their product $p_0(y)p_e(y)$, for $y_e = y_0 = 10$, $s_0 = s_e = 0.5$. The kurtosis of the posterior PDF is > 3 because the probability density at higher and lower values relative to the mean value is enhanced by the influence of the Laplace PDF. A Laplace PDF exhibits broader "wings" than a Gaussian PDF with a comparable standard deviation.*

This analysis leads to the following two general conclusions for the Laplace – Gaussian case that are comparable to those expressed in Section 4 for the Uniform – Gaussian case:

- ***Conclusion***: *The posterior PDF kurtosis appears to fall between the kurtosis of the prior PDF and the likelihood PDF. The largest difference between the least-squares and Bayesian evaluated standard deviations corresponds to a posterior PDF kurtosis that is about midway between the prior PDF kurtosis and the likelihood PDF kurtosis.*

- ***Conclusion***: *Differences in the posterior PDF kurtosis relative to that of a Gaussian can have a significant effect on the differences between evaluated results obtained by the least-squares and Bayesian methods, even if the input data mean values are equal. The Bayes standard deviations appear to be systematically less than or equal to the least-squares values whenever the prior PDF is the Laplace function.*

The present investigation also considered a number of examples corresponding to $y_0 = 10$ and $s_0 = 0.5$ (5%), but with various values of s_e and $y_e \neq y_0$. A graphical rendition of the PDFs for a typical example, i.e., $s_e = 1$ and $y_e = 11$, appears in Fig. 15. The product PDF shape (shown in red in Fig. 15) appears as a slightly skewed version of a Laplace PDF shape. Notice that this appearance is not as strange as the comparable result for the Uniform – Gaussian case shown in Fig. 6 since the Laplace shape is not as severe (i.e., truncated) as the Uniform PDF shape.

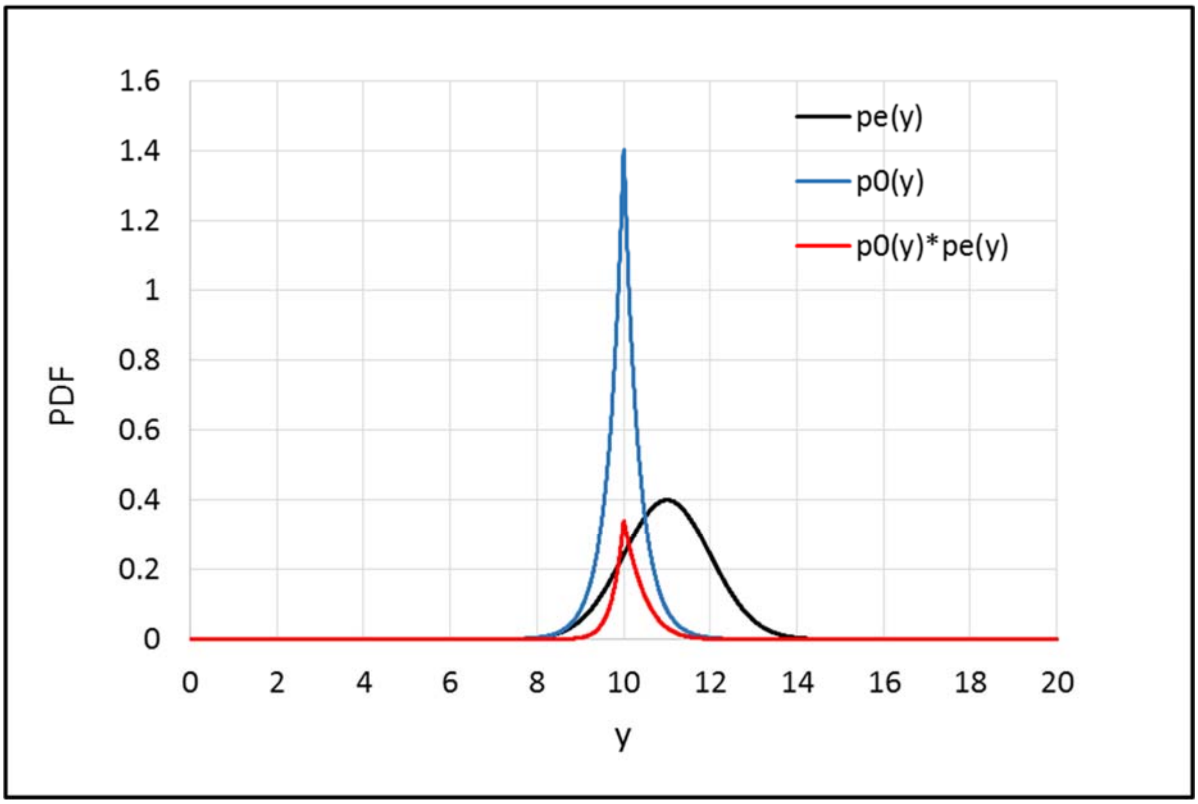


***Fig. 15**: Plots of $p_e(y)$, $p_0(y)$, and the product PDF $p_0(y)p_e(y)$ for $y_0 = 10$, $s_0 = 0.5$, $y_e = 11$, and $s_e = 1$. The prior PDF is Laplace and the likelihood is Gaussian.*

The considered examples are grouped into five categories based on standard deviation values $s_e = 0.1, 0.2, 0.5, 1$, and 1.5, respectively. All numerical results are compiled in the Appendix. What is shown here, in the same fashion as for the Uniform – Gaussian cases in Section 4, are systematic dependencies on the scaled displacement factor $(y_e\text{-}y_0)/s_e$ of ratios of the Bayesian and least-squares evaluated mean values y_{sol} and standard deviations s_{sol}, along with posterior PDF's skewness and kurtosis values. These results are shown in tables and figures below, with accompanying discussions.

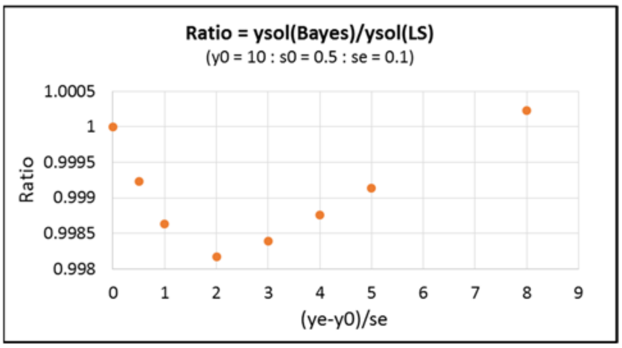
As stated before in Section 4, extreme displacement factors (indicated in red font) for the input mean values yield large evaluated chi-squares, small "Overlaps", and various other numerical anomalies due to the discrepant nature of the input data. Furthermore, there may be numerical biases associated with incomplete integration due to including only the range of variable "y" from 0 to 20. For these reasons, one should avoid drawing any conclusions from those tabulated values shown in red font even though they may be included in the tables and plots.

Laplace – Gaussian Examples with $s_e = 0.1$

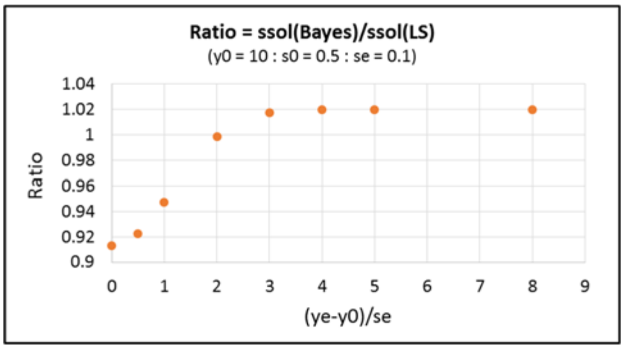
Results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.1$ are shown in Table 8 and Fig. 16 (A through D). Smooth variations with displacement $(y_e-y_0)/s_e$ are seen in the table and plots. The mean-value ratios agree very well with unity to < 0.2% for all $(y_e-y_0)/s_e$. The standard-deviation ratio differences from unity range from ≈ 0.2 to 9% for all $(y_e-y_0)/s_e$. Skewness values are all negligible. Kurtosis ranges systematically from ≈ 3 to 3.2.

***Table 8**: Tabulated Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.1$.*

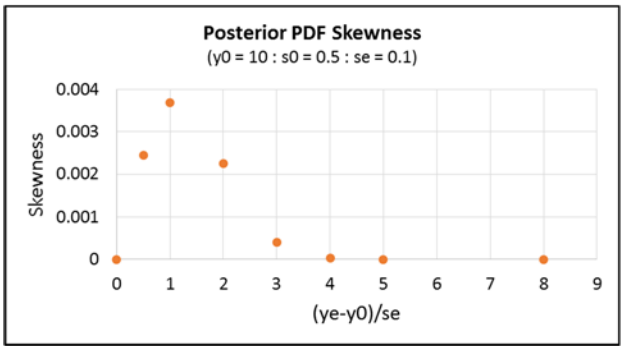
Examples with y0 = 10 : s0 = 0.5 : se = 0.1				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	0.912752121	7.97988E-15	3.224769698
0.5	0.999230737	0.922581605	0.002451288	3.167649471
1	0.998627786	0.94728445	0.0036858	3.045413572
2	0.998174379	0.998535854	0.002244878	2.919025964
3	0.998387812	1.017384887	0.000401466	2.970771383
4	0.998758295	1.019703236	2.30128E-05	2.997517152
5	0.999136185	1.01980238	4.43485E-07	2.999937156
8	1.000230746	1.019803903	1.40251E-14	3



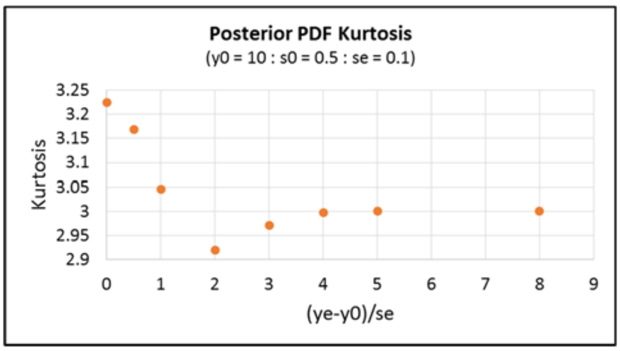
(A)



(B)



(C)



(D)

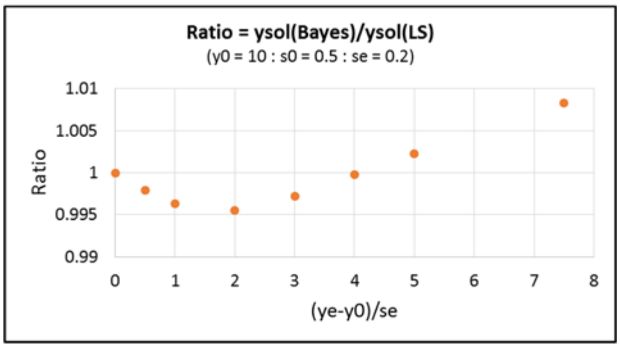
***Fig. 16**: Plots of Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.1$.*

Laplace – Gaussian Examples with $s_e = 0.2$

Results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.2$ are shown in Table 9 and Fig. 17 (A through D). Smooth variations with displacement $(y_e\text{-}y_0)/s_e$ are observed in the table and plots. The mean-value ratios agree with unity to within ≈ 0.2% for $(y_e\text{-}y_0)/s_e \leq 5$. The standard-deviation ratio differences from unity range from ≈ 2 to 13% for $(y_e\text{-}y_0)/s_e \leq 5$. Skewness is very small for all $(y_e\text{-}y_0)/s_e$. Kurtosis ranges systematically from ≈ 3 to 3.4.

***Table 9**: Tabulated Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.2$.*

Examples with y0 = 10 : s0 = 0.5 : se = 0.2				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	0.866421504	-2.14785E-15	3.445048739
0.5	0.997945106	0.881546349	0.010765613	3.360759258
1	0.996353037	0.9215827	0.017365124	3.165175906
2	0.99550824	1.019499629	0.01374437	2.877854055
3	0.997228113	1.068123763	0.003495564	2.92397044
4	0.999743657	1.076523351	0.000285535	2.990195951
5	1.002282695	1.077022472	7.65252E-06	2.999643968
7.5	1.008302365	1.077032961	8.74647E-12	2.999999999



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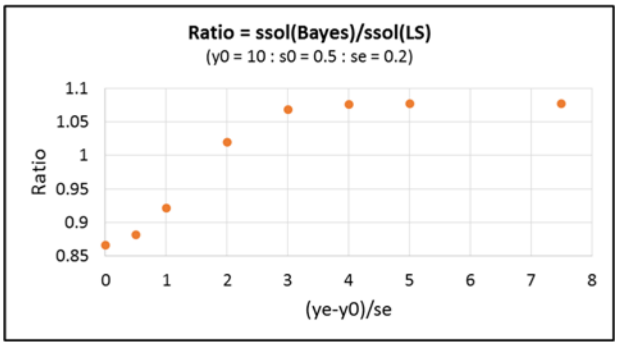
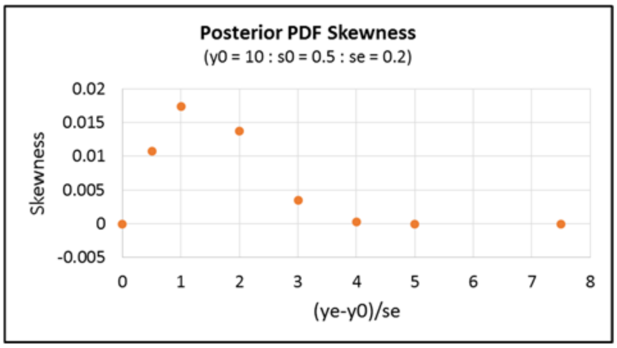


(B)



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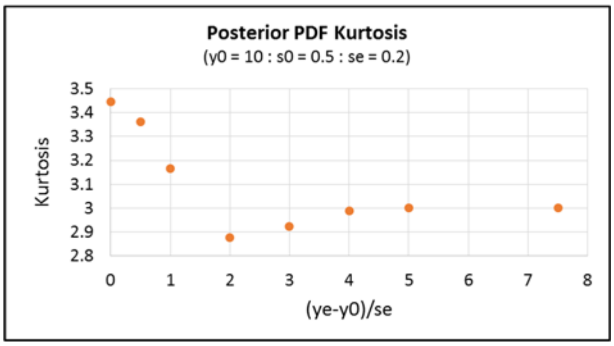


(D)

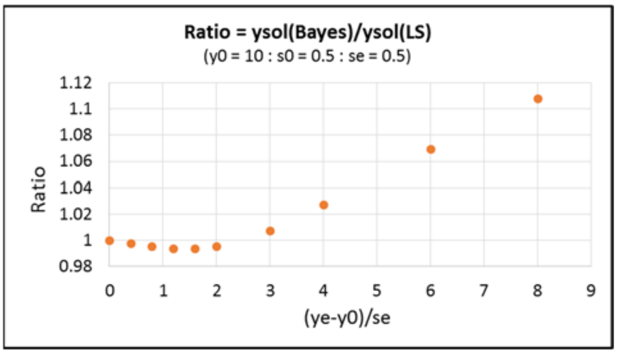
***Fig. 17**: Plots of Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.2$.*

Laplace – Gaussian Examples with $s_e = 0.5$

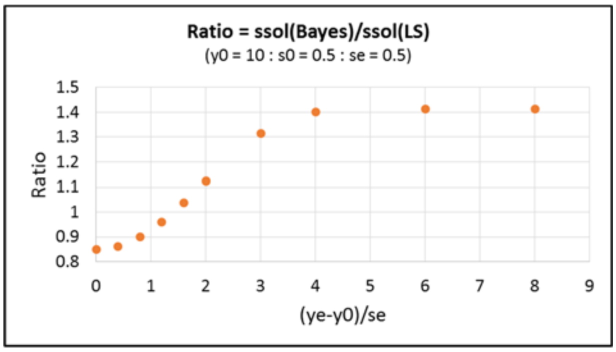
Results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.5$ are shown in Table 10 and Fig. 18 (A through D). Smooth variations with displacement $(y_e\text{-}y_0)/s_e$ are observed in the table and plots. The mean-value ratios agree with unity to within ≈ 0.7% for $(y_e\text{-}y_0)/s_e \leq 3$. The standard-deviation ratio differences from unity range from ≈ 4 to 32% for $(y_e\text{-}y_0)/s_e \leq 3$. Negligible to modest skewness values are seen for all $(y_e\text{-}y_0)/s_e$. Kurtosis ranges from ≈ 2.8 to 4 for $(y_e\text{-}y_0)/s_e \leq 3$.

Table 10: *Tabulated Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.5$.*

Examples with y0 = 10 : s0 = 0.5 : se = 0.5				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	0.849748734	-2.61885E-14	4.044833356
0.4	0.997320259	0.862503108	0.040612617	3.996444104
0.8	0.995122873	0.89993779	0.076883256	3.854314994
1.2	0.993825481	0.959421044	0.104872774	3.634404472
1.6	0.993824346	1.036112257	0.121259646	3.373428629
2	0.995451248	1.122548849	0.123709994	3.122285262
3	1.007325864	1.316229351	0.074912969	2.802108168
4	1.026951672	1.400154904	0.017201839	2.90074505
6	1.068947413	1.414200027	3.0506E-05	2.999630703
8	1.107741102	1.414213562	7.00348E-10	2.999999987



(A)



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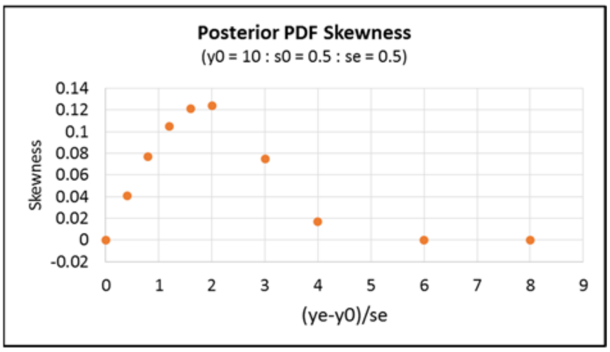

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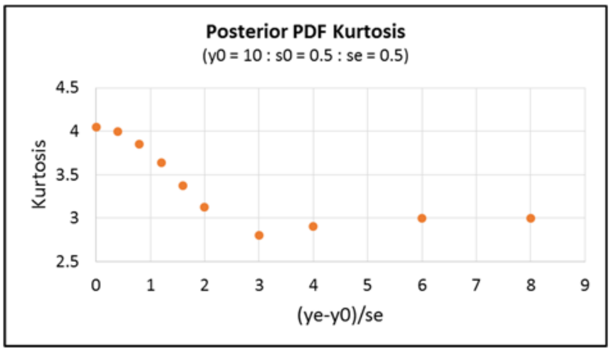


(D)

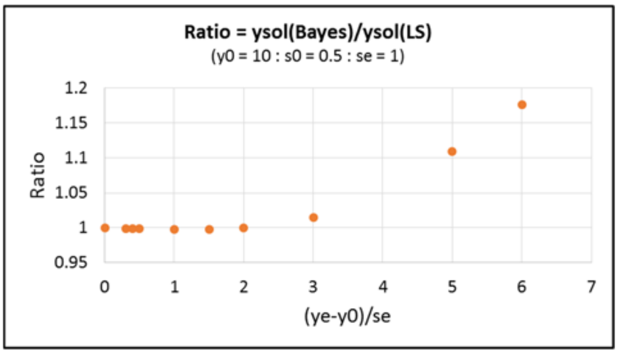
Fig. 18: *Plots of Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 0.5$.*

Laplace – Gaussian Examples with $s_e = 1$

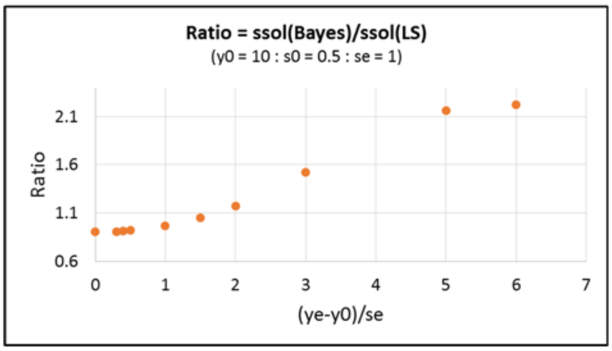
Results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1$ are shown in Table 11 and Fig. 19 (A through D). Smooth variations with displacement $(y_e\text{-}y_0)/s_e$ are observed in the table and plots. The mean-value ratios differ from unity by < 0.3% for $(y_e\text{-}y_0)/s_e \leq 2$. The standard-deviation ratio differences from unity range from ≈ 3 to 18% for $(y_e\text{-}y_0)/s_e \leq 2$. Noticeable skewness is seen for $(y_e\text{-}y_0)/s_e \geq 0.3$. Kurtosis ranges from ≈ 4.3 to 4.8 for $(y_e\text{-}y_0)/s_e \leq 2$.

***Table 11**: Tabulated Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1$.*

Examples with y0 = 10 : s0 = 0.5 : se = 1				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	0.904654339	4.32002E-14	4.770994208
0.3	0.998938173	0.910560161	0.055236153	4.768973698
0.4	0.998609159	0.915161107	0.073567666	4.766923094
0.5	0.998300529	0.921086208	0.091829874	4.763694149
1	0.99723627	0.970867578	0.181458713	4.714246512
1.5	0.997466476	1.055285448	0.266262596	4.567303279
2	0.999786963	1.175722398	0.341994059	4.291590832
3	1.014837975	1.518484919	0.42956845	3.486497854
5	1.108972955	2.164474085	0.127399471	2.807526153
6	1.176180842	2.229992624	0.016261884	2.956985745



(A)



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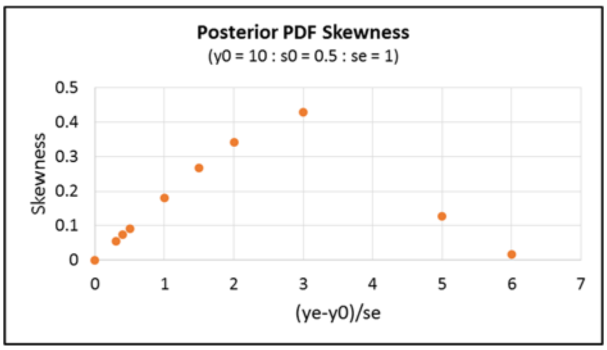

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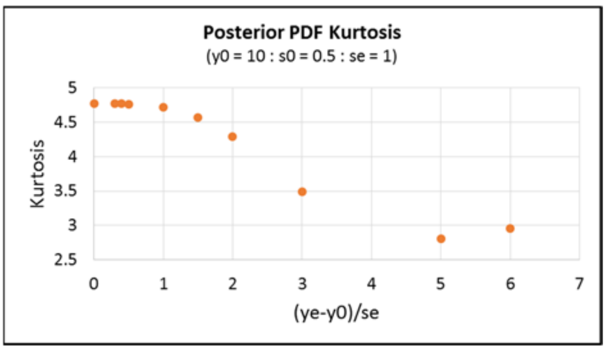

(D)

***Fig. 19**: Plots of Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1$.*

Laplace – Gaussian Examples with $s_e = 1.5$

Results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1.5$ are shown in Table 12 and Fig. 20 (A through D). Smooth variations with displacement $(y_e-y_0)/s_e$ are observed in the table and plots. The mean-value ratio differences from unity are < 0.2% for $(y_e-y_0)/s_e \leq 2$. The standard-deviation ratio differences from unity range from ≈ 1 to 14% for $(y_e-y_0)/s_e \leq 2$. Skewness increases noticeably for $(y_e-y_0)/s_e > 0.4$. Kurtosis ranges from ≈ 5.2 to 5.3 for $(y_e-y_0)/s_e \leq 2$.

***Table 12**: Tabulated Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1.5$.*

Examples with y0 = 10 : s0 = 0.5 : se = 1.5				
(ye-y0)/se	ysol(Bayes)/ysol(LS)	ssol(Bayes)/ssol(LS)	Skewness	Kurtosis
0	1	0.942017569	-2.54033E-14	5.205992566
0.2	0.999666669	0.943862824	0.039268487	5.210021623
0.4	0.999356148	0.949412645	0.07860543	5.221579412
0.6	0.999089398	0.958709355	0.118080452	5.239122188
1	0.998774311	0.988856897	0.19773268	5.281750267
1.5	0.999036589	1.04951517	0.299365839	5.313639764
2	1.000441535	1.138460322	0.404175656	5.26955386
3	1.009180877	1.418410461	0.620947963	4.815428794
5	1.082874312	2.441729547	0.778350477	3.085125698

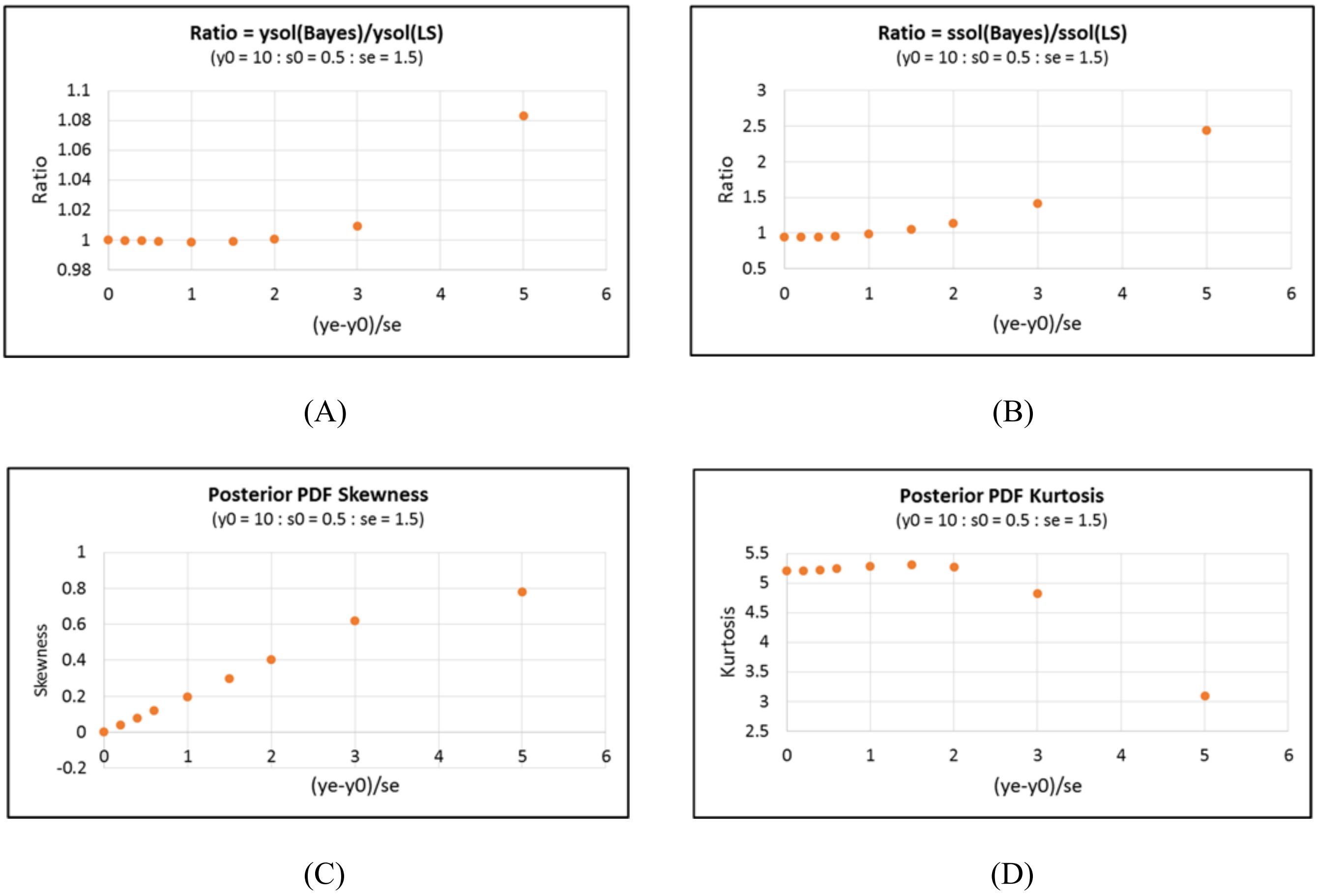

***Fig. 20**: Plots of Laplace – Gaussian results for $y_0 = 10$, $s_0 = 0.5$, and $s_e = 1.5$.*

An examination of the collected results from the preceding five hypothetical Laplace- Gaussian examples offers the possibility to arrive at some general conclusions about the evaluation of the two differently distributed data, one that is normal (Gaussian) and the other one that is Laplace. These conclusions are identical those reached from consideration of the Uniform - Gaussian examples in Section 3. They are repeated in this section for the reader's convenience, with "Laplace" replacing "Uniform" in the text.

- ***Conclusion***: *The derived Bayesian-to-least-squares mean-value and standard-deviation ratios, as well as skewness and kurtosis, for the collection of Laplace – Gaussian examples, are seen to vary smoothly with the scaled displacement* $(y_e\text{-}y_0)/s_e$ *for the five cases of* s_e *in these examples, as is seen in the tables and plots.*

- ***Conclusion***: *Generally, the derived mean-value ratios are reasonably close to unity in most of these examples except for extreme displacements.*

- ***Conclusion***: *Substantial differences are observed for the standard-deviation ratios. This is correlated with observed significant variations in the kurtosis of the posterior PDF rather than being traceable to skewness effects. The observed skewness values, if not always negligible, tend to be fairly modest for most of the studied examples.*

- ***Conclusion***: *Extreme values of displacement* $(y_e\text{-}y_0)/s_e$ *in the affected examples are generally seen to lead to unreasonable derived values for all four quantities. This phenomenon is traceable in some of these examples to the aforementioned incomplete integration phenomenon that occurs when the integration range is limited to 0 to 20.*

- ***Conclusion***: *For those examples that consider input data which are reasonably consistent, i.e., modest displacements* $(y_e\text{-}y_0)/s_e$*, the results of the present analysis, and the interpretations of these results, can be viewed as trustworthy.*

Analyses for all of the examples considered in the present investigation for the Laplace – Gaussian category generated chi-square values from the least-squares calculations and corresponding "Overlap" values from the Bayesian calculations, as described in Section 2. These results are presented in the Appendix in Tables A.5 and A.6.

Those examples for which very large chi-square values (chi-square/d.f. > 7) and very small "Overlap" values (Overlap < 0.03) were produced, correspond to hypothetical data that are clearly discrepant. The entries in all the tables for these discrepant cases are identified by means of red font. As indicated in the discussion above, one should not trust the evaluated results, nor draw any general conclusions from them, for these discrepant cases.

6. Summary and Conclusions

The present investigation was conducted to explore the influence of PDF kurtosis on evaluated data outcomes. Two independent data values are analyzed to produce an evaluated mean value and standard deviation as well as skewness and kurtosis. Three different combinations of input data PDFs are examined: Gaussian – Gaussian, Uniform – Gaussian, and Laplace – Gaussian. Each of the three assumed input data PDF types have symmetric shapes. Two different evaluation approaches are used. One is the conventional least-square approach which involves no explicit assumptions about the PDFs. The second is the Bayesian approach, in which PDF shape information is incorporated in evaluating the integrals involved in applying the method. Although the input data PDFs are symmetric, the posterior PDFs are asymmetric if the input data mean values differ.

If the two input data values that are used to produce an evaluated result are both assumed to be normally distributed (Gaussian – Gaussian), then the evaluated mean value and standard deviation produced by the Bayesian procedure will be identical to the least-squares results. Then, there is no advantage to using the more computationally intensive Bayesian approach. Otherwise, the Bayesian approach should be used since it accounts for PDF shape features.

If one of the input data values is assumed to have a Uniform PDF and the other a Gaussian PDF (Uniform – Gaussian), then significant differences may be observed between the results obtained from applying the two considered evaluation methods, depending on the details of the input data. Even when the two input data being evaluated have identical mean values, differences will arise in the standard deviations produced by the Bayesian and least-squares procedures. These differences are most pronounced when the two data points are equally weighted, as a consequence of equal input data standard deviation values, and these differences appear to be associated with differences in the input data PDF kurtosis values. If the input data mean values differ (data discrepancies), then matters become more complicated. However, differences in evaluated mean values tend to be modest, even for fairly discrepant input data. The evaluated standard deviation, skewness, and kurtosis values will vary noticeably with the discrepancy between the two input data values, although skewness tends to remain fairly modest. This is reflected in modest asymmetries observed in plots of the posterior PDFs.

The conclusions for the Laplace – Gaussian case are similar in most ways to those for the Uniform – Gaussian case. However, differences in the ratios of Bayesian and least-square standard deviations (greater than or less than unity) are observed when comparing the Uniform – Gaussian and Laplace – Gaussian data pairs. When the two input data mean values are equal, then the solution standard deviation ratios for Uniform – Gaussian data pairs are ≥ 1 whereas the corresponding ratios are ≤ 1 for Laplace – Gaussian data pairs. This appears to correlate with the fact that the kurtosis for a Uniform PDF is smaller than for a Gaussian PDF, whereas the Laplace PDF has a larger kurtosis than the Gaussian.

It was already demonstrated in earlier work that PDF shapes can have a significant impact on evaluated results obtained by the Bayesian method when compared to use of the least-squares technique [1,2]. The present investigation sheds further light on this matter by demonstrating that kurtosis impacts mainly the standard deviation whereas skewness affects the mean value.

Acknowledgments

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Appendix: Compiled Numerical Results

The numerical results of the evaluation exercises conducted in this investigation are tabulated here. They are organized in the three categories, as discussed in the main text of this report: Gaussian – Gaussian, Uniform – Gaussian, and Laplace – Gaussian. The values in red font correspond to those examples where the data sets (y_0,s_0) and (y_e,s_e) are very discrepant, as evidenced by the large chi-square values and very small values of "Overlap". Under these conditions, the solution values y_{sol} and s_{sol} should be treated as unreliable.

Gaussian Prior and Gaussian Likelihood Examples

Table A.1*: Evaluated results from least-squares and Bayesian analysis for various y_0, s_0, y_e, and s_e and the Gaussian – Gaussian assumption for the prior and likelihood PDFs.*

Input Data				Least-squares Solution			Bayesian Solution			
y0	s0	ye	se	ysol	ssol	Chi2/d.f.	ysol	ssol	skewsol	kurtsol
12	1	8	1	10	0.707107	8	10	0.707107	1.97E-14	3
12	2	8	2	10	1.414214	2	10	1.414214	-1.2E-13	3
11	1	9	1	10	0.707107	2	10	0.707107	4.64E-14	3
11	0.5	9	0.5	10	0.353553	8	10	0.353553	-4.4E-14	3
11	1	9	0.5	9.4	0.447214	3.2	9.4	0.447214	-9.8E-15	3
11	0.5	9	1	10.6	0.447214	3.2	10.6	0.447214	4.42E-15	3
11	2	9	2	10	1.414214	0.5	10	1.414214	-6.9E-14	3
11	2	9	0.2	9.019802	0.199007	0.990099	9.019802	0.199007	9.51E-15	3
10.5	1	9.5	1	10	0.707107	0.5	10	0.707107	1.37E-14	3
10.5	0.5	9.5	0.5	10	0.353553	2	10	0.353553	9.6E-15	3
10.2	0.2	9.8	0.2	10	0.141421	2	10	0.141421	-6E-15	3
10.1	0.2	9.9	0.2	10	0.141421	0.5	10	0.141421	-6E-15	3

Table A.2*: These results supplement those in Table A.1. The integrals are defined in Sections 1 and 2 of the main text. "Formula ∫$p_0p_e(y)$" corresponds to "Overlap" from Eq. (17).*

Input Data				Calculated			Formula	Ratio
y0	s0	ye	se	∫pedy	∫p0dy	∫p0pedy	∫p0pedy	∫p0pedy
12	1	8	1	1	1	0.005167	0.005167	1
12	2	8	2	0.999963	0.999968	0.051888	0.051888	1
11	1	9	1	1	1	0.103777	0.103777	1
11	0.5	9	0.5	1	1	0.010333	0.010333	1
11	1	9	0.5	1	1	0.072042	0.072042	1
11	0.5	9	1	1	1	0.072042	0.072042	1
11	2	9	2	0.999996	0.999997	0.109848	0.109848	1
11	2	9	0.2	1	0.999997	0.120982	0.120982	1
10.5	1	9.5	1	1	1	0.219696	0.219696	1
10.5	0.5	9.5	0.5	1	1	0.207554	0.207554	1
10.2	0.2	9.8	0.2	1	1	0.518884	0.518884	1
10.1	0.2	9.9	0.2	1	1	1.098478	1.098478	1

Uniform Prior and Gaussian Likelihood Examples

***Table A.3**: Evaluated results from least-squares and Bayesian analysis for various y_0, s_0, y_e, and s_e and the Uniform – Gaussian assumption for the prior and likelihood PDFs.*

Input Data				Least-squares Solution			Bayesian Solution			
y0	s0	ye	se	ysol	ssol	Chi2/d.f.	ysol	ssol	skewsol	kurtsol
10	0.5774	10	0.1	10	0.098058	0	10	0.1	3.36E-15	3
10	0.5774	10.05	0.1	10.04854	0.098533	0.007282	10.05	0.1	6.74E-15	3
10	0.5774	10.1	0.1	10.09709	0.098533	0.029126	10.1	0.1	-3.4E-15	3
10	0.5774	10.2	0.1	10.19417	0.098533	0.116505	10.2	0.1	-1.6E-14	3
10	0.5774	10.3	0.1	10.29126	0.098533	0.262136	10.3	0.1	-1.4E-11	3
10	0.5774	10.4	0.1	10.38835	0.098533	0.466019	10.4	0.1	-6.7E-09	2.999999
10	0.5774	10.5	0.1	10.48544	0.098533	0.728155	10.5	0.1	-1.1E-06	2.999837
10	0.5774	11	0.1	10.97087	0.098533	2.912621	10.9202	0.06027	-0.01474	3.870175
10	0.5774	10	0.2	10	0.188982	0	10	0.199999	-1.2E-14	2.999673
10	0.5774	10.1	0.2	10.08929	0.188982	0.026786	10.1	0.199993	-2.7E-05	2.998743
10	0.5774	10.2	0.2	10.17857	0.188982	0.107143	10.19997	0.199946	-0.00018	2.993032
10	0.5774	10.4	0.2	10.35714	0.188982	0.428571	10.39911	0.198662	-0.00322	2.916751
10	0.5774	10.6	0.2	10.53571	0.188982	0.964286	10.58895	0.188304	-0.01805	2.760912
10	0.5774	10.8	0.2	10.71429	0.188982	1.714286	10.74248	0.158704	-0.03742	3.001366
10	0.5774	11	0.2	10.89286	0.188982	2.678571	10.84042	0.120557	-0.04167	3.869427
10	0.5774	11.5	0.2	11.33929	0.188982	6.026786	10.93542	0.05964	-0.02355	6.445149
10	0.5774	10	0.5	10	0.377964	2.21E-29	10	0.439813	5.31E-14	2.365527
10	0.5774	10.2	0.5	10.11429	0.377964	0.068571	10.15276	0.431367	-0.06289	2.43796
10	0.5774	10.4	0.5	10.22857	0.377964	0.274286	10.29445	0.408115	-0.11384	2.646199
10	0.5774	10.6	0.5	10.34286	0.377964	0.617143	10.41757	0.375128	-0.14661	2.966416
10	0.5774	10.8	0.5	10.45714	0.377964	1.097143	10.51946	0.338013	-0.16162	3.367538
10	0.5774	11	0.5	10.57143	0.377964	1.714286	10.60116	0.301102	-0.16318	3.818156
10	0.5774	11.5	0.5	10.85714	0.377964	3.857143	10.73743	0.223084	-0.13862	4.990496
10	0.5774	12	0.5	11.14286	0.377964	6.857143	10.81338	0.16902	-0.10677	6.018423
10	0.5774	12.5	0.5	11.42857	0.377964	10.71429	10.85844	0.132807	-0.08128	6.803252
10	0.5774	13	0.5	11.71429	0.377964	15.42857	10.88718	0.10801	-0.06288	7.365675
10	0.5774	10	1	10	0.5	0	10	0.539559	3.56E-14	1.940905
10	0.5774	10.3	1	10.075	0.5	0.0675	10.08694	0.535845	-0.06712	1.983356
10	0.5774	10.4	1	10.1	0.5	0.12	10.1155	0.532996	-0.08865	2.016232
10	0.5774	10.5	1	10.125	0.5	0.1875	10.14373	0.529384	-0.10949	2.058319
10	0.5774	11	1	10.25	0.5	0.75	10.27721	0.501313	-0.19866	2.400818
10	0.5774	11.5	1	10.375	0.5	1.6875	10.39346	0.461397	-0.2561	2.937489
10	0.5774	12	1	10.5	0.5	3	10.48995	0.416475	-0.28166	3.614657
10	0.5774	13	1	10.75	0.5	6.75	10.62936	0.33103	-0.26799	5.102577
10	0.5774	15	1	11.25	0.5	18.75	10.77444	0.215766	-0.17596	7.194155
10	0.5774	16	1	11.5	0.5	27	10.81349	0.180769	-0.14043	7.705729
10	0.5774	10	1.5	10	0.538816	0	10	0.560364	3.78E-14	1.861773
10	0.5774	10.3	1.5	10.03871	0.538816	0.034839	10.04182	0.559476	-0.03558	1.871975
10	0.5774	10.6	1.5	10.07742	0.538816	0.139355	10.08338	0.556829	-0.07058	1.902534
10	0.5774	10.9	1.5	10.11613	0.538816	0.313548	10.12442	0.552478	-0.10445	1.953306
10	0.5774	11.5	1.5	10.19355	0.538816	0.870968	10.204	0.53905	-0.16683	2.114398
10	0.5774	12.25	1.5	10.29032	0.538816	1.959677	10.29678	0.514816	-0.23092	2.422553
10	0.5774	13	1.5	10.3871	0.538816	3.483871	10.38019	0.484731	-0.27666	2.838262
10	0.5774	14.5	1.5	10.58065	0.538816	7.83871	10.51624	0.417335	-0.3151	3.913507
10	0.5774	17.5	1.5	10.96774	0.538816	21.77419	10.68543	0.298403	-0.26788	6.248474

Table A.4: *These results supplement those in Table A.3. The integrals are defined in Sections 1 and 2 of the main text. "Formula ∫p0pe(y)" corresponds to "Overlap" from Eq. (17).*

Input Data				Calculated Integral			Formula	Ratio
y0	s0	ye	se	∫pe(y)	∫p0(y)	∫p0pe(y)	∫p0pe(y)	∫p0pe(y)
10	0.5774	10	0.1	1	1	0.5	0.680851	0.734375
10	0.5774	10.05	0.1	1	1	0.5	0.678377	0.737054
10	0.5774	10.1	0.1	1	1	0.5	0.671008	0.745148
10	0.5774	10.2	0.1	1	1	0.5	0.642323	0.778425
10	0.5774	10.3	0.1	1	1	0.5	0.597214	0.837221
10	0.5774	10.4	0.1	1	1	0.5	0.539333	0.92707
10	0.5774	10.5	0.1	1	1	0.5	0.473081	1.056902
10	0.5774	11	0.1	1	1	0.25	0.158703	1.575272
10	0.5774	10	0.2	1	1	0.5	0.652923	0.765787
10	0.5774	10.1	0.2	1	1	0.499998	0.644236	0.77611
10	0.5774	10.2	0.2	1	1	0.499984	0.618865	0.807905
10	0.5774	10.4	0.2	1	1	0.499325	0.526985	0.947512
10	0.5774	10.6	0.2	1	1	0.488626	0.403153	1.212013
10	0.5774	10.8	0.2	1	1	0.420676	0.277083	1.518231
10	0.5774	11	0.2	1	1	0.25	0.171087	1.461244
10	0.5774	11.5	0.2	1	1	0.003104	0.032075	0.096782
10	0.5774	10	0.5	1	1	0.47725	0.522338	0.913681
10	0.5774	10.2	0.5	1	1	0.468502	0.504733	0.928218
10	0.5774	10.4	0.5	1	1	0.441188	0.455398	0.968796
10	0.5774	10.6	0.5	1	1	0.393729	0.383655	1.026259
10	0.5774	10.8	0.5	1	1	0.327632	0.301794	1.085615
10	0.5774	11	0.5	1	1	0.249984	0.221666	1.127751
10	0.5774	11.5	0.5	1	1	0.079327	0.075925	1.04481
10	0.5774	12	0.5	1	1	0.011375	0.016941	0.671434
10	0.5774	12.5	0.5	1	1	0.000675	0.002462	0.274081
10	0.5774	13	0.5	1	1	1.58E-05	0.000233	0.067909
10	0.5774	10	1	1	1	0.341345	0.345494	0.987991
10	0.5774	10.3	1	1	1	0.330618	0.334028	0.989791
10	0.5774	10.4	1	1	1	0.322495	0.325374	0.991152
10	0.5774	10.5	1	1	1	0.312328	0.314576	0.992853
10	0.5774	11	1	1	1	0.238625	0.237454	1.00493
10	0.5774	11.5	1	1	1	0.151164	0.148595	1.017286
10	0.5774	12	1	1	1	0.078653	0.07709	1.020267
10	0.5774	13	1	1	1	0.011359	0.011822	0.960838
10	0.5774	15	1	1	1	1.58E-05	2.93E-05	0.540361
10	0.5774	16	1	0.999968	1	1.43E-07	4.74E-07	0.302584
10	0.5774	10	1.5	1	1	0.247508	0.24821	0.997169
10	0.5774	10.3	1.5	1	1	0.243284	0.243924	0.997377
10	0.5774	10.6	1.5	1	1	0.231038	0.231504	0.997985
10	0.5774	10.9	1.5	1	1	0.21197	0.212194	0.998941
10	0.5774	11.5	1.5	1	1	0.160826	0.16058	1.001529
10	0.5774	12.25	1.5	1	1	0.093599	0.093171	1.004593
10	0.5774	13	1.5	0.999998	1	0.04369	0.043482	1.004798
10	0.5774	14.5	1.5	0.999877	1	0.004846	0.004928	0.983416
10	0.5774	17.5	1.5	0.95221	1	3.67E-06	4.64E-06	0.790351

Laplace Prior and Gaussian Likelihood Examples

***Table A.5**: Evaluated results from least-squares and Bayesian analysis for various y_0, s_0, y_e, and s_e and the Laplace – Gaussian assumption for the prior and likelihood PDFs.*

Input Data				Least-squares Solution			Bayesian Solution			
y0	s0	ye	se	ysol	ssol	Chi2/d.f.	ysol	ssol	skewsol	kurtsol
10	0.5	10	0.1	10	0.098058	0	10	0.089503	7.98E-15	3.22477
10	0.5	10.05	0.1	10.04808	0.098058	0.009615	10.04035	0.090467	0.002451	3.167649
10	0.5	10.1	0.1	10.09615	0.098058	0.038462	10.0823	0.092889	0.003686	3.045414
10	0.5	10.2	0.1	10.19231	0.098058	0.153846	10.1737	0.097914	0.002245	2.919026
10	0.5	10.3	0.1	10.28846	0.098058	0.346154	10.27187	0.099763	0.000401	2.970771
10	0.5	10.4	0.1	10.38462	0.098058	0.615385	10.37172	0.09999	2.3E-05	2.997517
10	0.5	10.5	0.1	10.48077	0.098058	0.961538	10.47172	0.1	4.43E-07	2.999937
10	0.5	10.8	0.1	10.76923	0.098058	2.461538	10.77172	0.1	1.4E-14	3
10	0.5	10	0.2	10	0.185695	0	10	0.16089	-2.1E-15	3.445049
10	0.5	10.1	0.2	10.08621	0.185695	0.034483	10.06548	0.163699	0.010766	3.360759
10	0.5	10.2	0.2	10.17241	0.185695	0.137931	10.13532	0.171134	0.017365	3.165176
10	0.5	10.4	0.2	10.34483	0.185695	0.551724	10.29836	0.189316	0.013744	2.877854
10	0.5	10.6	0.2	10.51724	0.185695	1.241379	10.48809	0.198346	0.003496	2.92397
10	0.5	10.8	0.2	10.68966	0.185695	2.206897	10.68691	0.199905	0.000286	2.990196
10	0.5	11	0.2	10.86207	0.185695	3.448276	10.88686	0.199998	7.65E-06	2.999644
10	0.5	11.5	0.2	11.2931	0.185695	7.758621	11.38686	0.2	8.75E-12	3
10	0.5	10	0.5	10	0.353553	0	10	0.300432	-2.6E-14	4.044833
10	0.5	10.2	0.5	10.1	0.353553	0.08	10.07293	0.304941	0.040613	3.996444
10	0.5	10.4	0.5	10.2	0.353553	0.32	10.15025	0.318176	0.076883	3.854315
10	0.5	10.6	0.5	10.3	0.353553	0.72	10.2364	0.339207	0.104873	3.634404
10	0.5	10.8	0.5	10.4	0.353553	1.28	10.33577	0.366321	0.12126	3.373429
10	0.5	11	0.5	10.5	0.353553	2	10.45224	0.396881	0.12371	3.122285
10	0.5	11.5	0.5	10.75	0.353553	4.5	10.82875	0.465357	0.074913	2.802108
10	0.5	12	0.5	11	0.353553	8	11.29647	0.49503	0.017202	2.900745
10	0.5	13	0.5	11.5	0.353553	18	12.2929	0.499995	3.05E-05	2.999631
10	0.5	14	0.5	12	0.353553	32	13.29289	0.5	7E-10	3
10	0.5	10	1	10	0.447214	0	10	0.404574	4.32E-14	4.770994
10	0.5	10.3	1	10.06	0.447214	0.072	10.04932	0.407215	0.055236	4.768974
10	0.5	10.4	1	10.08	0.447214	0.128	10.06598	0.409272	0.073568	4.766923
10	0.5	10.5	1	10.1	0.447214	0.2	10.08284	0.411922	0.09183	4.763694
10	0.5	11	1	10.2	0.447214	0.8	10.17181	0.434185	0.181459	4.714247
10	0.5	11.5	1	10.3	0.447214	1.8	10.2739	0.471938	0.266263	4.567303
10	0.5	12	1	10.4	0.447214	3.2	10.39778	0.525799	0.341994	4.291591
10	0.5	13	1	10.6	0.447214	7.2	10.75728	0.679087	0.429568	3.486498
10	0.5	15	1	11	0.447214	20	12.1987	0.967982	0.127399	2.807526
10	0.5	16	1	11.2	0.447214	28.8	13.17323	0.997283	0.016262	2.956986
10	0.5	10	1.5	10	0.474342	0	10	0.446838	-2.5E-14	5.205993
10	0.5	10.3	1.5	10.03	0.474342	0.036	10.02666	0.447713	0.039268	5.210022
10	0.5	10.6	1.5	10.06	0.474342	0.144	10.05352	0.450346	0.078605	5.221579
10	0.5	10.9	1.5	10.09	0.474342	0.324	10.08081	0.454756	0.11808	5.239122
10	0.5	11.5	1.5	10.15	0.474342	0.9	10.13756	0.469056	0.197733	5.28175
10	0.5	12.25	1.5	10.225	0.474342	2.025	10.21515	0.497829	0.299366	5.31364
10	0.5	13	1.5	10.3	0.474342	3.6	10.30455	0.540019	0.404176	5.269554
10	0.5	14.5	1.5	10.45	0.474342	8.1	10.54594	0.672811	0.620948	4.815429
10	0.5	17.5	1.5	10.75	0.474342	22.5	11.6409	1.158214	0.77835	3.085126

Table A.6: *These results supplement those in Table A.5. The integrals are defined in Sections 1 and 2 of the main text. "Formula ∫p_0p_e(y)" corresponds to "Overlap" from Eq. (17).*

Input Data				Calculated			Formula	Ratio
y0	s0	ye	se	∫pedy	∫p0dy	∫p0pedy	∫p0pedy	∫p0pedy
10	0.5	10	0.1	1	0.999992	1.144093	0.78239	1.462305
10	0.5	10.05	0.1	1	0.999992	1.116408	0.778638	1.433796
10	0.5	10.1	0.1	1	0.999992	1.041646	0.767488	1.357214
10	0.5	10.2	0.1	1	0.999992	0.829157	0.724463	1.144513
10	0.5	10.3	0.1	1	0.999992	0.629743	0.658047	0.956988
10	0.5	10.4	0.1	1	0.999992	0.474825	0.575167	0.825542
10	0.5	10.5	0.1	1	0.999992	0.35785	0.483758	0.739731
10	0.5	10.8	0.1	1	0.999992	0.153176	0.228511	0.670322
10	0.5	10	0.2	1	0.999992	0.948621	0.740817	1.280506
10	0.5	10.1	0.2	1	0.999992	0.908126	0.728154	1.247162
10	0.5	10.2	0.2	1	0.999992	0.801172	0.691448	1.158686
10	0.5	10.4	0.2	1	0.999992	0.521306	0.562219	0.927228
10	0.5	10.6	0.2	1	0.999992	0.303449	0.398244	0.761969
10	0.5	10.8	0.2	1	0.999992	0.172694	0.245748	0.702728
10	0.5	11	0.2	1	0.999992	0.098092	0.132108	0.742513
10	0.5	11.5	0.2	1	0.999992	0.023848	0.015309	1.557753
10	0.5	10	0.5	1	0.999992	0.604688	0.56419	1.071781
10	0.5	10.2	0.5	1	0.999992	0.574638	0.542067	1.060086
10	0.5	10.4	0.5	1	0.999992	0.49402	0.480771	1.027559
10	0.5	10.6	0.5	1	0.999992	0.386256	0.393622	0.981288
10	0.5	10.8	0.5	1	0.999992	0.277092	0.297493	0.931424
10	0.5	11	0.5	1	0.999992	0.184622	0.207554	0.889512
10	0.5	11.5	0.5	1	0.999992	0.053481	0.059465	0.899364
10	0.5	12	0.5	1	0.999992	0.013398	0.010333	1.2966
10	0.5	13	0.5	1	0.999992	0.000794	6.96E-05	11.40049
10	0.5	14	0.5	1	0.999992	4.69E-05	6.35E-08	738.9498
10	0.5	10	1	1	0.999992	0.361181	0.356825	1.012207
10	0.5	10.3	1	1	0.999992	0.347846	0.344208	1.010571
10	0.5	10.4	1	1	0.999992	0.337823	0.334703	1.009322
10	0.5	10.5	1	1	0.999992	0.32537	0.322868	1.007747
10	0.5	11	1	1	0.999992	0.238229	0.239187	0.995995
10	0.5	11.5	1	1	0.999992	0.142444	0.145074	0.981871
10	0.5	12	1	1	0.999992	0.070158	0.072042	0.973857
10	0.5	13	1	1	0.999992	0.010104	0.00975	1.036309
10	0.5	15	1	1	0.999992	5.51E-05	1.62E-05	3.40314
10	0.5	16	1	0.999968	0.999992	3.29E-06	1.99E-07	16.54452
10	0.5	10	1.5	1	0.999992	0.253144	0.252313	1.003291
10	0.5	10.3	1.5	1	0.999992	0.248572	0.247812	1.003066
10	0.5	10.6	1.5	1	0.999992	0.23535	0.234785	1.002406
10	0.5	10.9	1.5	1	0.999992	0.214868	0.214578	1.001354
10	0.5	11.5	1.5	1	0.999992	0.160623	0.160882	0.998392
10	0.5	12.25	1.5	1	0.999992	0.091158	0.091668	0.994434
10	0.5	13	1.5	0.999998	0.999992	0.041422	0.041707	0.993154
10	0.5	14.5	1.5	0.999877	0.999992	0.004488	0.004396	1.020927
10	0.5	17.5	1.5	0.95221	0.999992	5.67E-06	3.28E-06	1.72769

Nuclear Data Section	E-mail: nds.contact-point@iaea.org
International Atomic Energy Agency	Fax: (43-1) 26007
Vienna International Centre, P.O. Box 100	Telephone: (43-1) 2600 21725
A-1400 Vienna, Austria	Web: http://nds.iaea.org