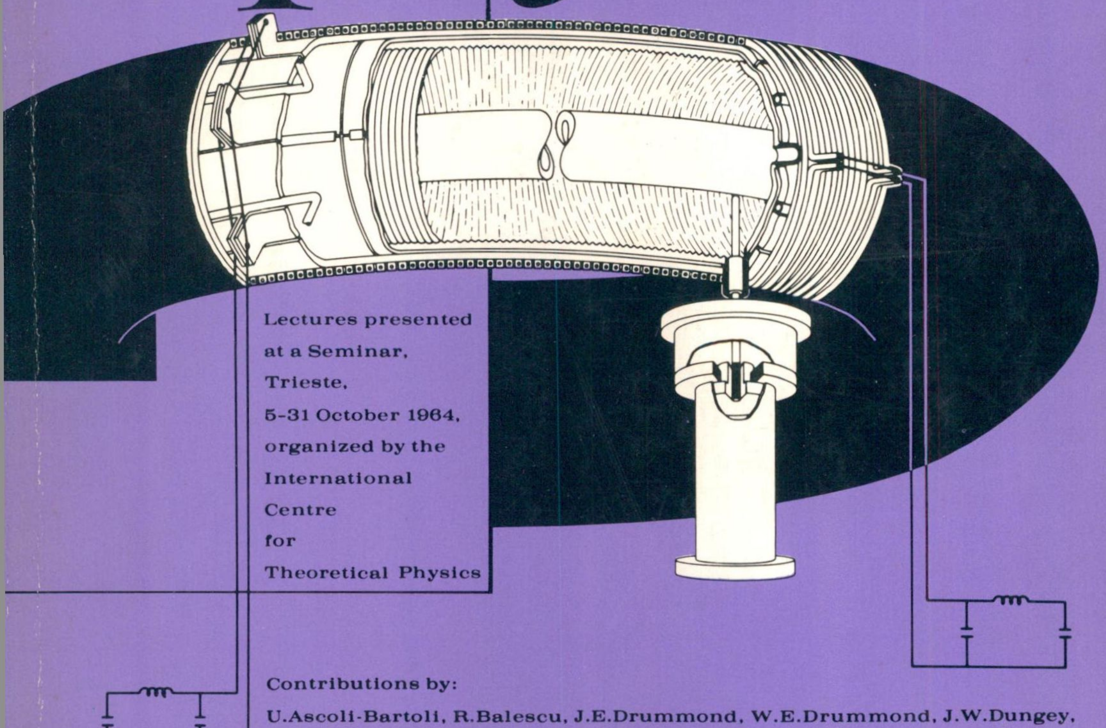
plasma physics



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INTERNATIONAL ATOMIC ENERGY AGENCY, VIENNA, 1965

PLASMA PHYSICS

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PLASMA PHYSICS

LECTURES PRESENTED AT THE SEMINAR ON
PLASMA PHYSICS ORGANIZED BY AND HELD AT THE
INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS, TRIESTE
FROM 5 - 31 OCTOBER 1964

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FOREWORD

The International Seminar on Plasma Physics held in Trieste during 5 - 31 October 1964 was the first major activity of the International Atomic Energy Agency's new International Centre for Theoretical Physics. In bringing together plasma physicists belonging to three distinct schools, the American, West European and the Soviet schools, the Seminar provided a unique opportunity for extended contacts between physicists in this field. It is hoped that these Proceedings will be of permanent value in the literature of the subject.

EDITORIAL NOTE

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CONTENTS

INTRODUCTION

In an attempt to be true to the principles upon which the International Atomic Energy Agency's new International Centre for Theoretical Physics at Trieste is founded, the Seminar programme was ambitiously designed to serve three purposes: (a) to introduce the subject to students, primarily from developing countries, who, while scientifically skilled, lacked previous specialized experience in plasma physics; (b) to introduce young researchers in this field to the paramount problems and the techniques proposed for their solution; and (c) to provide an opportunity for colloquy among the experts.

To meet these needs, the first part of this material is primarily didactic and attempts to give the reader an adequate introduction to basic plasma physics. This section includes an introduction to the basic concepts and fundamental processes in a plasma, the theory of plasma waves both in the macroscopic, or magnetohydrodynamic (MHD) approximation, and in the more fundamental, self-consistent Vlasov approximation, and a detailed description of the motion of a charged particle in an electromagnetic field. With this as a basis, the theory of plasma equilibrium and stability is formulated. Several lectures are devoted to experimental and observational aspects of laboratory and astrophysical plasmas. The topics of synchrotron radiation and bremsstrahlung, and MHD flow are treated.

The rest of the material is concerned principally with two topics of outstanding interest, a recently discovered class of highly stabilizing magnetic field configurations and the subject of turbulence in plasmas. The latter includes such topics as the quasi-linear theory of weakly turbulent systems and its application to enhanced diffusion and to collisionless shock structure, the kinetic theory of weakly unstable systems, as well as a novel approach to the problem of fully developed turbulence.

In addition, there are discussions on the generalizations of idealized descriptions of the plasma, including the effects of dissipation and of velocity-space instabilities.

It is hoped that this work will serve its intended purpose, providing a rapid, but fairly comprehensive introduction to the field of plasma theory, as well as exposing some of the important and unsolved problems and the techniques developed in attempting their solution.

INTRODUCTION

Le programme ambitieux de ces semaines d'études était fidèle à l'esprit qui a présidé à la création du nouveau Centre international de physique théorique de l'Agence internationale de l'énergie atomique à Trieste, puisqu'il visait les trois objectifs suivants: a) initier à la physique des plasmas les étudiants venus principalement des pays en voie de développement et qui, tout en possédant les bases scientifiques nécessaires, étaient dépourvues d'expérience spécialisée dans ce domaine; b) présenter les problèmes fondamentaux aux jeunes chercheurs et leur décrire les méthodes proposées pour leurs solutions; c) fournir l'occasion d'échanges de vue entre spécialistes.

A ces fins, la première partie du présent ouvrage est essentiellement didactique et a pour objet d'initier le lecteur à la physique fondamentale des plasmas. Elle expose succinctement les notions sur les concepts fondamentaux et les processus essentiels qui interviennent dans un plasma, sur la théorie des ondes du plasma selon l'approximation macroscopique, ou magnétohydrodynamique (MHD), et l'approximation plus cohérente et plus fondamentale encore de Vlassov, et donne une description détaillée du mouvement d'une particule chargée dans un champ magnétique. En prenant ces notions pour base, on formule la théorie de l'équilibre et de la stabilité du plasma. Plusieurs chapitres sont consacrés aux expériences et aux observations fondées sur les plasmas de laboratoire et les plasmas astrophysiques. On expose ensuite la théorie du rayonnement de synchrotron, du rayonnement de freinage et du flux MHD.

Le reste de l'ouvrage est consacré principalement à deux sujets d'intérêt capital: une catégorie récemment découverte de configurations de champs magnétiques qui ont un fort effet stabilisateur, et la turbulence dans le plasma. Ce dernier sujet englobe la théorie quasi linéaire des systèmes faiblement turbulents et son application à la diffusion accrue et à la structure de choc sans collision, la théorie cinétique des systèmes faiblement instables et une nouvelle manière d'aborder le problème de la turbulence complètement développée.

En outre, on trouvera une étude critique des généralisations des descriptions idéalisées du plasma, notamment sur les effets de la dissipation et des instabilités vitesse-espace.

On espère que cet ouvrage répondra à ses fins, car il offre une introduction succincte mais assez complète à la théorie des plasmas, et en expose certains des problèmes les plus importants non encore résolus ainsi que les méthodes imaginées pour tenter de leur trouver une solution.

ВВЕДЕНИЕ

В соблюдение принципов, на которых был создан Международный центр теоретической физики Агентства в Триесте, программа семинара была построена таким образом, чтобы служить трем целям: а) довести предмет до студентов главным образом из развивающихся стран, которые, хотя и имеют научную подготовку, испытывают недостаток в специальном опыте по физике плазмы; б) ввести молодых исследователей в эту область больших проблем и способов их решения; и в) предоставить возможность провести коллоквиум среди экспертов.

Вследствие этого первая часть настоящего материала является главным образом дидактической и представляет собой попытку ввести читателя в курс вопросов по основам физики плазмы. Этот раздел включает введение в основные концепции и фундаментальные процессы по плазме, теории волн плазмы как в макроскопическом или магнитогидродинамическом приближении, так и в наиболее фундаментальном, самосогласующемся приближении Власова, а также детальное описание движения заряженных частиц в электромагнитном поле. На основе этого формулируется теория равновесия плазмы и стабильности. Несколько лекций посвящено экспериментальным и обзорным аспектам лаборатории и астрофизических плазм. Трактуются вопросы синхротронной радиации и тормозного излучения и потока МГД.

Остальной материал касается главным образом двух вопросов, представляющих наибольший интерес, а именно — недавно открытого класса магнитного поля, обладающего высокостабилизованной конфигурацией, и турбулентности в плазме. Последний включает такие вопросы, как квазилинейная теория слабо турбулентных систем и ее применение к усиленной диффузии и к ударным структурам, не имеющим коллизий, кинетической теории слегка нестабильных систем, а также новый подход к проблеме полностью развитой турбулентности.

Кроме того, помещены дискуссии по вопросам обобщения идеальных описаний плазмы, в том числе эффектов диссипации и неустойчивостей за счет скоростей и пространственного распределения.

Выражается надежда, что данная работа послужит намеченным целям, обеспечив наиболее полную информацию по введению в область теории плазмы, а также укажет на некоторые наиболее важные проблемы, а также на методы, разрабатываемые с целью их решения.

INTRODUCCION

El extenso programa del Seminario correspondía fielmente a los principios que rigieron la fundación del nuevo Centro Internacional de Física Teórica de Trieste, del Organismo Internacional de Energía Atómica, ya que sus tres principales objetivos eran los siguientes: a) iniciar en la física del plasma a estudiantes, principalmente de países en desarrollo, que habían recibido una buena formación científica pero que carecían de conocimientos especializados en esta materia; b) exponer los problemas fundamentales de la física del plasma a los jóvenes investigadores y describirles los métodos propuestos para resolverlos; y c) dar a los especialistas la oportunidad de cambiar impresiones.

La primera parte de esta obra es esencialmente didáctica y su finalidad es iniciar al lector en los fundamentos de la física del plasma. Expone brevemente nociones sobre los conceptos básicos y los procesos fundamentales que se producen en un plasma, sobre la teoría de las ondas del plasma según la aproximación macroscópica o magnetohidrodinámica (MHD), y la aproximación aún más coherente y fundamental de Vlasov, y describe detalladamente el movimiento de una partícula cargada en un campo magnético. Basándose en esas nociones se formula la teoría del equilibrio y de la estabilidad del plasma. Varios capítulos están dedicados a las experiencias y a las observaciones fundadas en los plasmas de laboratorio y en los plasmas astrofísicos. A continuación se expone la teoría de la radiación sincrotrónica, de la de frenado y del flujo MHD.

El resto de la obra trata principalmente de dos temas de interés excepcional: una nueva clase de configuraciones de campos magnéticos con efectos estabilizadores muy pronunciados, descubierta recientemente, y la turbulencia en el plasma. Este último tema engloba la teoría cuasilineal de los sistemas de poca turbulencia y su aplicación a la difusión incrementada y a la estructura de choque sin colisiones, la teoría cinética de sistemas de poca inestabilidad, y una nueva manera de abordar el problema de la turbulencia totalmente desarrollada.

También se hace un análisis de las generalizaciones de las descripciones idealizadas del plasma, y muy especialmente de los efectos de la disipación y de las inestabilidades velocidad-espacio.

Se confía en que la obra resulte de gran utilidad, pues constituye una introducción breve pero bastante completa a la teoría de los plasmas, de la cual se exponen algunos de los problemas más importantes que están aún sin resolver, así como los métodos ideados para tratar de solucionarlos.

I

INTRODUCTION TO PLASMA PHYSICS

INTRODUCTION TO PLASMA PHYSICS

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I. DEFINITION

1. *Plasma physics*

Study of the property of matter so highly ionized that the dynamical behaviour of free charges dominates its behaviour usually refers to classical plasmas, in which ionization is due to high temperature and low density, and where particles behave classically, although the study of quantum plasmas in solids (metals, semi-conductors) is also important. In characteristic plasma systems ionization may often be assumed complete, so that inelastic collisions are unimportant.

II. RANGE OF APPLICATION

1. *Natural plasmas*

Ionized gases are found naturally throughout most of the universe except on the surface of cold planets, such as the earth. Ionization, however, begins in the upper layers of the earth's atmosphere where ionization produced by solar radiation was first detected by its effects on radio transmission and has since been intensively studied – this is the ionosphere. Above the ionosphere is a layer of diffuse ionized gas which can be studied through its effects on the earth's magnetic field, a study which has been supplemented by direct investigation by rocket and satellite flights. In the magnetosphere, satellite and rocket observations have revealed belts of high-energy particles (the van Allen belts) trapped in the earth's magnetic field.

Beyond the earth's environment, which might best be defined by the geometry of the magnetic field, satellite observations have confirmed what had been inferred from the occurrence of the aurora, that streams of plasma from the sun are impinging upon the magnetosphere. What was perhaps less expected is the presence of a small (10^{-4}–10^{-5} Gauss) magnetic field in interplanetary space, a field with small-scale variations which require a current carrying plasma stretching between the sun and the earth. On the sun itself temperatures are so high that throughout most of its volume matter is completely ionized, the major exception being the photosphere, where the principal constituent, hydrogen, has recombined. On the other hand, low-frequency electromagnetic interactions are unimportant except in sun-spots of magnetically active regions. Magnetic interactions appear to be responsible for solar flares of prominence – vast storms on the sun's surface – and determine many of the properties of the diffuse hot corona.

Magnetic fields have been detected on certain stars, but it is from radio astronomical observations that plasma behaviour has been revealed on a wide scale in the universe. Many strong radio sources have spectra characteristics, not of atomic transitions, but of magnetic bremsstrahlung, the radiation emitted by energetic electrons moving in a magnetic field.* Such radiation appears to come from interstellar regions in our galaxy, from neighbouring galaxies. It also comes strongly from gaseous nebulae which represent the residue of supernovae explosions, and finally, most strongly of all, from those remarkable and mysterious objects, the quasi-stellar radio sources.

2. *Technological plasmas*

Plasma physics is clearly essential for anything other than the most parochial understanding of nature, and must be central to any study of natural philosophy. Does it, on the other hand have any relevance to applied science? Here we must confess that promise has not yet been matched by fulfillment and that most of the spectacular applications seem to lie in the future.

Present applications of ionized gases depend simply on the fact that, when ionized, the gas will conduct electricity, and that ionization is produced rapidly once a critical electric field is exceeded. Major applications are in lighting where collisions between atoms and electrons provide an efficient method of converting electrical energy into light, in switching and voltage stabilization, where the breakdown of a gas when a critical field is exceeded is important, and in rectifying alternating current which depends on the same phenomenon.

One possible application of the characteristic dynamical properties of plasma is in the direct conversion of kinetic into electrical energy. This possibility arises, since an attempt to force a moving conductor through a magnetic field gives rise to an electric field $\vec{E} = -\vec{v} \times \vec{B}/c$, and if suitable contacts are provided this is capable of driving a current. If the conductor takes the form of a long wire wound on an armature, this is the familiar dynamo, but if it is a hot partially-ionized gas or flame flowing between electrodes, it is a magnetohydrodynamic generator which, in principle, should extract power from a high-temperature gas, hence at high thermodynamic efficiency.

Much research has gone into devices intended to exploit this principle, experimental devices using shock tubes have shown high efficiency, and generators producing large powers for short periods (a few minutes) have been developed.

A second possible application is in the provision of drive for interplanetary vehicles. Once clear of the earth's atmosphere the force needed to propel a vehicle through space becomes small, but the ultimate attainable velocity increases only logarithmically with the amount of propellant ejected, but linearly with its velocity of expulsion. Since chemical velocities are limited it seems practical to think of acquiring energy from the sun and ac-

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* This radiation forms a continuum and most important is polarized with its electric vector perpendicular to the magnetic field.

celerating the propellant to high velocities by electromagnetic forces. This makes sense only if the energy of ejection greatly exceeds the energy required to ionize the fluid. None the less, the advantage of high ejection velocity (specific impulse) seems to outweigh these difficulties and much study of plasma propulsion systems has been made. There is no doubt that specific impulse in the correct range can be fairly easily reached in plasma devices, but questions of weight, efficiency and above all, reliability, must be answered before any such device is flown.

Finally, the most spectacular and important application of plasma physics, and a major stimulus to its study, is the possibility of the controlled release of thermonuclear energy in a magnetically confined deuterium plasma. It seems, in principle, possible to confine a plasma by inducing currents in it, and allowing these currents to interact with a magnetic field in such a way that a pressure gradient is balanced by a gradient in magnetic energy, and the plasma is confined. When so confined by a non-material wall, the plasma can be heated by induced currents, and since the only mechanism for heat loss is radiation, which is inefficient from a transparent, fully ionized gas, the gas can be expected to reach temperatures of the order of 10^7 °K. At such high temperatures the energy of inter-particle collisions is sufficient to overcome the Coulomb barrier so that a significant nuclear reaction rate can be expected. At low energies the D–D reactions are exothermic and, if conditions are right, there may be a net energy gain. A thermonuclear reactor would differ significantly from most energy sources, since the energy circulating through the reactor is comparable to the energy produced, and such a device is a power source only if losses of the circulating energy can be held down and the device operated at high efficiency.

III. BASIC PROPERTIES OF PLASMA AND METHODS OF INVESTIGATION

1. *Electrical properties of plasma*

(a) Screening

Since a plasma contains free charges one expects it to act as a conductor and screen electric fields from its interior. Consider the plasma as two inter-penetrating gases of ions and electrons, and think of their equilibrium in a potential φ.

For electrons,

$$\vec{\nabla} p_- = kT\vec{\nabla} n_- = (n\,e)_- \vec{E} = -n\,|e|\vec{E} = n|e|\vec{\nabla}\varphi. \tag{1}$$

For ions,

$$\vec{\nabla} p_+ = kT\vec{\nabla} n_+ = (n\,e)_+\vec{E} = -ne\vec{\nabla}\varphi \tag{2}$$

for singly charged ions. Hence, the electron and the ion densities are given by the Boltzmann distribution law:

$$n_- = n_0 \exp e\varphi/kT \tag{3}$$

and

$$n_+ = n_0 \exp - e\varphi / kT . \tag{4}$$

The potential is determined by Poisson's equation

$$\nabla^2\varphi = -4\pi q = -4\pi(n_+e_+ - n_-e_-)$$

$$= 4\pi n_0 e[\exp(e\varphi/kT) - \exp(-e\varphi/kT)] \tag{5}$$

or

$$\nabla^2\varphi e/kT = (8\pi n_0 e^2/kT)\sinh(e\varphi/kT) = k_0^2 \sinh(e\varphi/kT) \tag{6}$$

where

$$k_0^2 = 8\pi n_0 e^2/kT . \tag{7}$$

For small $e\varphi/kT$

$$e\varphi/kT\ k_0^2 \sinh\ e\varphi/kT \simeq k_0^2 e\varphi/kT \tag{8}$$

and for a point source Ze, the solution of Eq.(6) is given by

$$\varphi = \frac{Ze}{r} e^{-k_0 r} . \tag{9}$$

If $r \gg Ze^2/kT$ charge is screened in distance of order $\lambda_0 \simeq (kT/8\pi n_0 e^2)^{\frac{1}{2}}$ where $\lambda_0 = 740(W/n)^{\frac{1}{2}}$ is the Debye screening length, W = temperature (in eV), n = density (in cm^{-3}).

(b) Langmuir Probe

Suppose a cold probe, i.e. an absorbing plate, is immersed in plasma, then ions and electrons will recombine on it. If there are no electric fields and the plasma is uncharged, $n_+ = n_-$ then a net current will flow to the plate given by

$$j_s = (ne)_+\left(\frac{kT}{2\pi m_+}\right)^{\frac{1}{2}} + (ne)_-\left(\frac{kT}{2\pi m_-}\right)^{\frac{1}{2}} = -ne\left(\frac{kT}{2\pi m_-}\right)^{\frac{1}{2}}\left[1 - \left(\frac{m_-}{m_+}\right)^{\frac{1}{2}}\right] . \tag{10}$$

If the plate draws no current, it must sit at some negative potential $-V_0$ with respect to the neutral plasma. All ions entering a region of thickness $\sim\lambda_0$ about the plate will reach it, and if the plate width is much greater than λ_0, the current density due to ions is $\simeq n_+(kT/2\pi m_+)^{\frac{1}{2}}$. Electrons must be held back by the potential and their density at the probe reduced to give the same flux as the ions, i.e.,

$$j_- = ne(kT/2\pi m_-)^{\frac{1}{2}} \exp(-eV_0/kT) . \tag{11}$$

Therefore, we must have

$$\exp(-eV_0/kT) = \left(\frac{m_+}{m_-}\right)^{\frac{1}{2}} \quad \text{or} \quad V_0 = \frac{1}{2}\frac{kT}{e}\log\frac{m_+}{m_-}. \tag{12}$$

This means that in a narrow region λ_0 about the plate, there is a strong electric field, k_0V_0 such that $E^2/8\pi \simeq n_0kT\log^2(m_+/m_-)$ in which there moves a stream of ions to the wall.

If now the potential of the wall is altered from $-V_0$, a current will be drawn of density

$$j = -ne(kT/2\pi m_-)^{\frac{1}{2}}[\exp(eV/kT) - \exp(-eV_0/kT)]$$

$$= ne(kT/2\pi m_-)^{\frac{1}{2}}\exp(eV_0/kT)[1 - \exp\Delta V/kT]; \tag{13}$$

hence $\log(1 - j/j_s) = e\Delta V/kT$, and the slope of a semi-logarithmic plot of the current drawn versus the applied voltage yields the electron temperature T. This semi-logarithmic plot will be straight only until all particles striking the wall are collected, when $j = j_s$ and the current saturates. The saturation current gives an estimate of the electron density n.

(c) Plasma conductivity

If the plasma is not in thermal equilibrium, the two component gases may move with respect to one another, and a current will flow. The simplest case occurs if the field is uniform and steady, for then the relative velocity is determined by the collisional interchange of momentum between the two species:

$$nm_-\nu(\vec{v}_- - \vec{v}_+) = (ne)_-\vec{E},$$

$$\vec{j} = ne_-\vec{v}_+ + ne_+\vec{v}_+ = (ne)_-(\vec{v}_- - \vec{v}_+)$$

$$= \frac{(ne_-)^2}{nm_-\nu}\vec{E} = \frac{ne_-^2}{m_-\nu}\vec{E} = \frac{1}{4\pi\nu}\omega_p^2\vec{E}. \tag{14}$$

Here, ν is a collision frequency for momentum interchange

$$\nu = n\sigma_0 v_- = n\sigma_0(kT_-/m_-)^{\frac{1}{2}}, \tag{15}$$

where σ_0 is the effective momentum transfer cross-section. Hence if the important collisions are between electrons and ions

$$\vec{j} = \frac{e^2}{m_-\sigma_0}(kT_-/m_-)^{-\frac{1}{2}}\vec{E}. \tag{16}$$

Thus in a fully ionized plasma the conductivity is (to a fair approximation) a function of temperature alone. In a slightly ionized gas where collisions are between electrons and neutrals this is not true. $\vec{j} = (n_-e^2/n_0\sigma_0)(kT_-/m)^{\frac{1}{2}}\vec{E}$ is a function of the fractional ionization (n_-/n_0). The quantity ω_p, $\omega_p^2 = (4\pi ne^2/m)$ is the Langmuir plasma frequency. $\nu_p = \omega_p/2\pi \simeq 8920\, n^{\frac{1}{2}}$.

(d) High frequency response of plasma

Consider the response of a plasma to a field varying slowly in space, but rapidly in time, and suppose the temporal variation to be sinusoidal, and the fields small, then the motion of electrons is given by

$$nm\frac{\partial \vec{v}_-}{\partial t} = -\vec{\nabla}p_- + (ne)_-\vec{E} + nm_-\nu(\vec{v}_+ - \vec{v}_-) . \tag{17}$$

If $\vec{v}_+$ and the pressure gradient may be neglected, we obtain

$$(i\omega + \nu)\, ne\vec{v}_- = (ne^2/m)\vec{E} , \tag{18}$$

where ω is the frequency of variation of the electric field. We thus obtain for the current density

$$\vec{j} = \frac{-i}{\omega - i\nu}\,\frac{ne^2}{m}\vec{E} ; \tag{19}$$

and from this, the polarization of the plasma

$$\vec{P} = \frac{\vec{j}}{i\omega} = -\frac{ne^2\vec{E}}{m\omega(\omega - i\nu)} . \tag{20}$$

Writing $\vec{D} = \vec{E} + 4\pi\vec{P} = \epsilon\vec{E}$, we obtain for the dielectric constant

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega - i\nu)} . \tag{21}$$

If the frequency $\omega \ll \nu$, the plasma is resistive, with a conductivity given by the d.c. value, but if the frequency is high, $\omega \gg \nu$, the plasma is reactive and

$$\epsilon = 1 - \frac{\omega_p^2}{\omega^2} . \tag{22}$$

By considering the behaviour of a plane parallel capacitor of area A, and plate separation L, for which $dV/dt = 4\pi LI/A\epsilon$, we see that the resistivity and reactivity are $-\mathrm{Im}\ \epsilon/\omega\epsilon\epsilon^*$ and $\mathrm{Re}\ \epsilon/\omega\epsilon\epsilon^*$ respectively. To study wave propagation, start from Maxwell's equations

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{23}$$

and

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}. \tag{24}$$

Assuming the space-time dependence of all quantities to be of the form $\exp i(\omega t+\vec{k}\cdot\vec{r})$, we obtain

$$i\vec{k}\times\vec{B} = \frac{4\pi}{c}\vec{j}+\frac{i\omega}{c}\vec{E} \tag{25}$$

and

$$i\vec{k}\times\vec{E} = -\frac{i\omega}{c}\vec{B}. \tag{26}$$

From these equations one readily obtains:

$$(k^2-\vec{k}\,\vec{k}\cdot)\vec{E}-\frac{\omega^2}{c^2}\vec{E} = -\frac{4\pi i\omega}{c^2}\vec{j} = -\frac{\omega_p^2}{c^2(1-i\nu/\omega)}\vec{E}. \tag{27}$$

For a solution ω and $\vec{k}$ must satisfy a dispersion relation. This is easily found by taking the scalar and vector products of Eq. (27) with $\vec{k}$. For longitudinal waves

$$\frac{\omega^2}{c^2}\vec{k}\cdot\vec{E} = \frac{\omega_p^2\,(\vec{k}\cdot\vec{E})}{c^2(1-i\nu/\omega)}. \tag{28}$$

hence

$$\omega^2/c^2\left[1-\frac{\omega_p^2}{\omega(\omega-i\nu)}\right]\vec{k}\;\vec{E}=0, \quad \text{i.e} \quad \epsilon=0$$

This leads to, if $\nu/\omega \ll 1$,

$$\omega^2 \simeq \omega_p^2\left(1+\frac{i\nu}{\omega}\right) \quad \text{and} \quad \epsilon=0. \tag{29}$$

For transverse waves

$$\left[k^2-\frac{\omega^2}{c^2}+\frac{\omega_p^2}{c^2}\frac{1}{(1-i\nu/\omega)}\right](\vec{k}\times\vec{E}) = 0 \tag{30}$$

and we obtain

$$k^2 \simeq \frac{\omega^2}{c^2}\left[1-\frac{\omega_p^2}{\omega^2}(1+i\nu/\omega)\right] = \frac{\omega^2\epsilon}{c^2} \tag{31}$$

If $\nu\to 0$ and $\omega>\omega_p$ transverse waves propagate with a phase velocity

$$\frac{\omega}{k} = \frac{c^2}{(1-\omega_p^2/\omega^2)^{\frac{1}{2}}} \tag{32}$$

and a group velocity

$$\frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}. \tag{33}$$

If $\omega < \omega_p$, transverse waves are damped in a distance of order $(1/k)$ where

$$\frac{1}{k} = \frac{c}{(\omega_p^2 - \omega^2)^{\frac{1}{2}}} \simeq \frac{c}{\omega_p} = \left(\frac{mc^2}{4\pi ne^2}\right)^{\frac{1}{2}} = \frac{1}{(4\pi n r_e)^{\frac{1}{2}}} = 6.6\times 10^5 n^{-\frac{1}{2}}, \tag{34}$$

where $r_e = e^2/mc^2$ is the classical electron radius. This is the collisionless screen length.

If $\omega < \nu$ the usual screening formula becomes valid. If $\omega = \omega_p$ the plasma will sustain longitudinal oscillations; these are Langmuir's plasma oscillations.

The plasma transmission characteristics, in particular the cut-off at $\omega = \omega_p$, provides a method of measuring electron density, and is so employed in the ionosonde which, by measuring the reflectivity of the ionosphere as a function of frequency, determines the increase of electron density with height.

IV. PLASMA PRODUCTION

1. *Thermal equilibrium of ionized gases*

Using the methods of statistical mechanics, it is possible to determine the degree of ionization in a gas in thermal equilibrium without considering the details of the ionization process, simply by observing that the probability of finding a configuration in a state i, in equilibrium at temperature T, is proportional to

$$P_i = g_i \exp - E_i/kT, \tag{35}$$

where E_i is the energy of the state i and g_i is the statistical weight of that state. We can then discover the fraction of the atoms which are ionized by using Eq. (35) to compare the probability of an electron and an ion existing as an independent pair with the probability of their existing as an atom.

The number of free particle states in the momentum range $d^3\vec{p}$ and configuration range $d^3\vec{x}$ is given by

$$dN = d^3\vec{x}\, d^3\vec{p}/h^3. \tag{36}$$

To get the effective statistical weight of a free particle of any momentum p we write

$$g_i = \frac{V}{h^3}(\exp - p^2/2mkT) d^3\vec{p}. \tag{37}$$

Integrating over all p, we obtain the partition function for a free particle in a volume V

$$g_i = \frac{V}{h^3} \int (\exp - p^2/2mkT)\, d^3\vec{p} = (2\pi mkT/h^2)^{3/2} V. \tag{38}$$

Now, if we neglect the small interaction potential between the free ion and electron, the difference in potential energy between the ion-electron pair and the atom is just the ionization energy eV_i. If the effective weights associated with internal degrees of freedom for the electron, ion, and neutral atoms are g_-, g_+ and g_0, and the ratio $g_- g_+/g_0 = g(T)$, then the ratio of the probability per unit volume of finding an electron and an ion to that of finding a neutral atom, which is equal to the ratio of the product of ion and electron density n_+ and n_- to the neutral density n_0, becomes

$$\frac{n_+ n_-}{n_0} = \left(\frac{2\pi m_+ kT}{h^2}\right)^{3/2} \left(\frac{2\pi m_- kT}{h^2}\right)^{3/2} \left(\frac{2\pi m_0 kT}{h^2}\right)^{-3/2} g \exp\left(-\frac{eV_i}{kT}\right). \tag{39}$$

Since the ion and atomic masses are almost equal, we may write

$$\frac{n_+ n_-}{n_0} = \left(\frac{2\pi m_- kT}{h^2}\right)^{3/2} g \exp\left(\frac{-eV_i}{kT}\right). \tag{40}$$

Equation (40) is a simple form of Saha's equation. If we express the temperature in electron volts and number densities in cm^{-3}, we may write this as

$$\frac{n_+ n_-}{n_0} = \gamma_i = \frac{3 \times 10^{21}}{n_0} T^{3/2} g \exp(-V_i/T). \tag{41}$$

For example, for caesium $V_i = 3.9$ V, it is not difficult to get a reasonable degree of ionization by heating caesium vapour to temperatures of the order of 2000°K.

2. Ionization behind shock waves

A shock tube is a device in which two parts of a long tube are separated by a diaphragm. On one side of the diaphragm is a hot compressed gas, and on the other side a diffuse gas. When the diaphragm is broken, the compressed gas expands with a speed $v_1 \simeq 2c_1/\gamma_1 - 1)$, where c_1 is the sound speed, $c_1^2 = \gamma_1 p_1/\rho_1 = \gamma_1 kT_1/m_1$, γ being the ratio of specific heats, m_1 the molecular weight, p_1, ρ_1, T_1, the pressure, density and temperature in the compressed gas. The expanding compressed gas pushes the rarified gas ahead of it with a velocity close to v_1, and ahead of the contact surface between the two gases runs a hydrodynamic discontinuity, a shock wave with speed $v_2 = (\gamma_2 + 1)v_1$, where γ_2 is the ratio of specific heats in the rarified gas. The temperature jump across the shock is such that the thermal speed behind it is of the order v_2, which is in term of order v_1, hence the temperature T_2 of the shocked gas is increased until $T_2/T_1 \sim m_2/m_1$. In fact, it may be shown that

$$\frac{T_2}{T_1} = \frac{\gamma_1(\gamma_2 - 1)}{2\gamma_2}\left(\frac{\gamma_2 + 1}{\gamma_1 - 1}\right)^2 \frac{m_2}{m_1}, \tag{42}$$

where m_2 is the molecular weight of the driver gas. If the driver gas is light, e.g. hydrogen, and the driven gas heavy, e.g. argon, then T_2/T_1 may be large, e.g. $\simeq 250$, and hydrogen at 1000°K would in the absence of ionization produce argon at 250 000°K, will above the ionization energy, hence the gas becomes fully ionized.

3. The positive column

If an electric current is passed through a gas confined to a tube, then ionization is produced in the body of the gas. The ions and electrons then diffuse to the walls and recombine thereon. Since the lifetime of the ion pairs is fairly short, the electrons which gain energy by falling in the electric field do not have time to share energy with the heavy ions. Diffusion rates are then determined by the electron temperature and the ion mass, while in a steady state the electron temperature is determined by the ionization rate.

Consider ions and electrons diffusing through a neutral gas, then if $nm\nu$ is the momentum transfer rate, and $\vec{E}$ is the electric field in plasma,

$$(nm\nu\vec{v})_- = -kT_-\vec{\nabla}n_- + (ne)_-\vec{E} \tag{43}$$

and

$$(nm\nu v)_+ = -kT_+\vec{\nabla}n_+ + (ne)_+\vec{E}. \tag{44}$$

Since the plasma is quasi neutral, i.e. $(ne)_- + (ne)_+ \simeq 0$, we may add (43) and (44), and for singly charged ions $n_+ = n_-$, hence

$$(nm\nu\vec{v})_- + (nm\nu\vec{v})_+ = -k(T_- + T_+)\vec{\nabla}n. \tag{45}$$

In most discharges ionization is produced by collisions between electrons and neutrals, and if σ_i is the ionization cross-section proceeds at a rate given by

$$\frac{dn}{dt} = 2\sqrt{\frac{2kT_-}{\pi m_-}}\, n_g n_- \frac{1}{(kT)^2}\int_{\epsilon_i}^{\infty} \sigma_i(\epsilon)\,(\exp - \epsilon/kT)\epsilon d\epsilon = \lambda(n_g, T_-)n_-. \tag{46}$$

In a steady state, the equation of continuity reads

$$\vec{\nabla}\cdot n\vec{v} = \lambda n, \tag{47}$$

and taking the divergence of (45) yields

$$\lambda n = -\frac{k(T_- + T_+)}{(m\nu)_- + (m\nu)_+} \nabla^2 n \tag{48}$$

or

$$D\nabla^2 n = \lambda n, \tag{49}$$

where D is the ambipolar diffusion coefficient.

$$D = k(T_- + T_+)/[(m\nu)_- + (m\nu)_+]. \tag{50}$$

If (49) is solved subject to suitable boundary conditions, i.e. $n > 0$, $n = 0$ on the walls of a containing vessel, then an eigenvalue problem is posed and the ratio λ/D determined by purely geometric conditions. The eigenvalue then determines the electron temperature as a function of neutral gas pressure and the size and shape of the containing vessel. This is the plasma balance equation; physically it implies that in a steady state the electron temperature must have that value which will produce ions fast enough to replace the diffusion loss to the walls of the containing vessel.

V. THE MAGNETIZED PLASMA

A plasma immersed in a magnetic field has much more complex properties than an unmagnetized plasma, e.g. it is anisotropic and has many resonant frequencies. For a brief survey of these phenomena let us first consider the motion of particles in a uniform magnetic field. We have

$$\dot{\vec{v}} = \frac{e}{mc}\vec{v}\times\vec{B}. \tag{51}$$

Since there is no force along the magnetic field,

$$v_{\parallel} = \text{constant}, \quad v_{\parallel} = \vec{v}\cdot\vec{b}, \tag{52}$$

where $\vec{b} = \vec{B}/|\vec{B}|$. We use a subscript $_\perp$ to denote the component of a vector perpendicular to the field lines. Then

$$\vec{v}_\perp = v_\perp[\cos(\Omega t + \varphi), \sin(\Omega t + \varphi)]. \tag{53}$$

Further integrations of Eqs. (52) and (53) give

$$x_{\parallel} = v_{\parallel} t \tag{54}$$

and

$$\vec{x}_\perp = \frac{v_\perp}{\Omega}[\sin(\Omega t + \varphi), -\cos(\Omega t + \varphi)] \tag{55}$$

2

where the constants of integration have been set equal to zero. The motion of the particle is a helix with radius $v_\perp/\Omega$.

The motion of each particle gives rise to a current loop with an associated dipole moment

$$\mu = IA = \frac{e\nu}{c}\pi r^2 = \frac{1}{2}\frac{mv_\perp^2}{B} \tag{56}$$

or

$$\vec{\mu} = -\frac{1}{2}\frac{mv_\perp^2}{B}\vec{b}. \tag{57}$$

Now, if the field is effectively uniform and the density varies with position, there is a variation in the magnetic moment density:

$$\vec{M} = -\frac{1}{2}\frac{nmv_\perp^2}{B}\vec{b}. \tag{58}$$

The corresponding current density is given by

$$\vec{j} = \vec{\nabla}\times\vec{M} = \frac{\vec{B}}{B^2}\times\vec{\nabla}\left(\frac{1}{2}nmv_\perp^2\right). \tag{59}$$

Therefore

$$\vec{j}\times\vec{B} = \vec{\nabla}_\perp\left(\frac{1}{2}nmv_\perp^2\right) = \vec{\nabla}_\perp p_\perp, \tag{60}$$

where $p_\perp = \frac{1}{2}nmv_\perp^2$ and $\vec{\nabla}_\perp = \vec{\nabla} - \vec{b}\vec{b}\cdot\vec{\nabla}$.

The pressure along the lines of force is given by $p_\parallel = mv^2{}_\parallel$. Thus the pressure is a tensor

$$\mathbb{P} = p_\parallel\vec{b}\vec{b} + p_\perp(\mathbb{I} - \vec{b}\vec{b}). \tag{61}$$

The condition of equilibrium is then given by

$$\vec{j}\times\vec{B} = \vec{\nabla}\cdot\mathbb{P}. \tag{62}$$

If now a steady electric field is applied to a plasma, the component of the electric field along $\vec{B}$ will accelerate particles along $\vec{B}$. The component of $\vec{E}$ normal to $\vec{B}$ will produce a drift velocity

$$\vec{v}_E = c\frac{\vec{E}\times\vec{B}}{B^2}. \tag{63}$$

Consequently $\vec{E} + (\vec{v}\times\vec{B})/c = 0$. This drift speed is shared by both ions and electrons, hence is the velocity of the plasma as a whole. The relation $\vec{E} + (\vec{v}\times\vec{B})/c = 0$ which demands that the force on a charge having velocity $\vec{v}$ shall vanish, is that held between fields in a perfect conductor. If the field E varies with time, $\vec{v}_E$ is not constant and contributes a term to the acceleration. This term may be annihilated by a further added velocity $\vec{v}_D$ such that

$$\vec{v}_E = \frac{e}{mc} \vec{v}_D \times \vec{B}, \tag{64}$$

$$\vec{v}_D = \frac{c}{\Omega B} \dot{\vec{E}} = \frac{mc^2}{eB^2} \dot{\vec{E}}. \tag{65}$$

This drift velocity gives rise to a current

$$\vec{j} = \Sigma ne\vec{v}_D = \Sigma \frac{nmc^2}{B^2} \dot{\vec{E}}_\perp . \tag{66}$$

The resulting polarization is given by

$$\vec{P} = \frac{\rho c^2}{B^2} \vec{E}_\perp . \tag{67}$$

Hence, for the low frequency fields directed across the magnetic field, the plasma has an effective dielectric constant

$$\epsilon = 1 + \frac{4\pi\rho c^2}{B^2}. \tag{68}$$

Thus, at low frequencies, a transverse wave propagates along the magnetic field with a phase velocity

$$V = \frac{c}{\epsilon^{1/2}} = c\left(1 + \frac{4\pi\rho c^2}{B^2}\right)^{-1/2}. \tag{69}$$

If $4\pi\rho c^2/B^2 \gg 1$, we get

$$V = \left(\frac{B^2}{4\pi\rho}\right)^{1/2} = c_A, \tag{70}$$

called the Alfvén speed. These slow waves may be thought of as vibrations of lines of magnetic force weighted by plasma. At higher frequencies, the propagation of transverse waves alter but is simply understood by considering the equation for transverse motion:

$$\dot{v}_x = \frac{e}{m} E_x + \Omega v_y \tag{71}$$

and

$$\dot{v}_y = \frac{e}{m} E_y - \Omega v_x . \tag{72}$$

If $E = E_0 \exp i\omega t$, then

$$v_x \pm iv_y = \frac{e}{m} \frac{1}{i(\omega \pm \Omega)} (E_x \pm iE_y), \tag{73}$$

and the polarization

$$4\pi P = -\omega_p^2 \frac{E_\pm}{\omega(\omega \pm \Omega)} . \tag{74}$$

Since the two circularly polarized components induce different polarization, the plasma is birefringent and Faraday rotation should be expected. Taking into account the contributions of both the ions and electrons, the polarization may be written

$$4\pi P_\pm = -\omega_p^2 \left[\frac{1}{\omega(\omega \pm \Omega_-)} + \frac{m}{M} \frac{1}{\omega(\omega \pm \Omega_+)}\right] E_\pm , \tag{75}$$

$$4\pi P_i = -\omega_p^2 \left\{\frac{1}{\omega[\omega + (-1)^i \Omega_-]} + \frac{m}{M} \frac{1}{\omega[\omega \pm (-1)^i \Omega_+]}\right\} E_i . \tag{76}$$

where $i = +1, -1, 0$ corresponds to right, left circularly polarized waves, and to polarization along B. From this, the dielectric coefficient may be readily obtained as a Hermitian tensor with three independent components. Taking the magnetic field along O_z

$$\epsilon_{zz} = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{m}{M}\right), \tag{77}$$

$$\epsilon_{xx} = \epsilon_{yy} = 1 - \omega_p^2 \left(\frac{1}{\omega^2 - \Omega_-^2} + \frac{m}{M} \frac{1}{\omega^2 - \Omega_+^2}\right) , \tag{78}$$

$$\epsilon_{xy} = -\epsilon_{yx} = i\omega_p^2 \left(\frac{\Omega_+}{\omega(\omega^2 - \Omega^2_-)} + \frac{m}{M} \frac{\Omega_+}{\omega(\omega^2 - \Omega^2_+)}\right). \tag{79}$$

To discuss wave propagation, this may be used in Maxwell's equations, in their reduced form

$$(k^2 - \vec{k} \cdot \vec{k})\vec{E} = \frac{\omega^2}{c^2} \vec{\epsilon} \cdot \vec{E} . \tag{80}$$

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MACROSCOPIC THEORY OF PLASMA WAVES

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It is of historical interest to note that the problem of plasma oscillations was first considered by Lord Rayleigh in 1906 [1] in connection with the electrical vibrations and the constitution of the atom. Rayleigh's formulation of the problem was in the following terms: "The cloud of electrons may then be assimilated to a fluid whose properties, however, must differ in many respects from those with which we are most familiar. We suppose that the whole quantities of positive and negative charges are equal. The difference between them is that the positive are constrained to remain undisplaced while the negative are free to move. In equilibrium, the negative distributes itself with uniformity throughout the sphere occupied by the positive so that the total density is everywhere zero. There is then no force at any point; but if the negative be displaced, a force is usually called into existence.....".

We then use these concepts to consider the simplest case of electron oscillations in a uniform plasma neglecting in the first instance the thermal motions of the particles. The ions are assumed to form a uniform fluid providing the neutralizing background for the electron fluid in equilibrium. Let N denote the electron density (which is equal to the ion density) in equilibrium. In the perturbed state let the density be denoted by $N + n$, where $n/N \ll 1$. The fluctuations in the particle density satisfy the continuity equation

$$\frac{\partial n}{\partial t} + N \vec{\nabla} \cdot \vec{v} = 0; \tag{1}$$

while $\vec{v}$ is given by the equation of motion

$$m \frac{\partial \vec{v}}{\partial t} = -e \vec{E}, \tag{2}$$

where $-e$ and m denote the charge and mass of the electron respectively. The electric field E which results due to the displacement of the particles, for longitudinal oscillations, is given by

$$\vec{\nabla} \cdot \vec{E} = -4\pi e n. \tag{3}$$

We thus have three equations describing the behaviour of the three unknowns $n, \vec{v}$ and E from which two of the variables can be eliminated to obtain for the equation governing the density fluctuations in the plasma:

$$\frac{\partial^2 n}{\partial t^2} = -\left(\frac{4\pi N e^2}{m}\right) n = -\omega_p^2 n, \tag{4}$$

where

$$\omega_p = (4\pi N e^2/m)^{\frac{1}{2}}, \tag{5}$$

is the so-called plasma frequency. It follows from Eq. (4) that

$$n(t) = n(0)\, e^{\pm i\omega_p t}. \tag{6}$$

Thus the density fluctuates sinusoidally with the characteristic frequency ω_p. The absence of space co-ordinates in Eq. (4) shows that these waves are non-dispersive — their group velocity vanishes — and there is no tendency for a wave packet of this type to propagate through the plasma. As a result we can specify the phases of the electron displacement in such a way as to obtain a travelling wave — a wave, however, which moves continuously through a fixed region without ever progressing beyond like the familiar barbar pole. Values of the plasma frequency for several plasmas of astronomical and laboratory interest are given in Table I. For comparison, the free space wavelengths of electromagnetic oscillations of the same frequency are given and it will be noted that electron plasma oscillations are a high frequency phenomenon.

1. EXPERIMENTAL EVIDENCE FOR PLASMA OSCILLATIONS

The theory of plasma oscillations was first given by Langmuir in 1929 who in the same paper presented experimental evidence for the occurrence of these oscillations in electric discharges. His measurements were made on a hot cathode mercury arc containing a rather complex electrode structure and designed for a survey of the possible oscillations of a plasma. In arc discharges, the electron density is 10^{11}-10^{12} cm^{-3} and the plasma frequency $\sim$ 100 Mc/s, so the high frequency signals were picked up on resonant Lecher wires, rectified by a crystal and detected by a galvanometer. Unfortunately, in the discharge used, it was not possible to make reliable measurements of the electron density, which had to be inferred from measurements of the gas pressure and the electric current, making use of a theory of the arc discharge; thus the published results while exhibiting oscillations in the correct range with roughly the correct relation between frequency and density, do not permit quantitative comparison between theory and experiment.

A later series of experiments by Merril and Webb (1939) were performed using a long mercury arc in which the electron density could be measured by the Langmuir probe technique. The oscillating signal was again detected by Lecher wires, crystal rectifier and galvanometer. Their results are presented in Table II.

TABLE I

PLASMA FREQUENCIES ($\nu = \omega_p/2\pi = 8920\ N^{1/2}$)

Plasma	Density (N)	Plasma frequency (ν) (sec^{-1})	Corresponding free space wavelength	Location in the electromagnetic spectrum
1. Interstellar gas	1 - 100	0.89×10^4 - 10^5	3×10^5 - 10^6 cm	long wave h. f.
2. Dense ionosphere, upper stellar atmosphere, tenuous laboratory plasma	10^{10} - 10^{12}	10^9 - 10^{10}	3 - 30 cm	u. h. f. μ-waves
3. Lower stellar atmosphere, laboratory plasma	10^{14} - 10^{16}	10^{11} - 10^{12}	0.03 - 0.3 cm	μ-waves, far infra-red
4. Dense laboratory plasma	10^{16} - 10^{18}	10^{12} - 10^{13}	0.003 - 0.03 cm	infra-red
5. Stellar interiors, metals	10^{22} - 10^{25}	10^{15} - 10^{16}	300 - 3000 A	visible, far ultra-violet

TABLE II

RESULTS OF PLASMA OSCILLATION MEASUREMENTS

N (probe)	Theoretical $\nu = 8920\ N^{1/2}$ (sec^{-1})	ν measured (sec^{-1})
1.77×10^{10}	1.2×10^{9}	1.18×10^{9}
2.56	1.44	1.44
3.33	1.64	1.50
1.93	1.25	1.17
3.09	1.58	1.34

2. EFFECT OF THERMAL MOTIONS

We now consider the effect of the finite temperature of the plasma on the electron oscillations. We shall assume that the electrons form a charged fluid obeying the basic hydrodynamical equations (cf. Ref. [2]):

$$\frac{\partial N}{\partial t} + \vec{\nabla} \cdot (N\vec{v}) = 0, \tag{7}$$

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} \cdot \mathbb{P} + \frac{1}{c} \vec{j} \times \vec{B} + \epsilon \vec{E}, \tag{8}$$

where $\rho = mn$, $\vec{j} = -Ne\vec{v}$, $\epsilon = -Ne$ and $\vec{E}$ and $\vec{B}$ are the electromagnetic fields which are, of course, governed by Maxwell's equations:

$$\vec{\nabla} \times \vec{B} = -\frac{4\pi}{c} Ne\vec{v} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \tag{9}$$

and

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}. \tag{10}$$

We need not consider the divergence equation for $\vec{E}$ as this is now a consequence of (7) and (9). The stress tensor $\mathbb{P}$ in the adiabatic approximation is governed by the equation:

$$\frac{\partial \mathbb{P}}{\partial t} + \vec{\nabla} \cdot (\vec{v}\mathbb{P}) + \mathbb{P} \cdot \vec{\nabla}\vec{v} + (\mathbb{P} \cdot \vec{\nabla}\vec{v})^{T} = 0. \tag{11}$$

We use the generalized equation of state instead of the usual adiabatic relation in the theory of neutral gases:

$$\frac{d}{dt}(p\rho^{-\gamma}) = 0. \tag{12}$$

The relation (11) reduces to (12) in case the pressure tensor is isotropic, i.e. it has the form

$$p_{ij} = p\delta_{ij}, \tag{13}$$

with $\gamma = 5/3$. The reason for the use of relation (11) instead of (12) in a plasma is that usually in most of the physical plasmas that one encounters, the densities are rather low. Thus while there may be some justification to assume the pressure to be a scalar in the initial state one must allow for the anisotropies in the pressure during the oscillations.

The equilibrium state is taken to be characterized by

$$\vec{v} = 0,\ N,\ \mathbb{P} = p\mathbb{I}, \tag{14}$$

where $p = N\Theta$, Θ being the kinetic temperature of the plasma measured in ergs. Let the perturbed state be characterized by

$$\vec{v},\ N + n,\ \mathbb{P} = p\mathbb{I} + \mathbb{p}_1, \tag{15}$$

where $\vec{v}$, n and $\mathbb{p}_1$ are all considered to be small so that terms quadratic in them may be neglected. The equations governing these quantities are the linearized forms of Eqs. (7) - (11). These are:

$$\frac{\partial n}{\partial t} + N\vec{\nabla}\cdot\vec{v} = 0, \tag{16}$$

$$mN\frac{\partial\vec{v}}{\partial t} = -\vec{\nabla}\cdot\mathbb{p}_1 - Ne\vec{E}, \tag{17}$$

$$\frac{\partial}{\partial t}\mathbb{p}_1 = -p(\vec{\nabla}\cdot\vec{v})\mathbb{I} - p\,[\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T], \tag{18}$$

$$\vec{\nabla}\times\vec{B} = -\frac{4\pi Ne}{c}\vec{v} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \tag{19}$$

and

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}. \tag{20}$$

Introducing $\vec{\xi}$, the Lagrangian displacement of an element of the electron fluid from its equilibrium position, to a first approximation we can write

$$\vec{v} = \frac{\partial \vec{\xi}}{\partial t}. \tag{21}$$

Eqs. (16) and (18) can then be readily integrated to give

$$n = -N\,\vec{\nabla}\cdot\vec{\xi}, \tag{22}$$

and

$$\mathbb{P}_1 = -p(\vec{\nabla}\cdot\vec{\xi})\mathbb{I} - p[\vec{\nabla}\vec{\xi} + (\vec{\nabla}\vec{\xi})^T], \tag{23}$$

where the constants of integration have, by definition, been set equal to zero. Since the coefficients of the linearized equations do not depend explicitly on time and space co-ordinates, one can carry out the normal mode analysis and, assuming the space time behaviour of all quantities to be of the form $\exp i(-\omega t + \vec{k}\cdot\vec{r})$, the foregoing equations then lead to:

$$\omega^2\vec{\xi} = 2\frac{\Theta}{m}\vec{k}\vec{k}\cdot\vec{\xi} + \frac{\Theta}{m}k^2\vec{\xi} + \frac{e}{m}\vec{E} \tag{24}$$

and

$$c^2\vec{k}\times(\vec{k}\times\vec{E}) = 4\pi Ne\,\omega^2\vec{\xi} - \omega^2\vec{E}. \tag{25}$$

For longitudinal oscillations $\vec{k}\times\vec{E} = 0$ and Eqs. (24) and (25) lead to the dispersion relation:

$$\omega^2 = \omega_p^2 + \frac{3\Theta}{m}k^2; \tag{26}$$

while for transverse oscillations $\vec{k}\cdot\vec{E}$ vanishes and we obtain

$$(\omega^2 - c^2k^2)\left(\omega^2 - \frac{\Theta}{m}k^2\right) - \omega^2\omega_p^2 = 0. \tag{27}$$

This equation leads to the following quadratic for ω^2

$$\omega^4 - \omega^2\left(\omega_p^2 + c^2k^2 + \frac{\Theta}{m}k^2\right) + \frac{\Theta}{m}c^2k^4 = 0. \tag{28}$$

The roots are given by

$$\omega^2 = \tfrac{1}{2}\left(\omega_p^2 + c^2k^2 + \frac{\Theta}{m}k^2\right) \pm \tfrac{1}{2}\left(\omega_p^2 + c^2k^2 + \frac{\Theta}{m}k^2\right)$$

$$\cdot \left\{1 - \frac{4(\Theta/m)c^2k^4}{[\omega_p^2 + c^2k^2 + \Theta k^2/m]^2}\right\}^{\frac{1}{2}}, \tag{29}$$

It is to be noted that if we neglect the temperature of the plasma, we obtain for transverse oscillations:

$$\omega^2 = \omega_p^2 + c^2k^2. \tag{30}$$

If we make the plausible assumption that

$$\omega^2 \gg k^2\frac{\Theta}{m}, \tag{31}$$

then to the lowest significant order, we obtain for the two transverse modes of oscillation (distinguished by the subscripts 1 and 2):

$$\omega_1^2 = \omega_p^2 + c^2k^2 + k^2\frac{\Theta}{m}\,\frac{\omega_p^2}{\omega_p^2 + c^2k^2} \tag{32}$$

and

$$\omega_2^2 = \frac{k^2\Theta}{m}\,\frac{1}{1 + (\omega_p^2/c^2k^2)}\,. \tag{33}$$

The root ω_1 corresponds to the transverse oscillation, however, the root ω_2 is incompatible with our basic approximation (31) and, therefore, must be discarded.

It may be remarked here that if, instead of using the equation for the stress tensor, we had used the classical adiabatic relation given by Eq. (12), we would have obtained for longitudinal oscillations the dispersion relation

$$\omega^2 = \omega_p^2 + \gamma\frac{\Theta}{m}k^2 \tag{34}$$

instead of the relation (26). However, in a dilute plasma, as we can show from the use of the kinetic equation, i.e. a Maxwellian plasma, the real part of the frequency is indeed given by Eq. (26). However, an essential consequence of the treatment from the kinetic equation in the absence of collisions, which is not recovered on using the truncated set of moment equations devoid of collision terms, is the phenomena of Landau damping. That is, the kinetic equation yields for isotropic equilibria a negative imaginary part effectively proportional to the number of particles moving with the phase velocity of the wave. This number, of course, tends to be exponentially small since, by assumption, the mean thermal speed is much less than the phase velocity of the wave.

It must be emphasized here that the longitudinal and transverse oscillations are strictly uncoupled only in the case of a non-relativistic plasma and in the absence of any external magnetic fields, temperature or density gradients. The presence of an external magnetic field or inhomogeneities in plasma density and/or temperature result in the coupling of longitudinal and transverse modes and the behaviour of the plasmas is, in general, quite complex.

3. ION OSCILLATIONS

So far we have considered only electron plasma oscillations on the assumption that these oscillations are too rapid for the heavy ions to follow, which implies that the ions, therefore, may be considered at rest. Another class of oscillations which are possible in a collision-free plasma is the so-called ion oscillations. These are so slow that electrons see them as quasi-static and consequently are distributed according to the Boltzmann distribution. During the oscillations it is appropriate to treat the ions and electrons as having different temperatures. We shall restrict here to the case of longitudinal oscillations only.

The linearized equations of motion now are:

$$\frac{\partial n_i}{\partial t} + N\vec{\nabla}\cdot\vec{v} = 0, \tag{35}$$

$$MN\frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\cdot\mathbb{P}_i + Ne\vec{E}, \tag{36}$$

$$\frac{\partial}{\partial t}\mathbb{P}_i = -p_i\,\vec{\nabla}\cdot\vec{v}\,\mathbb{I} - p_i\,[\vec{\nabla}\vec{v} + (\vec{\nabla}\vec{v})^T], \tag{37}$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi e(n_i - n_e) \tag{38}$$

and

$$n_e = N\left(\exp\frac{e\phi}{\Theta_e} - 1\right), \tag{39}$$

where n_i and n_e denote the perturbation in the ion and electron equilibrium densities (assumed equal) respectively, Θ_e is the electron temperature; $p_i = N\Theta_i$, where Θ_i is the ion temperature, $\mathbb{P}_i$ is the perturbation in the ion material stress tensor, and M is the ion mass. ϕ denotes the electrostatic potential defined as $\vec{E} = -\vec{\nabla}\phi$.

Introducing the Lagrangian displacement $\vec{\xi}$, we can integrate Eqs. (35) and (37) to obtain

$$n_i = -N\vec{\nabla}\cdot\vec{\xi} \tag{40}$$

and

$$\mathbb{P}_i = -p_i \vec{\nabla} \cdot \vec{\xi}\, \mathbb{I} - p_i [\vec{\nabla}\vec{\xi} + (\vec{\nabla}\vec{\xi})^T]; \qquad (41)$$

Eq. (38) can now be written as

$$\nabla^2 \phi = -4\pi e (n_i - n_e). \qquad (42)$$

If we assume that $e\phi/\Theta_e \ll 1$ (which is certainly true for small amplitude oscillations) then Eq. (39) can be written as

$$n_e = N\left(1 + \frac{e\phi}{\Theta_e} + \ldots\ldots - 1\right) = \frac{Ne\phi}{\Theta_e}. \qquad (43)$$

Assuming the space time dependence to be of the form $\exp i(-\omega t + \vec{k}\cdot\vec{r})$ we obtain from (42) and (43)

$$k^2 \phi = 4\pi e n_i - 4\pi \frac{Ne^2 \phi}{\Theta_e}. \qquad (44)$$

Introduce the Debye shielding distance as

$$\lambda_D = \left(\frac{\Theta_e}{4\pi N e^2}\right)^{\frac{1}{2}}; \qquad (45)$$

Eq. (44) can then be solved for ϕ to obtain

$$\phi = \frac{4\pi e n_i}{k^2[1 + (k\lambda_D)^{-2}]}. \qquad (46)$$

Eliminating n_i, $\mathbb{P}_i$ and $\vec{E}$ from Eqs. (36), (40), (41) and (46), we obtain the dispersion relation:

$$\omega^2 = \omega_{pi}^2 \frac{1}{1 + (k\lambda_D)^{-2}} + 3k^2 \frac{\Theta_i}{M}, \qquad (47)$$

where ω_{pi}, the ion plasma frequency, is given by

$$\omega_{pi} = \left(\frac{4\pi N e^2}{M}\right)^{\frac{1}{2}} = \left(\frac{m}{M}\right)^{\frac{1}{2}} \omega_p. \qquad (48)$$

If the wavelength of the disturbance is large compared to the Debye length, formula (47) reduces to

$$\omega^2 = \omega_{pi}^2 k^2 \lambda_D^2 + 3k^2 \frac{\Theta_i}{M}$$
$$= \frac{1}{M} (\Theta_e + 3\,\Theta_i)\, k^2. \tag{49}$$

These low frequency ion oscillations differ from the sound waves in the mechanism responsible for the organized motion. Ion oscillations are produced by long range Coulomb forces whereas sound waves are produced by short range collisions between the particles.

4. PLASMA IN AN EXTERNAL MAGNETIC FIELD

In the presence of an external magnetic field, the behaviour of a plasma, in general, is quite complex. There exist several resonance frequencies in the system. One can get a good physical picture of the various processes by neglecting the thermal motions of the particles, i.e., considering the case of a cold plasma. The motion of the constituents of the plasma, in general, will give rise to a current distribution $\vec{J}$. We shall, however, assume that the magnetic field which results from $\vec{J}$ is negligibly small compared with the external magnetic field $\vec{B}_0$.

With the neglect of the particle pressure, the linearized equations of motion are:

$$\frac{\partial \vec{v}}{\partial t} = \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}_0\right), \tag{50}$$

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (N\vec{v}) = 0, \tag{51}$$

where the perturbations in the electric ($\vec{E}$) and magnetic ($\vec{B}$) fields are given by the Maxwell equations

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}, \tag{52}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{53}$$

and

$$\vec{J} = \sum_{+,-} e N\vec{v}. \tag{54}$$

The summation in Eq. (54) is to be carried over both the species of the plasma. We define the cyclotron frequency to be

$$\omega_c = eB_0/mc\,; \tag{55}$$

the cyclotron frequency referring to the ions and electrons will be denoted by ω_i and $\omega_e = -(m/M)\omega_i$, respectively. We shall now assume that all quantities have space time dependence of the form $\exp i(-\omega t + \vec{k}\cdot\vec{r})$. Eqs. (50) and (51) then give

$$n = \frac{N}{\omega}\,(\vec{k}\cdot\vec{v}),\, -i\omega\vec{v} = \frac{e}{m}\,\vec{E} - \vec{\omega}_c \times \vec{v}. \tag{56}$$

From the equation for $\vec{v}$, one obtains

$$-i\omega\,\vec{\omega}_c \cdot \vec{v} = \frac{e}{m}\,\vec{\omega}_c \cdot \vec{E} \tag{57}$$

and

$$-i\omega\,\vec{\omega}_c \times \vec{v} = \frac{e}{m}\,\vec{\omega}_c \times \vec{E} - \vec{\omega}_c\,(\vec{\omega}_c \cdot \vec{v}) + \omega_c^2\,\vec{v}. \tag{58}$$

From Eqs. (56), (57) and (58), we then obtain

$$\vec{v} = \frac{e}{m}\,\frac{1}{\omega^2 - \omega_c^2}\left(i\omega\vec{E} + \vec{\omega}_c \times \vec{E} - \frac{i\vec{\omega}_c}{\omega}\,\vec{\omega}_c \cdot \vec{E}\right). \tag{59}$$

From Eqs. (52)-(54), we obtain

$$(\omega^2 - c^2k^2)\,\vec{E} + c^2\vec{k}\,(\vec{k}\cdot\vec{E}) + 4\pi i\omega \sum e N\vec{v} = 0, \tag{60}$$

where $\vec{v}$ is given by Eq. (59). On substituting for $\vec{v}$ in accordance with Eq.(59) into Eq. (60), we obtain

$$\mathbb{R}\cdot\vec{E} = 0, \tag{61}$$

where $\mathbb{R}$ is a 3×3 Hermitian matrix whose elements are given by

$$
\begin{aligned}
R_{11} &= \omega^2 - c^2k^2 - \sum \omega_p^2\left(1 - \frac{\omega_c^2 \cos^2\theta}{\omega_c^2 - \omega^2}\right),\\
R_{12} &= -R_{21} = \sum i\,\omega_p^2 \frac{\omega\omega_c \cos\theta}{\omega_c^2 - \omega^2},\\
R_{13} &= R_{31} = -\sum \omega_p^2 \frac{\omega_c^2 \sin\theta\cos\theta}{\omega_c^2 - \omega^2},\\
R_{22} &= \omega^2 - c^2k^2 + \sum \frac{\omega_p^2\omega^2}{\omega_c^2 - \omega^2},\\
R_{23} &= -R_{32} = i\sum \omega_p^2 \frac{\omega\omega_c \sin\theta}{\omega_c^2 - \omega^2},\\
R_{33} &= \omega^2 - \sum \omega_p^2\left(1 - \frac{\omega_c^2 \sin^2\theta}{\omega_c^2 - \omega^2}\right),
\end{aligned}
\tag{62}
$$

where $\omega_p^2 = 4\pi Ne^2/m$. In writing the foregoing expressions, we have taken $\vec{k}$ to be along the z-axis of a cartesian system of co-ordinates and we have assumed that the magnetic field lies in the xz-plane making an angle θ with the z-axis. In order that Eq. (31) has a non-trivial solution, we must demand that the secular equation

$$\|\mathbb{R}\| = 0 \tag{63}$$

be satisfied. This results in the required dispersion relation.

For any arbitrary direction of propagation vector $\vec{k}$ with respect to the magnetic field, the solutions of the dispersion relation are rather unwieldy and are given in [2]. The essential features of the problem can be obtained by considering the special cases of propagation along magnetic field and propagation transverse to the field.

4.1. Propagation along the magnetic field

For propagation along the magnetic field, $\theta = 0$ and Eq. (63) takes the particularly simple form

$$
\begin{vmatrix}
\omega^2 - c^2k^2 + \sum \frac{\omega_p^2\omega^2}{\omega_c^2 - \omega^2} & i\sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} & 0 \\
-i\sum \omega_p^2 \frac{\omega\omega_c}{\omega_c^2 - \omega^2} & \omega^2 - c^2k^2 + \sum \omega_p^2 \frac{\omega^2}{\omega_c^2 - \omega^2} & 0 \\
0 & 0 & \omega^2 - \sum \omega_p^2
\end{vmatrix}
\begin{vmatrix} E_x \\ E_y \\ E_z \end{vmatrix} = 0. \tag{64}
$$

It is clear that one of the roots of Eqs. (64) is

$$\left(\omega^2 - \sum \omega_p^2\right)E_z = 0. \tag{65}$$

This corresponds to the longitudinal oscillations; and for these the frequency of oscillation is given by

$$\omega^2 = \sum \omega_p^2 = \omega_{pe}^2 \left(1 + \frac{m}{M}\right) \simeq \omega_{pe}^2, \tag{66}$$

where ω_{pe} denotes the electron plasma frequency.

The transverse oscillations are circulary polarized and are described by the equation

$$\left(\omega^2 - c^2k^2 + \sum \omega_p^2 \frac{\omega}{\omega_c \mp \omega}\right)E_\pm = 0. \tag{67}$$

where

$$E_\pm = E_x \pm iE_y, \tag{68}$$

the plus or minus sign corresponding to the right- or left-handed circularly polarized waves, respectively. We shall henceforth consider only right-handed circularly polarized waves. It is clear that similar results will apply to the left-hand polarized waves also.

The oscillations are, therefore, governed by the dispersion relation

$$\omega^2 - c^2k^2 - \omega_{pe}^2 \frac{\omega}{\omega + \omega_e} - \omega_{pi}^2 \frac{\omega}{\omega - \omega_i} = 0. \tag{69}$$

Here the electron and the ion plasma frequencies are denoted by ω_{pe} and ω_{pi} respectively, while the electron and the ion cyclotron frequencies are denoted by $-\omega_e$ and $+\omega_i$ respectively. We now consider the following special cases.

(a) High frequency oscillations

Let us first consider the case when $\omega \gg \omega_{e,i}$. Then equation (69) yields, correct to the lowest significant order in ω_e/ω:

$$\omega^2 = c^2k^2 + \omega_{pe}^2\left(1 - \frac{\omega_e}{(\omega_{pe}^2 + c^2k^2)^{\frac{1}{2}}}\right). \tag{70}$$

The necessary condition for the validity of this result is that $\omega_e^2 \ll (\omega_{pe}^2 + c^2k^2)$.

The term containing ω_e on the right-hand side of Eq. (70) represents the correction term to the dispersion relation for transverse oscillations in a cold plasma in the weak magnetic field approximation.

(b) Oscillations near the electron cyclotron frequency

Another case of interest is the one when the frequency of oscillation is close to the electron cyclotron frequency, i.e. $\omega \simeq \omega_e \gg \omega_i$. Firstly we observe here that when $\omega \gg \omega_i$ Eq. (69) can be written as

$$\omega^2 - c^2k^2 - \omega_{pe}^2 \frac{\omega}{\omega + \omega_e} - \omega_{pi}^2 = 0. \tag{71}$$

For the root near $\omega = -\omega_e$, Eq. (71) yields

$$\omega = -\omega_e\left(1 + \frac{\omega_{pe}^2}{\omega_e^2 - (c^2k^2 + \omega_{pi}^2)}\right). \tag{72}$$

It is clear that the necessary condition for this approximation to be valid is that

$$\omega_{pe}^2 < \left|\omega_e^2 - (c^2k^2 + \omega_{pi}^2)\right|. \tag{73}$$

(c) Oscillations near the ion cyclotron frequency

Let us now consider the case when $\omega \simeq \omega_i$. This implies that $\omega < \omega_e$ and the dispersion relation (69) reduces to

$$\omega^2 - c^2k^2 - \omega_{pi}^2 \frac{\omega}{\omega - \omega_i} = 0. \tag{74}$$

This leads to the frequency of oscillation

$$\omega \simeq \omega_i \left(1 + \frac{\omega_{pi}^2}{\omega_i^2 - c^2k^2}\right). \tag{75}$$

The condition for the validity of this approximation is that $\omega_i^2 \gg c^2k^2$.

(d) Oscillations much below the ion cyclotron frequency

We now consider the low frequency oscillations such that the condition $\omega < \omega_i$ is satisfied. Then to the lowest significant order, we can write Eq. (69) as

$$\omega^2 - c^2k^2 + \omega_{pi}^2 \frac{\omega}{\omega_i}\left(1 + \frac{\omega}{\omega_i}\right) - \omega_{pe}^2 \frac{\omega}{\omega_e}\left(1 - \frac{\omega}{\omega_e}\right) = 0. \tag{76}$$

We now observe that $\omega_{pi}^2/\omega_i = \omega_{pe}^2/\omega_e$ and

$$\begin{aligned} \omega_{pi}^2 \frac{\omega^2}{\omega_i^2} &= \frac{4\pi Ne^2}{M}\frac{\omega^2 M^2 c^2}{e^2 B_0^2} = \frac{4\pi NMc^2\omega^2}{B_0^2}, \\ \omega_{pe}^2 \frac{\omega^2}{\omega_e^2} &= \frac{4\pi Nmc^2\omega^2}{B_0^2}. \end{aligned} \tag{77}$$

Thus Eq. (76) leads to

$$\omega^2 = \frac{c^2k^2}{1 + c^2/A^2}, \tag{78}$$

where $A = B_0/(4\pi\rho)^{\frac{1}{2}}$, $\rho = N(M + m)$ the mass density, denotes the Alfvén speed. Eq. (78) is the dispersion relation of Aströ m for the extraordinary hydromagnetic wave. It must be emphasized here that the hydromagnetic wave, in principle, is just a special case, in the appropriate frequency region, of the well-known transverse electromagnetic waves.

In the limit when $c/A \gg 1$ Eq. (78) reduces to

$$\left(\frac{\omega}{k}\right)^2 = A^2; \tag{79}$$

that is, the phase velocity of the wave is just the Alfvén speed. In the other limit when $c/A \ll 1$ we get

$$\omega = ck \tag{80}$$

which corresponds to the usual electromagnetic modes.

4.2. *Propagation transverse to the magnetic field*

For propagation perpendicular to the magnetic field, $\theta = \pi/2$ and the secular equation (63) reduces to

$$\begin{vmatrix} \omega^2 - c^2k^2 - \sum \omega_p^2 & 0 & 0 \\ 0 & \omega^2 - c^2k^2 + \sum \dfrac{\omega_p^2\omega^2}{\omega_c^2 - \omega^2} & +i\sum \omega_p^2 \dfrac{\omega\,\omega_c}{\omega_c^2 - \omega^2} \\ 0 & -i\sum \omega_p^2 \dfrac{\omega\,\omega_c}{\omega_c^2 - \omega^2} & \omega^2 + \sum \dfrac{\omega^2\omega_p^2}{\omega_c^2 - \omega^2} \end{vmatrix} = 0. \tag{81}$$

One of the modes of oscillation corresponds to the usual transverse oscillation with electric field along $\vec{B}_0$ and satisfies the dispersion relation

$$\omega^2 = c^2k^2 + \sum \omega_p^2$$

$$\simeq c^2k^2 + \omega_{pe}^2. \tag{82}$$

The other two modes of oscillation are determined as roots of the equation

$$\begin{vmatrix} \omega^2 - c^2k^2 + \sum \dfrac{\omega^2\omega_p^2}{\omega_c^2 - \omega^2} & + i\sum \omega_p^2 \dfrac{\omega\,\omega_c}{\omega_c^2 - \omega^2} \\ - i\sum \omega_p^2 \dfrac{\omega\,\omega_c}{\omega_c^2 - \omega^2} & \omega^2 + \sum \dfrac{\omega_p^2\omega^2}{\omega_c^2 - \omega^2} \end{vmatrix} = 0. \tag{83}$$

Consider first the case when $\omega \ll ck$. Then Eq. (83) requires that

$$c^2k^2\left(\omega^2 + \sum \frac{\omega^2\omega_p^2}{\omega_c^2 - \omega^2}\right) = 0. \tag{84}$$

This leads to the dispersion relation

$$1 + \frac{\omega_{pi}^2}{\omega_i^2 - \omega^2} + \frac{\omega_{pe}^2}{\omega_e^2 - \omega^2} = 0. \tag{85}$$

(a) In the limit when $\omega_i < \omega < \omega_e < \omega_{pe,i}$, Eq. (85) reduces to

$$-\frac{\omega_{pi}^2}{\omega^2} + \frac{\omega_{pe}^2}{\omega_e^2} = 0. \tag{86}$$

The frequency of oscillation is then determined by

$$\omega = \left(\frac{\omega_e^2\omega_{pi}^2}{\omega_{pe}^2}\right)^{\frac{1}{2}} = (\omega_e\omega_i)^{\frac{1}{2}}. \tag{87}$$

Thus the frequency of oscillation turns out to be the geometric mean of the electron and ion cyclotron frequencies. This frequency of oscillation is referred to as the lower hybrid frequency.

(b) Next we consider the other limiting case where the conditions $\omega^2 \gg \omega_i^2$ and $\omega^2 \gg \omega_{pi}^2$ are satisfied. Then Eq. (85) reduces to

$$1 + \frac{\omega_{pe}^2}{\omega_e^2 - \omega^2} = 0. \tag{88}$$

This yields the dispersion relation

$$\omega = (\omega_e^2 + \omega_{pe}^2)^{\frac{1}{2}}. \tag{89}$$

This frequency of oscillation is referred to as the upper hybrid frequency.
Several other cases can be discussed from the dispersion relation (63), e.g. waves propagating at any arbitrary angle with respect to the magnetic field or the coupling of the longitudinal and transverse oscillations. The essential resonances are, however, given by sections 4.1 (a) to (d) and 4.2 (a) and (b).

5. HYDROMAGNETIC WAVES IN A DISSIPATIVE MEDIUM

We have studied earlier the phenomena of wave propagation in a magnetized plasma neglecting the thermal motions of the particles. We shall now consider the plasma to be a hydromagnetic fluid characterized by a finite pressure p, viscosity ν and conductivity σ. The equations basic to our problem now are:

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{B}) \times \vec{B} + \nu \nabla^2 \vec{v} + \frac{1}{3}\nu \vec{\nabla}(\vec{\nabla} \cdot \vec{v}), \tag{90}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}, \tag{91}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \tag{92}$$

For the equation of state we shall assume the validity of the classical adiabatic law:

$$\frac{d}{dt}(p\rho^{-\gamma}) = 0. \tag{93}$$

The steady state is characterized by ρ = constant, p = constant, $\vec{B} = B_0 \vec{1}_z$ and the mean fluid velocity is assumed to vanish everywhere. Let the various fluctuating quantities be denoted by δp, $\delta\rho$, $\vec{v}$ and $\vec{b}$. The equations governing these are the linearized forms of Eqs. (90)-(93), and these are:

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla}\delta p + \frac{1}{4\pi}(\vec{\nabla} \times \vec{b}) \times \vec{B} + \nu \nabla^2 \vec{v} + \frac{1}{3}\nu \vec{\nabla}\vec{\nabla} \cdot \vec{v}, \tag{94}$$

$$\frac{\partial \vec{b}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 b, \tag{95}$$

$$\frac{\partial}{\partial t}\,\delta\rho + \rho\,\vec{\nabla}\cdot\vec{v} = 0, \tag{96}$$

and

$$\delta p = s^2\delta\rho, \tag{97}$$

where $s^2 = \gamma p/\rho$, denoting the sound speed in the medium. If we introduce the Lagrangian displacement $\vec{\xi}$ then to first order $\vec{v} = \partial\vec{\xi}/\partial t$ and Eq. (96) can be integrated to give

$$\delta\rho = -\rho\,\vec{\nabla}\cdot\vec{\xi}, \tag{98}$$

where the constant of integration has, by definition, been set equal to zero. We shall now assume that all fluctuating quantities have space time dependence of the form

$$\exp\,(-i\omega t + i\vec{k}\cdot\vec{r}). \tag{99}$$

Equation (95) then leads to

$$\vec{b} = \beta\, i\vec{k}\times(\vec{\xi}\times\vec{B}), \tag{100}$$

where

$$\beta = \frac{1}{1+i\,c^2k^2/4\pi\sigma\omega}\,. \tag{101}$$

Using the foregoing results in Eq. (94), we obtain

$$\rho\omega^2\vec{\xi} = s^2\rho\,\vec{k}\vec{k}\cdot\vec{\xi} + \frac{\beta}{4\pi}\,\{\vec{k}\times[\vec{k}\times(\vec{\xi}\times\vec{B})]\}\times\vec{B} - i\omega\nu k^2\vec{\xi} - i\omega\frac{\nu}{3}\vec{k}\vec{k}\cdot\vec{\xi}, \tag{102}$$

or

$$(\rho\omega^2 + i\omega\nu k^2)\vec{\xi} = \rho\left(s^2 - i\omega\frac{\nu}{3\rho}\right)\vec{k}\,(\vec{k}\cdot\vec{\xi}) + \frac{\beta}{4\pi}\,\{\vec{k}\times[\vec{k}\times(\vec{\xi}\times\vec{B})]\}\times\vec{B}. \tag{103}$$

We have assumed the magnetic field to be along the z-axis. We take $\vec{k}$ to be in the xz-plane making an angle θ with the z-axis. Eq. (103) then leads to

$$\omega^2\xi_x = \left(s^2 - \frac{i\omega}{3}\mu\right)k^2\sin\theta\left(\sin\theta\xi_x + \cos\theta\xi_z\right) + \beta k^2A^2\,\xi_x - i\mu\omega k^2\xi_x, \tag{104}$$

$$\omega^2\xi_y = \beta k^2A^2\cos^2\theta\;\xi_y - \mu\,k^2 i\omega\xi_y, \tag{105}$$

$$\omega^2\xi_z = \left(s^2 - \frac{i\omega}{3}\mu\right)k^2\cos\theta\left(\sin\theta\xi_x + \cos\theta\,\xi_z\right) - i\mu\omega k^2\xi_z, \tag{106}$$

where we have put $\mu = \nu/\rho$ and $A^2 = B_0^2/4\pi\rho$, A being the Alfvén speed. It is clear from Eqs. (104)-(106) that the ξ_x and ξ_z equations are coupled while the equation for ξ_y gives

$$\omega^2 = k^2A^2\cos^2\theta\left(1 + \frac{ic^2k^2}{4\pi\sigma\omega}\right)^{-1} - \mu k^2 i\omega. \tag{107}$$

If $c^2k^2/4\pi\sigma\omega \ll 1$ the damping is small — a condition necessary for wave propagation to be physically meaningful in a dissipative medium — we obtain from Eq. (107):

$$\omega^2 = k^2A^2\cos^2\theta\left(1 - \frac{ic^2k^2}{4\pi\sigma\omega}\right) - \mu k^2 i\omega \tag{108}$$

$$= k^2A^2\cos^2\theta - ik^2\left(\mu + \frac{c^2}{4\pi\sigma}\frac{k^2A^2\cos^2\theta}{\omega^2}\right)\omega. \tag{109}$$

This equation can be iterated to give for the frequency of oscillation:

$$\omega = kA\cos\theta - \frac{ik^2}{2}\left(\mu + \frac{c^2}{4\pi\sigma}\right). \tag{110}$$

We now consider Eqs. (104) and (106). For these equations to possess a non-trivial solution, we must have

$$\begin{vmatrix} \omega^2 - \beta k^2A^2 + i\mu k^2\omega - \left(s^2 - \frac{i\omega}{3}\mu\right)k^2\sin^2\theta & -\left(s^2 - \frac{i\omega}{3}\mu\right)k^2\sin\theta\cos\theta \\ -\left(s^2 - \frac{i\omega}{3}\mu\right)k^2\sin\theta\cos\theta & \omega^2 + i\mu k^2\omega - \left(s^2 - \frac{i}{3}\mu\omega\right)k^2\cos^2\theta \end{vmatrix} = 0. \tag{111}$$

We may here first consider the ideal case when $\sigma \to \infty$ and $\mu \to 0$. Eq. (111) then leads to the quadratic:

$$\omega^4 - \omega^2k^2(A^2 + s^2) + k^4A^2s^2\cos^2\theta = 0. \tag{112}$$

We now distinguish the two cases according to whether A is greater than or less than s, i.e. the strong field and the weak field cases, respectively. In the first case we get the two roots (distinguished by the subscripts 1 and 2)

$$\omega_1^0 = kA\left(1 + \tfrac{1}{2}\frac{s^2}{A^2}\sin^2\theta\right) \tag{113a}$$

and

$$\omega_2^0 = ks\cos\theta, \tag{113b}$$

while in the weak field case we obtain

$$\omega_3^0 = ks\left(1 + \tfrac{1}{2}\frac{A^2}{s^2}\sin^2\theta\right) \tag{114a}$$

and

$$\omega_4^0 = kA\cos\theta, \tag{114b}$$

where we have here distinguished the two roots by the subscripts 3 and 4. The superscript "0" has been added to indicate that these roots are in the zero'th approximation, namely when $\mu \to 0$ and $\sigma \to \infty$.

Now we proceed to solve Eq. (111) under the condition that $c^2k^2/4\pi\sigma\omega^0$ and $\mu k^2/\omega^0$ are both small, i.e. < 1. This enables us to replace ω by ω^0 in all terms which appear in the terms containing the transport coefficients and we carry out the calculations consistently up to the first order in the small parameter. First we note that

$$\beta \simeq 1 - \frac{ic^2k^2}{4\pi\sigma\omega^0}. \tag{115}$$

Eq. (111) now leads to the quadratic:

$$\omega^4 - \omega^2\left[k^2(A^2 + s^2) - ik^2\omega^0\left(\frac{1}{3}\mu + \frac{c^2}{4\pi\sigma}\frac{k^2A^2}{\omega^{02}}\right)\right]$$

$$+ k^4A^2s^2\cos^2\theta - ik^4\omega^0\left\{\mu\left[A^2(1+\cos^2\theta) + s^2\right]\right.$$

$$\left. + \frac{c^2k^2}{4\pi\sigma}\frac{A^2s^2}{\omega^{02}}\cos^2\theta\right\} = 0. \tag{116}$$

Solving this equation and retaining terms up to the first order, we obtain for the four cases:

$$\omega_1 = kA\left(1 + \frac{1}{2}\frac{s^2}{A^2}\sin^2\theta\right) - \frac{ik^2}{2}\left[\mu\left(\frac{1}{3} + \sin^2\theta\right) + \frac{c^2}{4\pi\sigma}\right], \tag{117a}$$

$$\omega_2 = sk\cos\theta - \frac{ik^2}{2}\left[\mu(1 + \cos^2\theta) + \frac{c^2}{4\pi\sigma}\right], \tag{117b}$$

$$\omega_3 = ks\left(1 + \frac{1}{2}\frac{A^2}{s^2}\sin^2\theta\right) - \frac{2ik^2}{3}\mu, \tag{118a}$$

$$\omega_4 = kA\cos\theta - \frac{ik^2}{2}\left(\mu + \frac{c^2}{4\pi\sigma}\right). \tag{118b}$$

We thus find that the presence of finite conductivity and viscosity leads to the damping of the hydromagnetic waves. It follows from Eqs. (117) and (118) that the time required for the wave amplitude to be reduced to a value $1/e$ of its initial amplitude is given by

$$\tau = \left[\frac{k^2}{2}\left(\frac{c^2}{4\pi\sigma} + \mu\right)\right]^{-1}. \tag{119}$$

The distance travelled by the wave during this interval of time is

$$L = A\left[\frac{k^2}{2}\left(\frac{c^2}{4\pi\sigma} + \mu\right)\right]^{-1}. \tag{120}$$

For these disturbances to be of any physical interest, it is clear that we must have

$$L \gg \lambda = \frac{2\pi}{k}. \tag{121}$$

This condition requires that $\lambda \gg \lambda_c$, where

$$\lambda_c = \frac{2\pi^2}{A}\left(\frac{c^2}{4\pi\sigma} + \mu\right). \tag{122}$$

6. THE EFFECT OF THERMAL CONDUCTIVITY

In our earlier discussion we have ignored the effect of finite thermal conductivity of the medium. If we take this into account, the general energy equation is given by

$$\frac{d}{dt}(p\rho^{-\gamma}) = \frac{2}{3}\vec{\nabla}(\kappa\vec{\nabla}T) + \frac{2}{3}\vec{J}\cdot\overset{\leftrightarrow}{\eta}\cdot\vec{J}, \tag{123}$$

where $\overset{\leftrightarrow}{\eta}$ is the resistivity tensor, and κ is the coefficient of thermometric conductivity. We shall restrict ourselves to the case when κ is a constant and the resistivity tensor a scalar.
Since

$$T = \frac{p}{nk} = \frac{M}{k\rho}p, \tag{124}$$

we can write (123) as

$$\frac{d}{dt}(p\rho^{-\gamma}) = \kappa\nabla^2\frac{p}{\rho} + \frac{2}{3}\frac{J^2}{\sigma}. \tag{125}$$

The linearized form of Eq. (125) is:

$$\frac{dp_1}{dt} = s^2\frac{d\rho_1}{dt} + \kappa\rho^{\gamma-1}\left(\nabla^2 p_1 - \frac{p}{\rho}\nabla^2\rho_1\right), \tag{126}$$

where we have now written p_1 and ρ_1 for the perturbations in p and ρ, respectively. Proceeding as before, we now obtain instead of (97):

$$p_1 = \phi s^2\rho_1, \tag{127}$$

where

$$\phi = \frac{1 + i\kappa k^2 \rho^{\gamma-2} p/\omega s^2}{1 + i\kappa k^2 \rho^{\gamma-1}/\omega}. \tag{128}$$

Having obtained p_1 in this form, we need only replace s^2 by $s^2\phi$ in our preceding discussion. If we assume that $\kappa k^2 \rho^{\gamma-1} \ll 1$ we obtain

$$\begin{aligned}\phi &= 1 - i\kappa k^2 \rho^{\gamma-1} \frac{1}{\omega s^2}\left(s^2 - \frac{p}{\rho}\right)\\ &= 1 - i\kappa k^2 \rho^{\gamma-1} \frac{1}{\omega}\left(\frac{\gamma - 1}{\gamma}\right).\end{aligned} \tag{129}$$

In the strong field case ($A/s \gg 1$) we have

$$\omega^2 = k^2 A^2 \left\{1 + \frac{s^2}{A^2}\sin^2\theta + \frac{ik}{A}\left[\frac{c^2}{4\pi\sigma} + \mu(1 + \sin^2\theta) + \kappa\rho^{\gamma-1}\frac{\gamma - 1}{\gamma}\right]\right\}, \tag{130}$$

while for the weak field case we get

$$\omega^2 = k^2 s^2 \left[1 + \frac{A^2}{s^2}\sin^2\theta + \frac{ik}{s}\left(\frac{4}{3}\mu + \kappa\rho^{\gamma-1}\frac{\gamma - 1}{\gamma}\right)\right]. \tag{131}$$

7. HYDROMAGNETIC WAVES FROM THE CGL THEORY

We now discuss the phenomena of wave propagation using the one fluid equations of Chew, Goldberger and Low. The basic assumptions underlying this approximation are:

(a) The magnetic field in the plasma is sufficiently strong so that the ion Larmor frequency is much larger than any other frequency;
(b) The heat flow along the lines of force is negligible; and
(c) The conductivity of the plasma is very high so that the simplified Ohms law can be used.

Under these circumstances, the pressure tensor is given by

$$\mathbb{P} = \begin{pmatrix} P_\perp & 0 & 0 \\ 0 & P_\perp & 0 \\ 0 & 0 & P_\parallel \end{pmatrix}, \tag{132}$$

where we assume the magnetic field to be along the z-axis. The variations of $P_\perp$ and $P_\parallel$ are governed by the two adiabatic relations:

$$\frac{d}{dt}\left(\frac{P_\perp}{\rho B}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{P_\parallel B^2}{\rho^3}\right) = 0. \tag{133}$$

The system is now governed by the closed set of moment equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{134}$$

$$\rho \frac{d\vec{v}}{dt} = \rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} \cdot \mathbb{P} + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B}, \tag{135}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \tag{136}$$

and

$$\mathbb{P} = P_{\perp} \mathbb{I} + (P_{\parallel} - P_{\perp})\vec{n}\vec{n}, \tag{137}$$

where $\vec{n} = \vec{B}/|\vec{B}|$.

Consider, then, a spatially homogeneous plasma in static equilibrium in a uniform magnetic field which we take to be along the z-axis of a cartesian system of co-ordinates. In the perturbed state let the various quantities be denoted by

$$\vec{v},\ \rho + \delta\rho,\ P_{\perp} + \delta P_{\perp},\ P_{\parallel} + \delta P_{\parallel},\ \vec{B} + \delta\vec{B}. \tag{138}$$

The equations governing the perturbed quantities are obtained by linearizing Eqs. (133)-(137). These are

$$\rho \frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} \cdot \delta \mathbb{P} + \frac{1}{4\pi} (\vec{\nabla} \times \delta\vec{B}) \times \vec{B}, \tag{139}$$

$$\frac{\partial}{\partial t} \delta\rho = -\rho \vec{\nabla} \cdot \vec{v}, \tag{140}$$

$$\frac{\partial}{\partial t} \delta\vec{B} = \vec{\nabla} \times (\vec{v} \times \vec{B}), \tag{141}$$

$$\delta P_{\perp} = P_{\perp} \frac{\delta B}{B} + P_{\perp} \frac{\delta\rho}{\rho}, \tag{142}$$

and

$$\delta P_{\parallel} = -2P_{\parallel} \frac{\delta B}{B} + 3P_{\parallel} \frac{\delta\rho}{\rho}, \tag{143}$$

where $\delta B = \vec{B} \cdot \delta\vec{B}/B$. Introducing the Lagrangian displacement $\vec{\xi}$, $\vec{v} = \partial\vec{\xi}/\partial t$ and Eqs. (140) and (141) can now be integrated to give

$$\delta\rho = -\rho \vec{\nabla} \cdot \vec{\xi} \tag{144}$$

and

$$\delta\vec{B} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}). \tag{145}$$

It follows readily from Eq. (145) that

$$\delta B = -\vec{\nabla}_\perp \cdot \vec{\xi}_\perp , \tag{146}$$

where $\vec{\nabla}_\perp = \vec{\nabla} - \vec{n}\vec{n} \cdot \vec{\nabla}$. As in the preceding discussion, we assume the space-time dependence of all the quantities to be of the form $\exp i(-\omega t + \vec{k}\cdot\vec{r})$. Using the foregoing results, Eqs. (142) and (143) reduce to

$$\delta P_\perp = -P_\perp(2\vec{\nabla}_\perp \cdot \vec{\xi}_\perp + ik_\parallel \xi_\parallel) \tag{147}$$

and

$$\delta P_\parallel = -P_\parallel (\vec{\nabla}_\perp \cdot \vec{\xi}_\perp + 3\, ik_\parallel \xi_\parallel). \tag{148}$$

From Eq. (137), we obtain

$$\vec{\nabla} \cdot \delta \mathbb{P} = \vec{\nabla}_\perp \delta P_\perp - i(P_\perp - P_\parallel)k_\parallel \delta\vec{n} + \vec{n}\,[(P_\parallel - P_\perp)\vec{\nabla} \cdot \delta\vec{n}] + ik_\parallel \delta p_\parallel . \tag{149}$$

The change $\delta\vec{n}$ in the unit vector along the lines of force is obtained readily from Eq. (145):

$$\delta\vec{n} = \vec{n}\cdot\vec{\nabla}\vec{\xi}_\perp + [\,(\vec{n}\cdot\vec{\nabla})\vec{n}\vec{n}] \cdot \vec{\xi}. \tag{150}$$

We thus obtain

$$\vec{\nabla} \cdot \delta \mathbb{P} = \nabla_\perp \delta P_\perp + (P_\perp - P_\parallel)k_\parallel^2 \vec{\xi}_\perp + ik_\parallel\, [(P_\parallel - P_\perp)\vec{\nabla}_\perp \cdot \vec{\xi}_\perp + P_\parallel]\vec{n}. \tag{151}$$

From Equation (145) it readily follows that

$$(\vec{\nabla} \times \delta\vec{B}) \times \vec{B} = B^2(\vec{\nabla}_\perp \vec{\nabla}_\perp \cdot \vec{\xi}_\perp - k_\parallel^2 \vec{\xi}_\perp). \tag{152}$$

On using the foregoing relations we find that the equation of motion (139) splits up into the two equations:

$$\left[\rho\,\omega^2 - k_\parallel^2\left(\frac{B^2}{4\pi} + P_\perp - P_\parallel\right)\right]\vec{\xi}_\perp = \vec{k}_\perp\left[2(\vec{k}_\perp \cdot \vec{\xi}_\perp)\left(P_\perp + \frac{B^2}{8\pi}\right) + k_\parallel P_\perp \xi_\parallel\right] \tag{153}$$

and

$$(\rho\omega^2 - 3P_\parallel k_\parallel^2)\xi_\parallel = k_\parallel P_\perp (k_\perp \cdot \vec{\xi}_\perp). \tag{154}$$

From these equations we obtain the relation

$$\rho\,\omega^2 - k_\parallel^2\left(\frac{B^2}{4\pi} + P_\perp - P_\parallel\right) - 2k_\perp^2\left(P_\perp + \frac{B^2}{8\pi}\right) = k_\parallel^2 P_\perp^2 \frac{k_\perp^2}{\rho\omega^2 - 3P_\parallel k_\parallel^2}\,. \tag{155}$$

If θ denotes the angle which the propagation vector makes with the z-axis, then

$$k_\parallel = k\cos\theta, \quad k_\perp = k\sin\theta. \tag{156}$$

After some reductions, Eq. (155) reduces to the following quadratic for ω^2

$$\omega^4 - \frac{\omega^2}{\rho} k^2 \left(\frac{B^2}{4\pi} + P_\perp + 2P_\parallel \cos^2\theta + P_\perp \sin^2\theta \right)$$

$$+ \frac{3k^4}{\rho^2} P_\parallel \cos^2\theta \left[\frac{B^2}{4\pi} + P_\perp (1 + \sin^2\theta) - P_\parallel \cos^2\theta \right]$$

$$- \frac{k^4}{\rho^2} P_\perp \sin^2\theta \cos^2\theta = 0. \tag{157}$$

The roots of this equation are given by

$$\omega^2 = \frac{k^2}{2\rho} \left(\frac{B^2}{4\pi} + P_\perp + 2P_\parallel \cos^2\theta + P_\perp \sin^2\theta \right)$$

$$+ \frac{k^2}{2\rho} \left\{ \left[\frac{B^2}{4\pi} + P_\perp (1 + \sin^2\theta) - 4P_\parallel \cos^2\theta \right]^2 + 4P_\perp^2 \sin^2\theta \cos^2\theta \right\}^{\frac{1}{2}}. \tag{158}$$

For propagation along the lines of force, $\theta = 0$ and Eq. (158) leads to the following two modes of oscillation (distinguished by the subscripts 1 and 2):

$$\omega_1^2 = \frac{k^2}{\rho} \left(\frac{B^2}{4\pi} + P_\parallel - P_\perp \right) \tag{159}$$

and

$$\omega_2^2 = \frac{3k^2}{\rho} P_\parallel . \tag{160}$$

It is clear that the mode of oscillation given by Eq. (159) is unstable if

$$P_\parallel > \frac{B^2}{4\pi} + P_\perp . \tag{161}$$

That is, if the distribution function for the ions and/or electrons is strongly peaked along the lines of force, the hydromagnetic waves propagating along the field lines are unstable.

For propagation transverse to the field lines $\theta = \pi/2$ and we obtain

$$\omega^2 = \frac{2k^2}{\rho} \left(\frac{B^2}{8\pi} + P_\perp \right). \tag{162}$$

The corresponding phase velocity of the wave is given by

$$v_p = \frac{\omega}{k} = \left(\frac{2P}{\rho} \right)^{\frac{1}{2}}, \tag{163}$$

where $P = P_\perp + B^2/8\pi$ denotes the total transverse pressure.

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MAGNETOHYDRODYNAMICS

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I. INTRODUCTION

If a conducting fluid moves in a magnetic field, electric fields are induced in the fluid and electric currents will flow. The magnetic field exerts forces on these currents which can modify the flow, at the same time the currents may substantially alter the magnetic field. The study of these complicated interactions is the science of MHD (magnetohydrodynamics) which combines the electromagnetic field equations of Maxwell with the equations of fluid dynamics.

MHD in its widest sense has relevant applications in applied science, geophysics and cosmic physics. In these cases the fluid media may be a conducting liquid metal (earth's core, electro-magnetic pumps), or a weakly ionized gas (MHD generators, upper atmosphere), or a highly ionized gas, or plasma (laboratory systems and cosmic physics). The approximations used in the fluid equations depend very much on the problem at hand, but the regimes can be conveniently discussed in terms of dimensionless numbers.

First I shall revise some basic electromagnetic concepts; I shall then derive the dimensional analysis in a simple physical way and, finally, I shall give the basic MHD equations in various approximations, and the boundary conditions.

Secondly I shall consider certain solutions of the equations which are relevant to laboratory plasma physics, i.e. I shall talk about the equilibrium and dynamics of highly conducting plasma in magnetic fields from an elementary point of view.

Thirdly I consider MHD wave propagation and high speed MHD flows. These include phenomena in which small perturbations are propagated according to linearized approximate theories and also phenomena in which finite amplitude, quasi discontinuities, occur.

II. MOTION OF A CONDUCTOR IN A MAGNETIC FIELD

We start with either of two experimental laws.

1. *Faraday's law*

The change of flux ϕ through a circuit is given by

$$-\frac{d\phi}{cdt} = \oint \vec{E}' \cdot \vec{dl} = \mathcal{E}, \tag{1}$$

where $\vec{E}'$ is the electric field in the circuit which may be in motion, and

$$\phi = \int \vec{B} \cdot d\vec{s}, \tag{2}$$

the integral being taken over the surface and $\mathcal{E}$ is the EMF around the loop.

If the circuit is a wire loop, then $\mathcal{E} = JR$, where R is the resistance and J the current in the loop.

Since

$$\oint \vec{E}' \cdot d\vec{l} = -\frac{d\phi}{cdt} = -\frac{1}{c}\frac{d}{dt}\int \vec{B} \cdot d\vec{s} = -\frac{1}{c}\int \frac{D\vec{B}}{Dt} \cdot d\vec{s}, \tag{3}$$

then by Stokes' theorem we get

$$\vec{\nabla} \times \vec{E}' = -\frac{D\vec{B}}{cDt} \tag{4}$$

where*

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\vec{v} \cdot \vec{\nabla}). \tag{5}$$

Or for a stationary loop,

$$\oint \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{\partial \phi}{\partial t}, \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{c\partial t}, \tag{6}$$

where $\vec{E}$ is now the field in the laboratory frame. Let us compute $D\vec{B}/Dt$. For a moving loop, $d\phi/dt$ arises due to (a) the change of $\vec{B}(t)$ within the loop, (b) the motion of the boundary. Consider a loop moving from 1 to 2 in time Δt, then

$$\frac{d\phi}{dt} = \lim \frac{1}{\Delta t}\left[\int \vec{B}(t+\Delta t) \cdot d\vec{s}_2 - \int \vec{B}(t) \cdot d\vec{s}_1\right]. \tag{7}$$

Since $\vec{\nabla} \cdot \vec{B} = 0$ (Maxwell's equations),

$$\int_{\text{Vol. 1-2}} \vec{\nabla} \cdot \vec{B}\, d\tau = \int \vec{B}(t) \cdot d\vec{s}_2 - \int \vec{B}(t) \cdot d\vec{s}_1 + \int_{\text{Wall}} \vec{B}(t) \cdot d\vec{s}_W, \tag{8}$$

where the volume is the small element swept out by the moving loop and the integrals are carried out over this element (Fig. 1).

Using a Taylor's expansion,

$$\vec{B}(t+\Delta t) = \vec{B}(t) + \frac{\partial \vec{B}}{\partial t}\Delta t + \ldots \tag{9}$$

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* D/Dt is called the convective derivative and measures the rate of change for a moving observer.

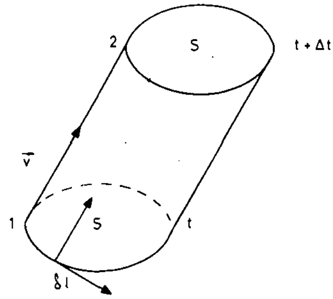

Fig. 1

Element swept out by the moving loop

Consequently,

$$\frac{d\phi}{dt} = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} + \int_{\text{Wall}} \vec{B} \cdot \vec{v} \times d\vec{l} = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s} - \int \vec{v} \times \vec{B} \cdot d\vec{l} \ . \tag{10}$$

Comparing (3) with (10) we get

$$-\frac{d\phi}{cdt} = \oint \vec{E}' \cdot d\vec{l} = \oint (\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \cdot d\vec{l} \, . \tag{11}$$

Consequently

$$\vec{E}' = \vec{E} + \frac{\vec{v} \times \vec{B}}{c} \ . \tag{12}$$

2. Lorentz's law

The expression (12) for the transformed field $\vec{E}'$ could be derived also from Lorentz's law for a moving observer. The force exerted on a charge q is given by

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) = q\vec{E}' . \tag{13}$$

(Note that to order v/c, $\vec{B}' = \vec{B}$.)

Ohms' law for a moving conductor is

$$\vec{j} = \sigma\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) = \sigma \vec{E}' . \tag{14}$$

For a highly conducting fluid $\sigma \to \infty$, therefore $\left(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}\right) \to 0$, and

$$\vec{\nabla} \times (\vec{v} \times \vec{B}) = \frac{\partial \vec{B}}{\partial t} \ . \tag{15}$$

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Otherwise

$$\frac{c^2}{4\pi\sigma}\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\frac{\partial\vec{B}}{\partial t} + \vec{\nabla}\times(\vec{v}\times\vec{B}). \tag{16}$$

In a highly conducting fluid let us consider a ring of fluid particles enclosing a surface Σ moving with the fluid motion. Then,

$$\frac{D}{Dt}\int_{\Sigma}\vec{B}\cdot d\vec{s} = 0, \tag{17}$$

i.e. the flux through Σ is conserved. Thus, the field and the fluid are frozen together.

By combining,

$$\frac{\partial\vec{B}}{\partial t} = \vec{\nabla}\times(\vec{v}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}\cdot\vec{v}) \tag{18}$$

with the continuity equation

$$\frac{D\rho}{Dt} + \rho(\vec{\nabla}\cdot\vec{v}) = 0, \tag{19}$$

and writing it as

$$\frac{D\vec{B}}{Dt} - (\vec{B}\cdot\vec{\nabla})\vec{v} = \vec{B}(\vec{\nabla}\cdot\vec{v}), \tag{20}$$

it can be shown that, for an incompressible fluid with $\vec{\nabla}\cdot\vec{v} = 0$,

$$\frac{D}{Dt}\left(\frac{\vec{B}}{\rho}\right) = \left(\frac{\vec{B}}{\rho}\cdot\vec{\nabla}\right)\vec{v}. \tag{21}$$

Consider a line element $\delta\vec{l}$ joining two particles initially very close together (Fig. 2). Then,

$$\frac{D\delta\vec{l}}{Dt} = (\delta\vec{l}\cdot\vec{\nabla})\vec{v}. \tag{22}$$

If two particles are close together on a field line then $\delta\vec{l}$ is parallel to $\vec{B}$ and $|\delta\vec{l}| \propto |\vec{B}/\rho|$.

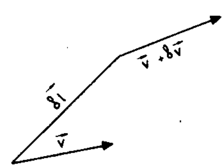


Fig. 2

Line element joining two particles

If the fluid is incompressible, increasing $|\vec{\delta l}|$ increases $|\vec{B}|$. Stretching the magnetic field lines increases the field strength and may be a means of energy equipartition between a turbulent fluid and a field.

Magnetic energy $(B^2/8\pi) \leftrightharpoons$ fluid turbulent energy $(\rho v^2/2)$.

3. *Conditions for the validity of Ohms' law*

The Ohms' law is valid if:

(a) The period of the field variation $t \gg t_e$, where t_e is the collision times of the electrons;

(b) The electron mean free path $\ll$ gyro radius, or equivalently if,

$$t_e \Omega_e \ll 1, \tag{23}$$

where $\Omega_e \equiv eB/mc$. Otherwise $\vec{j}$ is not parallel to $\vec{E}$ (i.e. σ has tensor properties) and the Hall current must be included in Ohm's law. In fact, the Ohms' law can be modified for the range when $\Omega_e t_e > 1$, but then

$$\vec{E} + \frac{\vec{v}_e \times \vec{B}}{c} = \vec{j}/\sigma, \tag{24}$$

where $\vec{v}_e$ is the electrons' velocity considering the ions and electrons as two fluids. Equation (24) is the equation of motion for the electron fluid neglecting the electron inertia.

There is then the supplementary equation

$$\vec{j} = ne(\vec{v} - \vec{v}_e), \tag{25}$$

where $\vec{v} = \vec{v}_i$.

Hence, eliminating $\vec{v}_e$, we get

$$\vec{E} + \frac{\vec{v} \times \vec{B}}{c} = \frac{\vec{j} \times \vec{B}}{nec} + \frac{\vec{j}}{\sigma}, \tag{26}$$

where the third term represents the Hall effect. The ratio of the magnitude of the third to the fourth term is $\sim |\Omega_e t_e| > 1$.

So this equation extends, in an approximate fashion at least, the realm of the validity of MHD into a regime where $\Omega_e t_e > 1$

4. *Conditions for MHD behaviour*

Let us consider a conducting fluid moving across a magnetic field (Fig.3). The induced current is

$$j = \sigma E' = \sigma v B/c. \tag{27}$$

The force on the unit area over an interaction length L is

$$\frac{\vec{j} \times \vec{B}}{c} L \simeq \frac{\sigma}{c^2} \vec{v} B^2 L. \tag{28}$$

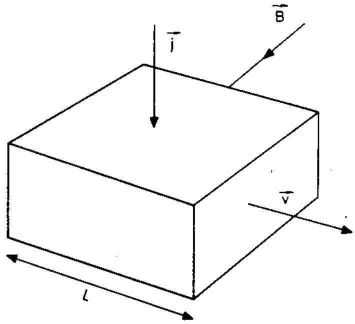


Fig. 3

Conducting fluid moving across a magnetic field

The ratio to the dynamic pressure (which is $\sim \rho v^2$) gives the interaction

$$\simeq \frac{\sigma}{c^2} B^2 \frac{L}{v} \equiv M. \tag{29}$$

The magnetic field ΔB induced is given by $\Delta B/L \sim 4\pi j/c$, hence,

$$\frac{\Delta B}{B} \simeq \frac{4\pi\sigma v L}{c^2} = R_M, \tag{30}$$

where R_M is the magnetic Reynolds number.

If $R_M > 1$, the magnetic field is strongly perturbed, whereas if $M > 1$, the fluid is strongly perturbed.

Combining these conditions so as to eliminate v, one obtains

$$\sqrt{MR_M} \equiv S = \frac{4\pi\sigma L}{c^2}\left(\frac{B^2}{4\pi\rho}\right)^{1/2}. \tag{31}$$

If both R_M and $M > 1$, then $S \gg 1$.

Some examples in the laboratory ($B = 1$ kG, $L = 10$ cm) are:

(1) For a liquid sodium experiment, $\sigma/c^2 \simeq 10^{-3}$, $\rho \simeq 1$ g cm^{-3} then $S = 40$.

(2) For a hot plasma, $T = 10^6$K, $n = 10^{15}$ cm^{-3}, $\rho \simeq 10^{-19}$ g cm^{-3}, $\sigma/c^2 \simeq 10^{-6}$, then $S \simeq 10^4$.

III. EQUATIONS OF MOTION OF A FLUID IN A MAGNETIC FIELD

We shall assume that the magnetic permeability $\mu = 1$, and we shall not distinguish (numerically) between $\vec{B}$ and $\vec{H}$. Then

$$\begin{gathered} \nabla \cdot B = 0, \\ \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}, \end{gathered} \tag{32}$$

where often the last term is neglected (i.e. $\sigma = \infty$).

The general equations become very complicated when viscosity and compressibility are considered* and it is simpler to consider approximate equations suitable for each problem as it arises. We give two representative cases.

1. Incompressible flow

The equations describing such a flow are

$$\vec{\nabla}\cdot\vec{B} = 0, \qquad \vec{\nabla}\cdot\vec{v} = 0, \qquad \frac{D\rho}{Dt} = 0,$$

$$\frac{\partial\vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B} - (\vec{B}\cdot\vec{\nabla})\vec{v} = \frac{c^2}{4\pi\sigma}\nabla^2\vec{B}, \tag{33}$$

$$\rho\frac{Dv}{Dt} = \rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}\right) = \frac{\vec{j}\times\vec{B}}{c} - \vec{\nabla}p + \mu\nabla^2\vec{v} = -\vec{\nabla}(p+\frac{B^2}{8\pi}) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} + \mu\nabla^2\vec{v}.$$

The energy equation is not needed.

2. Isentropic, inviscid flow

The equations

$$\vec{\nabla}\cdot\vec{B} = 0,$$

$$\frac{\partial\rho}{\partial t} + (\vec{v}\cdot\vec{\nabla})\rho + \rho\vec{\nabla}\cdot\vec{v} = 0, \tag{34}$$

plus the Ohms' law and the momentum equation with $\mu = 0$, are needed to describe an isentropic and inviscid flow.

To close the system of equations, we need the conservation of entropy $s = \text{const.}\log p\rho^{-\gamma}$ (where γ is the ratio of the specific heats), namely

$$\frac{Ds}{Dt} = 0, \tag{35}$$

or equivalently to (35)

$$\frac{Dp}{Dt} = \frac{\gamma p}{p}\frac{D\rho}{Dt}. \tag{36}$$

We have neglected electrical dissipation. Let us suppose that

L is a characteristic length,
U is a characteristic velocity,
a is a characteristic sound speed ($a^2 = \gamma p/\rho$)
A is a characteristic speed ($A^2 = B^2/4\pi\rho$)

---

* LANDAU, L.D. and LIFSHITZ, E.M., Electrodynamics of Continuous Media, Pergamon Press, London (1960).

If $U \ll a$ or A, the fluid is incompressible.
If $R \equiv UL\rho/\mu >> 1$, the fluid is inviscid.
If $R_m \equiv 4\pi UL\sigma/c^2 >> 1$, the fluid is a good conductor.
This dimensional approach has to be applied with care and experimental insight (cf. boundary layers in fluids and resistive layers in resistive instabilities).

Generally, the great difficulty in solving these equations is their non-linearity, in contrast to Maxwell's equations by themselves. They can be linearized to discuss such questions as (a) small perturbations of flows, (b) waves and (c) stability. Non-linearity is essential to discussions of turbulence and large amplitude waves (shocks etc.). If the solutions remain well behaved (single-valued, finite, etc.) then computational procedures are possible. But it is important to remember that the final appeal is always to experiment.

IV. BOUNDARY CONDITIONS FOR A PERFECT CONDUCTOR

Let us now discuss the boundary conditions at fluid-fluid-wall or at fluid-vacuum-wall interfaces. If $\vec{n}$ is the outward pointing unit normal, [G] the increment in any quantity G across a surface and $\vec{J}$ the surface current on a conductor, then by using Maxwell's equations across the surface of the conductor one obtains

$$\vec{n}\cdot[\vec{B}] = 0 \tag{37}$$

and

$$\vec{n}\times[\vec{B}] = \frac{4\pi}{c}\vec{J}. \tag{38}$$

At a fluid-vacuum interface with $\sigma = \infty$,

$$\vec{n}\times\left[\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right] = 0 \tag{39}$$

gives

$$\vec{n}\times[\vec{E}] = \vec{n}\cdot\frac{\vec{v}}{c}[\vec{B}]. \tag{40}$$

At a fluid-fluid interface,

$$\vec{n}\cdot[\vec{v}] = 0 \tag{41}$$

and

$$\left[p+\frac{B^2}{8\pi}\right] = 0. \tag{42}$$

At a perfectly conducting wall,

$$E_{\parallel} = 0, \qquad \vec{n}\times\vec{E} = 0,$$

$$\vec{n}\cdot\vec{v} = 0, \qquad \vec{n}\cdot\frac{\partial\vec{B}}{\partial t} = 0. \tag{43}$$

If $\vec{B} = 0$ in the wall at $t = 0$,

$$\vec{n} \cdot \vec{B} = 0. \tag{44}$$

The boundary conditions can be obtained by integrating the original equations across a finite discontinuity ϵ and then letting $\epsilon \to 0$ using Stokes' and Gauss' theorems to do the integration. Let us now apply the equations and boundary conditions to some simple (in principle!) experiments.

V. THE DYNAMIC PINCH

This experiment has demonstrated the validity of the MHD equations in the $\sigma = \infty$ approximation. A simple theory (by M. Rosenbluth) is as follows: an electric field E_z is applied to a conducting cylindrical plasma density ρ_0 at time $t = 0$. A skin current J_z and a magnetic field B_θ occur, where $B_\theta = 4\pi J_z / c = 2I_z / rc$, I_z being the total axial current. Since $\sigma = \infty$ in the skin,

$$E_z + \frac{\dot{r}}{c} B_\theta = 0. \tag{45}$$

We now suppose the skin collapses uniformly inwards sweeping up the gas within as it goes. The collapse velocity of the skin $(-\dot{r})$ depends on the magnetic pressure as follows:

$$\rho_0 (-\dot{r})^2 = K B_\theta^2 / 8\pi, \tag{46}$$

where $K \sim 1$ is a constant depending on the precise mechanism of the collapse. If the potential V across the tube remains constant then

$$V = \frac{d}{c\,dt}(L I_z) = \frac{d}{dt}\left(\frac{2 l I_z}{c^2} \log \frac{R}{r}\right), \tag{47}$$

where l is the tube length, $L(t)$ the inductance of the collapsing plasma column in the tube. (The current returns immediately outside the wall.) Hence

$$\frac{c^2 V}{2l} = -I_z \frac{\dot{r}}{r} + \dot{I}_z \log \frac{R}{r}. \tag{48}$$

Now

$$-\dot{r} = (K/8\pi\rho_0)^{1/2} \cdot \frac{2I_z}{rc}. \tag{49}$$

Hence we can solve numerically for I_z and r as functions of z. However, the scaling laws are evident immediately. When $r \approx R$

$$-\dot{r} = \frac{c^2 V}{2l} \frac{R}{I_z} = \frac{cV}{2l}\left(\frac{K}{2\pi\rho_0}\right)^{1/2}\left(\frac{1}{-\dot{r}}\right). \tag{50}$$

Hence

$$|\dot{r}| \propto E_0^{1/2}/\rho_0^{1/4}, \tag{51}$$

where E_0 is the applied electric field. Because the collapse time is so rapid, instabilities play little part. The same principles of MHD acceleration have been used in different geometries, for example the co-axial plasma gun (see Fig. 4). Here current flows radially from a central electrode along the z-axis, the magnetic field is in the θ-direction and the plasma is accelerated along the z-direction, being swept up just ahead of the advancing radial current sheet.

VI. STATIC EQUILIBRIUM WITH CYLINDRICAL SYMMETRY

Consider a constricted or "pinched" plasma where $\vec{B} = B_\theta(r)$ and $p = p(r)$ satisfy

$$\frac{dp}{dr} = -j_z \frac{B_\theta}{c}, \tag{52}$$

$$\frac{1}{r}\frac{d}{dr}(r B_\theta) = \frac{4\pi}{c} j_z, \tag{53}$$

with the boundary condition $p \to 0$ as $r \to \infty$, $B_\theta r = 2I_z/c \to$ finite value. The variations along z are neglected.

At constant temperature T, line density N length^{-1} and particle density n volume^{-1},

$$\int_0^\infty 2\pi r n T dr = NT = \int_0^\infty 2\pi\, p r dr$$

$$= [\pi p r^2]_0^\infty - \pi \int_0^\infty r^2 \frac{dp}{dr} dr$$

$$= \frac{1}{4}\int_0^\infty r^2 \frac{B_\theta}{r}\frac{d}{dr}(rB_\theta) dr$$

$$= \frac{[rB_\theta]^2}{8} \quad \text{at } r = \infty$$

$$= \left[\int_0^\infty 2\pi r j_z\, dr\right]^2 / 2c^2 = I_z^2/2c^2.$$

Hence

$$I_z^2 = 2NTc^2, \tag{54}$$

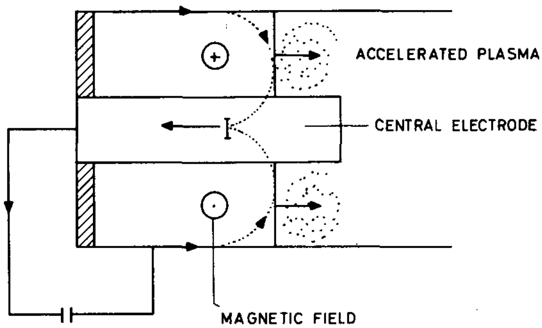


Fig. 4

Co-axial plasma gun

which is Bennett's relation for so-called "z-pinch". In experiments this equilibrium is highly unstable and is readily destroyed.

VII. HYDROMAGNETIC WAVES

Consider a uniform plasma of density ρ_0, pressure p_0 in a uniform field B_θ. In this paragraph we only consider small amplitude waves propagating as $\exp i(\vec{k}\cdot\vec{r} - \omega t)$.

Let us substitute the quantities $v = \hat{v}$, $B = B_0 + \hat{b}$ and $p = p_0 + \hat{p}$ into the basic equations

$$\dot{\vec{B}} = \vec{\nabla} \times (\vec{v} \times \vec{B})$$

$$\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0$$

$$\rho \dot{\vec{v}} = -\vec{\nabla} p + \frac{1}{4\pi} (\vec{\nabla} \times \vec{B}) \times \vec{B}, \tag{55}$$

$$\frac{dp}{d\rho} = \frac{\gamma p_0}{\rho_0} = a^2 .$$

The linearized equations are

$$-\omega \hat{b} = \vec{k} \times (\hat{v} \times \vec{B}).$$

$$\omega \hat{\rho} = \rho (\vec{k} \cdot \hat{v}),$$

$$-\omega \hat{v} + a^2 \vec{k} \frac{\hat{\rho}}{\rho} = -\vec{B} \times \frac{(\vec{k} \times \hat{b})}{4\pi\rho}, \tag{56}$$

$$\hat{p} = a^2 \hat{\rho} .$$

If $\vec{u} = \omega/\vec{k} \equiv$ phase velocity and $A^2 \equiv B_0^2/4\pi\rho_0$ then the solutions separate into two types, the first being an effectively incompressible shearing motion (see Fig. 5).

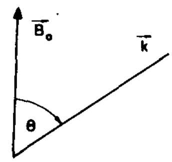


Fig. 5

$\vec{k}$-vector direction versus magnetic field direction

1. Type I

$\hat{\rho} = 0$ (a shear Alfven wave) with $\hat{b}$, $\hat{v} \perp \vec{B}_0, \vec{k}$.

$$|\omega/k| = u = A \cos\theta. \qquad (57)$$

The group velocity $\partial\omega/\partial\vec{k} = A\vec{B}/|\vec{B}|$ is always directed along the field.

2. Type II

$$u_{2,3} = \tfrac{1}{2}\,[(a^2 + A^2 + 2aA\cos\theta)^{1/2} \pm (a^2 + A^2 - 2aA\cos\theta)^{1/2}]. \qquad (58)$$

in which compressibility is important.

Note that $u_2 \geqslant u_1 \geqslant u_3$ and that $\mu_1 = A\cos\theta$, $u_2 \geqslant \max(a, A)$, $\min(A, a) \geqslant u_3$.

u_1 = shear Alfvén wave speed, effectively incompressible and therefore independent of a.

u_2 = compressional Alfvén wave speed
u_3 = electro acoustic wave speed
} fast and slow magnetosonic waves.

If $A >> a$, $u_2 \simeq A$, $u_3 = a\cos\theta$

The phase velocity diagrams are shown in Fig. 6 (for $A > a$) and Fig. 7 for ($A<a$). Really, they are sections cut through three-dimensional figures. The line $\alpha\beta$ is the normal at P to the phase velocity vector $\overrightarrow{OP}$ and represents the direction of the wave front. Group velocity diagrams (envelopes of lines $\alpha\beta$) are the surfaces of maximum interference of the wave fronts and surfaces of the propagating disturbance. A typical group velocity diagram is Fig. 8. The velocities corresponding to QRST are

OQ: larger (A, a), OR: smaller (a, A)

OS: $\left(\frac{1}{a} + \frac{1}{A}\right)^{-1}$, OT: $(a^2 + A^2)^{1/2}$

The construction of the wave cones for a body moving with velocity $\vec{U}$ (corresponding to a vector OU) is to drop tangents from the tip U to the group velocity surfaces. If U lies outside the surface corresponding to u_2 we have a familiar swept-back cone projecting from the body. However, if U lies inside one of the small ogival surfaces corresponding to u_3 one finds a swept-forward cone (see Fig. 9). Notice that the surfaces of u_3 correspond to a small lateral disturbance perpendicular to $\vec{B}_0$ while the surfaces corresponding to u_1 degenerate to two points moving with velocity $\pm A$ along $\vec{B}_0$.

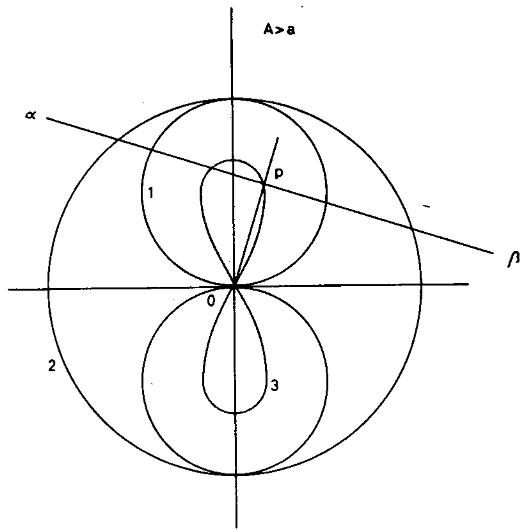


Fig. 6

Phase velocity diagram (A > a)

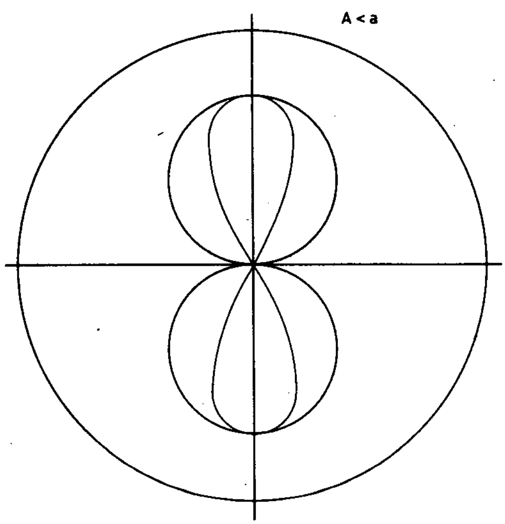


Fig. 7

Phase velocity diagram (A< a)

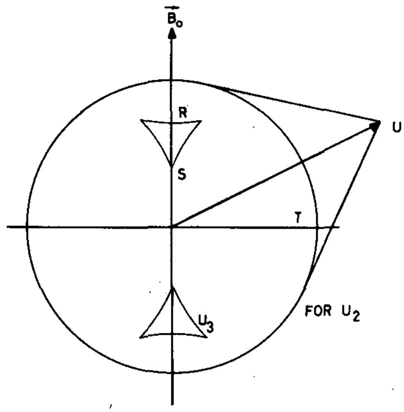


Fig. 8

Typical group velocity diagram

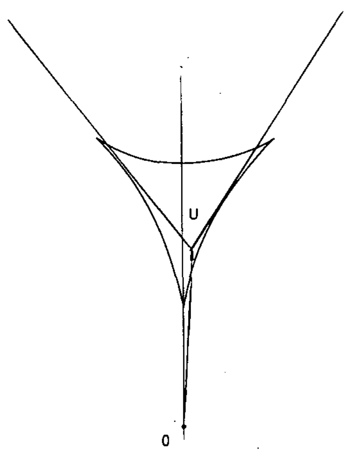


Fig. 9

Construction of the wave cone if U lies inside one of the small ogival surfaces corresponding to u_3

3. *Non-linear waves (finite amplitude)*

In an incompressible fluid the plane MHD Alfvén wave is an exact solution with the phase velocity

$$\vec{v}_p = -\vec{b}\frac{A}{|\vec{B}|} \tag{59}$$

regardless of amplitude. Large amplitude waves in a compressible gas have a more complicated behaviour.

If $\vec{k} \perp \vec{B}$, $u_2 = (a^2 + A^2)^{1/2}$

$$\dot{B} + \frac{\partial}{\partial x}(v_x B) = \dot{\rho} + \frac{\partial}{\partial x}(v_x \rho) = 0$$

$$\dot{v}_x + v_x \frac{\partial v_x}{\partial x} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(p + \frac{B^2}{8\pi}\right) \tag{60}$$

Thus

$$\frac{D}{Dt}(B/\rho) = 0$$

and

$$\rho \frac{Dv_x}{Dt} = -\frac{\partial}{\partial x}\left(p + \frac{B^2}{8\pi}\right) \tag{61}$$

Replace $p(\rho) + \frac{B^2}{8\pi} \equiv p^*(\rho)$, since $B \propto \rho$.

These equations are equivalent to a one-dimensional motion of an ordinary compressible gas with $p^*(\rho)$ replacing $p(\rho)$. Thus we can apply Riemann's method of solution using a pseudo velocity of "sound" $a^* \equiv (A^2 + a^2)^{1/2}$. For large amplitude perturbations one is led often to discontinuities, which physically occur as shock waves.

VIII. DISCONTINUITIES

We shall consider possible discontinuities without regard to their structure or their physical stability.

Let us choose co-ordinates in which the discontinuity is at rest. $\vec{n} \equiv$ normal to surface, $\vec{j} \equiv \rho v_n$: the continuity of mass gives

$$[\rho v_n] = 0. \tag{62}$$

The continuity of momentum gives

$$[\Pi_{ik} n_k] = 0 \quad \text{and} \quad [\vec{n} \cdot \vec{\vec{\Pi}} \cdot \vec{n}] = 0, \tag{63}$$

where Π is the momentum-stress tensor (neglecting v^2/c^2),

$$\Pi_{ik} \equiv \rho v_i v_k + p\delta_{ik} - (B_i B_k - \tfrac{1}{2}\delta_{ik} B^2)/4\pi . \tag{64}$$

Thus (63) gives

$$[p + \rho v^2 + (B_t^2 - B_n^2)/8\pi] = 0,$$

and

$$[\rho v_n \vec{v}_t - B_n \vec{B}_t/4\pi] = 0. \tag{65}$$

The continuity of energy gives

$$[q_n] = [\rho v_n (\tfrac{1}{2} v^2 + e + \frac{p}{\rho}) + \vec{S}\cdot\vec{n}] = 0, \tag{66}$$

where $\vec{S}$ is the energy flux of the electromagnetic field (Poynting vector).

$$\vec{S} = \frac{c\vec{E}\times\vec{B}}{4\pi} = -\frac{(\vec{v}\times\vec{B})\times\vec{B}}{4\pi} \tag{67}$$

(neglecting resistivity)

hence

$$\begin{aligned}\vec{S}\cdot\vec{n} &= \frac{1}{4\pi}[B^2\vec{v} - (\vec{B}\cdot\vec{v})\vec{B}]\cdot\vec{n} \\ &= v_n\frac{B^2}{4\pi} - \frac{(\vec{B}\cdot\vec{v})B_n}{4\pi}.\end{aligned} \tag{68}$$

$e = p/(\gamma-1)\rho$ is the internal energy of the fluid. Thus

$$[\rho v_n\left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\ \frac{p}{\rho}\right) + \frac{1}{4\pi}(v_n B^2 - (\vec{B}\cdot\vec{v})B_n)] = 0. \tag{69}$$

Finally,

$$c[\vec{E}_t] = -[(\vec{v}\times\vec{B})_t] = 0, \tag{70}$$

thus

$$[B_n\vec{v}_t - \vec{B}_t v_n] = 0. \tag{71}$$

We summarize four possible discontinuities as follows:

(1) Contact discontinuity (plasma A in contact with plasma B)

$$j = 0, \quad [v_t] = 0, \quad [\rho] \neq 0, \quad [p] = 0$$
$$[B_t] = 0, \quad B_n \neq 0$$

(2) Tangential discontinuity (sharp plasma boundary; plasma jet across a magnetic field)

$$j = 0, \quad [v_t] \neq 0, \quad [\rho] \neq 0, \quad [p + B_t^2/8\pi] = 0$$
$$[B_t] \neq 0, \quad B_n = 0$$

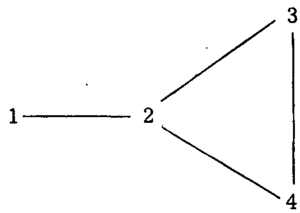
(3) Rotational discontinuities (e.g. large amplitude Alfven wave)

$$j \neq 0, \quad [\vec{v}_t] \neq 0, \quad [\rho] = 0, \quad [p] = 0, \quad B_n \neq 0$$
$$\vec{B}_{t1} \longrightarrow -\vec{B}_{t2}$$

(4) Shock waves (e.g. large amplitude magneto-sonic compression)

$$j \neq 0, \qquad [\rho] \neq 0, \qquad \vec{B}_1,\ \vec{B}_2 \text{ and } \vec{n} \text{ coplanar.}$$

The diagram



gives possible transitions between the four discontinuous states.

IX. SHOCK WAVES

These are discontinuities in which $j \equiv \rho v_n \neq 0$ and $[\rho] \neq 0$.
Since

$$j[\vec{v}_t] = \frac{B_n}{4\pi}[\vec{B}_t] \tag{72}$$

and

$$B_n[\vec{v}_t] = j\left[\frac{\vec{B}_t}{\rho}\right] \tag{73}$$

it follows that the three vectors $\vec{B}_{t2} - \vec{B}_{t1}$, $\vec{B}_{t2}/\rho_2 - \vec{B}_{t1}/\rho_2$ and $\vec{v}_{2t} - \vec{v}_{1t}$ are all parallel, and hence $\vec{B}_{t1}$ is parallel to $\vec{B}_{t2}$ from which it follows that $\vec{B}_1$, $\vec{B}_2$ and $\vec{n}$ be all coplanar. Since $(\vec{v}_{t2} - \vec{v}_{t1})$ lies in this plane $\vec{v}_1$ and $\vec{v}_2$ can also lie in this plane, i.e. the shock is two-dimensional.

Two special cases arise if $\vec{B}_{t1} \equiv 0$.
Then

$$\frac{B_n^2}{4\pi} B_{t2} = j B_n [\vec{v}_t] = \frac{j^2}{\rho^2} B_{t2} \tag{74}$$

and it follows that either (i) $B_{t2} = 0$ or (ii) $j^2 = B_n^2 \rho_2/4\pi$. In the second case, B_{t2} can take any value, i.e. field changes direction and is propagated with velocity $v_{n2} = j/\rho_2 = B_n/(4\pi\rho_2)^{1/2}$ relative to gas behind it. This case has been investigated experimentally by using an annular electromagnetic shock tube which initially contains an axial magnetic field. The shock wave is then driven down the tube in the axial direction, the field taking on a θ-component behind the shock front.

However, most experiments to date have concentrated on the simpler, normal shock geometry in which $B_n = 0$, and $B_t \neq 0$. This geometry arises naturally in the cylindrical imploding pinch, for example where the unshocked gas initially contains a magnetic field in the z-direction.

MAGNETOHYDRODYNAMIC CHARACTERISTICS AND SHOCK WAVES

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Dr. H.E. Petschek delivered a series of lectures on this topic. Since the material covered by these lectures will be published elsewhere, it is not included in these proceedings.

ELEMENTARY ORBIT AND DRIFT THEORY*

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A. GENERAL EQUATIONS OF CLASSICAL PLASMA

A completely ionized plasma consists of a large number of charged particles moving in an electromagnetic field consistent with their own presence. Given the electric field $\vec{E}(\vec{r}, t)$ and the magnetic field $\vec{B}(\vec{r}, t)$, the motion of the i-th particle is determined by the equations

$$\dot{\vec{r}}_i = \vec{v}_i, \tag{1}$$

$$m_i \dot{\vec{v}}_i = q_i [\vec{E}(\vec{r}_i, t) + \vec{v}_i \times \vec{B}(\vec{r}_i, t)], \tag{2}$$

where $\vec{r}_i$, $\vec{v}_i$, m_i and q_i are the position, velocity, mass and charge of the particle. On the other hand, given the motions of all the particles, the electric charge and current densities $\sigma(\vec{r}, t)$ and $\vec{j}(\vec{r}, t)$ are given by:

$$\sigma = \sum_i q_i \, \delta(\vec{r} - \vec{r}_i), \tag{3}$$

$$\vec{j} = \sum_i q_i \vec{v}_i \, \delta(\vec{r} - \vec{r}_i), \tag{4}$$

where δ denotes the Dirac delta function; and then the fields are determined by (boundary and initial conditions and) Maxwell's equations:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \kappa_0 \vec{E}_t, \tag{5}$$

$$\vec{\nabla} \times \vec{E} = -\vec{B}_t, \tag{6}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{7}$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\sigma}{\kappa_0}, \tag{8}$$

where μ_0 and κ_0 are the permeability and permittivity of free space. (We use MKS units throughout). Of these last four equations, only (5) and (6) are needed to advance $\vec{E}$ and $\vec{B}$ in time, while (7) and (8) are consistent side

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* The material of this Paper is substantially that contained in the Lecture "Notes on Orbit Theory" by E.A. Frieman, Summer Institute on Plasma Physics, Princeton University, 25 June - 3 August 1962, which in turn followed C. Longmire (LA - 2055) closely, and C. Longmire, Elementary Plasma Physics, Interscience, New-York (1963).

conditions, remaining valid by virtue of the other equations if they are valid at some initial time.

Because of the non-linearity of these equations, the coupling between the particles and the fields, the large number of particles, and the singularities of the fields (at the particle positions), it is hopeless to try to solve them without considerable approximation. The first approximation is the usual one of statistical mechanics, treating the particles as infinitely but smoothly dense - instead of individual particles we then have (usually several) density distribution functions, with the sums replaced by density-weighted integrals, while the fields become smooth in the limit. Although the ordinary differential equations for the particle motions are replaced by partial differential equations for the distribution functions, this represents a tremendous simplification. The resulting system consists of what are called the Vlasov equations.

B. SINGLE PARTICLE MOTION

In this limit no one particle has any influence on the fields. It is therefore all the more worthwhile to study the motion of a single particle in given (smooth) fields $\vec{B}$ and $\vec{E}$. Dropping the indices, (1) and (2) may be written

$$\dot{\vec{r}} = \vec{v}, \tag{9}$$

$$\epsilon\dot{\vec{v}} = \vec{E}(\vec{r}, t) + \vec{v} \times \vec{B}(\vec{r}, t), \tag{10}$$

where we have introduced the ratio

$$\epsilon \equiv \frac{m}{q} \tag{11}$$

as the one combination of the two particle parameters which really enters. The only Maxwell equations which we must keep in mind (because they hold independently of the distribution of particles) are (6) and (7).

(We may note parenthetically that all our equations can be transformed to emu by formally replacing $\vec{B}$ by $c^{-1}\vec{B}$, μ_0 by $4\pi c^{-2}$, and κ_0 by $(4\pi)^{-1}$, where c is the speed of light in vacuum. Our results can therefore be easily rewritten in emu).

1. Constant uniform fields

We start by studying the motion of a particle in the important special case of (temporally) constant (spatially) uniform fields. The uniformity in particular means that $\vec{E}$ and $\vec{B}$ are independent of $\vec{r}$, so (10) decouples from (9) and can be solved for $\vec{v}$ by itself, with $\vec{r}$ then obtained from (9).

(a) Only electric field

If $\vec{B} = 0$, then $\vec{v}$ changes linearly with time, and $\vec{r}$ quadratically. The particle accelerates freely under a constant force.

(b) Only magnetic field

If, instead, $\vec{E} = 0$, then the component of $\vec{v}$ parallel to $\vec{B}$ remains constant. For taking the inner product of (10) with $\vec{B}$ gives

$$\epsilon \dot{\vec{v}} \cdot \vec{B} = (\vec{v} \times \vec{B}) \cdot \vec{B} = 0, \tag{12}$$

since a triple scalar product of three vectors vanishes if two of them are equal (or even merely parallel). Since we will have much occasion to separate vectors into components parallel and perpendicular to $\vec{B}$ we introduce the notations

$$B \equiv |\vec{B}|,$$

$$\vec{n} \equiv \frac{\vec{B}}{B} \tag{13}$$

$$\vec{v}_{\parallel} \equiv \vec{v} \cdot \vec{n}\, \vec{n}, \quad \vec{v}_{\perp} \equiv \vec{v} - \vec{v}_{\parallel} = \vec{n} \times (\vec{v} \times \vec{n}), \tag{14}$$

$$v_{\parallel} \equiv |\vec{v}_{\parallel}| = \vec{v} \cdot \vec{n}, \quad v_{\perp} \equiv |\vec{v}_{\perp}| = |\vec{v} \times \vec{n}|, \tag{15}$$

and similarly with other vectors (notably $\vec{E}$). From (12) we therefore have

$$\dot{\vec{v}}_{\parallel} = 0, \quad \vec{v}_{\parallel} = \text{constant}. \tag{16}$$

Substituting $\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$ into (10) then gives

$$\epsilon \dot{\vec{v}}_{\perp} = \vec{v}_{\perp} \times \vec{B}. \tag{17}$$

From (17), "dotting" with $\vec{v}_{\perp}$ (i.e. taking the inner product with $\vec{v}_{\perp}$) we have

$$\epsilon \vec{v}_{\perp} \cdot \dot{\vec{v}}_{\perp} = \vec{v}_{\perp} \cdot (\vec{v}_{\perp} \times \vec{B}) = 0, \tag{18}$$

$$(\vec{v}_{\perp}^{\,2})^{\cdot} = 0, \quad \vec{v}_{\perp}^{\,2} = \text{constant}. \tag{19}$$

It follows that $\vec{v}_{\perp}$ remains on a sphere with centre at the origin. Since $\vec{v}_{\perp}$ also lies on the plane through the origin perpendicular to $\vec{B}$ (since $\vec{B} \cdot \vec{v}_{\perp} = 0$), $\vec{v}_{\perp}$ actually remains on a circle centred at the origin.

Because the operation of "crossing" with $\vec{B}$ (taking the cross-product with $\vec{B}$) is a rotation around the axis $\vec{B}$, (17) indeed says that $\vec{v}_{\perp}$ rotates around $\vec{B}$, and in fact uniformly. To verify this and obtain the frequency we use (17) twice to obtain

$$\epsilon^2 \ddot{\vec{v}}_\perp = \epsilon\,(\vec{v}_\perp \times \vec{B})^{\cdot} = \epsilon\,\dot{\vec{v}}_\perp \times \vec{B}$$

$$= (\vec{v}_\perp \times \vec{B}) \times \vec{B} = \vec{B}\,\vec{v}_\perp \cdot \vec{B} - \vec{v}_\perp\,\vec{B} \cdot \vec{B} \tag{20}$$

$$= -B^2 \vec{v}_\perp .$$

Thus each component of $\vec{v}_\perp$ satisfies the equation of a simple harmonic oscillator. Introducing co-ordinates x, y, z with the z axis in the direction of $\vec{B}$, we have

$$v_x = a \cos\left(\frac{B}{\epsilon} t + \varphi\right), \tag{21}$$

where the amplitude a and initial phase φ are arbitrary constants. We could write a similar result for v_y, but v_x and v_y are not independent: Taking say the x component of (17) gives

$$\epsilon \dot{v}_x = v_y B_z = v_y B, \tag{22}$$

where we have taken $B_z = B$ (i.e. $B_z > 0$). Thus

$$v_y = -a \sin\left(\frac{B}{\epsilon} t + \varphi\right). \tag{23}$$

Together (21) and (23) give

$$v_\perp = (v_x^2 + v_y^2)^{\frac{1}{2}} = [a^2 (\cos^2 + \sin^2)]^{\frac{1}{2}} = a. \tag{24}$$

We see that the motion of $\vec{v}_\perp$ is indeed uniformly circular, with angular or "Larmor" frequency

$$\Omega = \frac{B}{\epsilon} = \frac{qB}{m} \tag{25}$$

and period

$$T = \frac{2\pi|\epsilon|}{B} = \frac{2\pi m}{|q|B}. \tag{26}$$

From (9) it now follows that $\vec{r}_\perp$ also undergoes uniform circular motion perpendicular to $\vec{B}$, but about an arbitrary centre $\vec{R}$:

$$x = R_x + \epsilon \frac{v_\perp}{B} \sin\left(\frac{B}{\epsilon} t + \varphi\right), \tag{27}$$

$$y = R_y + \frac{\epsilon v_\perp}{B} \cos\left(\frac{B}{\epsilon} t + \varphi\right). \tag{28}$$

The frequency and period of this "gyration" are also given by (25) and (26), while the radius of gyration, the "Larmor" radius, is

$$R_L \equiv |\vec{r} - \vec{R}| = \frac{|\epsilon| v_\perp}{B} = \frac{m v_\perp}{|q| B} . \qquad (29)$$

(To obtain formulas (25), (26), and (29) in emu, merely replace B by c^{-1} B).

It should be noted that the direction of rotation (of both $\vec{r}$ and $\vec{v}$), or equivalently the sign of Ω, depends on the sign of q, the charge of the particle. With $B_z > 0$, a positive particle (i.e., a positively charged one) gyrates clockwise in the xy plane. More generally, we have a "left-hand rule": with the left thumb in the direction of $\vec{B}$, the left fingers curl in the direction of gyration of a positive particle. For negative particles the corresponding right-hand rule applies. In both cases the gyrating particle constitutes a circulating electric current in the same direction, given by the left-hand rule.

Now the direction of the magnetic field produced by a current is given by a right-hand rule, as may be verified from (5): right thumb along current, fingers curl in direction of $\vec{B}$ produced. Using both hands we see that a gyrating particle tends to produce a magnetic field contrary to the field in which it is gyrating. So a collection of many charged particles (a plasma) is diamagnetic. This may be thought of as an exemplification of Le Chatelier's Principle: if we increase the pressure of a plasma in a magnetic field by adding more (gyrating) particles, these decrease the magnetic field and hence the magnetic pressure $B^2/2\mu_0$, so that the (total fluid and magnetic) pressure increases less than would be naively expected.

(c) Both electric and magnetic fields

Even with both $\vec{B}$ and $\vec{E}$ present (constant and uniform) the parallel and perpendicular velocity components still decouple. The parallel part of (10) gives

$$\epsilon \dot{\vec{v}}_\parallel = \vec{E}_\parallel , \qquad (30)$$

representing free acceleration as in sub-section B.1(a).

The perpendicular part gives

$$\epsilon \dot{\vec{v}}_\perp = \vec{E}_\perp + \vec{v}_\perp \times \vec{B}. \qquad (31)$$

Now $\vec{v}_\perp$ undergoes simultaneous translation and rotation, the superposition of which amounts to rotation about a shifted centre. To show this we first express $\vec{E}_\perp$ as a cross-product by $\vec{B}$.

$$\vec{E}_\perp = -\vec{\alpha} \times \vec{B}; \qquad (32)$$

assuming that $\vec{\alpha}_\parallel = 0$ this requires

$$\vec{E}_\perp \times \vec{B} = B^2\vec{\alpha}, \tag{33}$$

and if we use this to define $\vec{\alpha}$,

$$\vec{\alpha} \equiv \frac{\vec{E} \times \vec{B}}{B^2} \quad , \tag{34}$$

then (32) does indeed hold. Using (32) in (31) gives what may be written

$$\epsilon(\vec{v}_\perp - \vec{\alpha})^{\cdot} = (\vec{v}_\perp - \vec{\alpha}) \times \vec{B}, \tag{35}$$

since $\dot{\vec{\alpha}} = 0$. We see that $\vec{v}_\perp - \vec{\alpha}$ rotates uniformly about the origin just as $\vec{v}_\perp$ was found to do in sub-section B. 1 (b).

Denoting the average over one gyration period by angle brackets we have

$$\langle \vec{v}_\perp - \vec{\alpha} \rangle = 0\,, \quad \langle \vec{v}_\perp \rangle = \vec{\alpha}, \tag{36}$$

$$\langle \vec{v} \rangle = \vec{v}_\parallel + \vec{\alpha}. \tag{37}$$

Thus $\vec{\alpha}$ is the average perpendicular velocity, called the (perpendicular) "drift velocity".

Assuming for the rest of this lecture that $\vec{E}_\parallel = 0$, the motion of the particle itself is the superposition of a uniform gyration on a rectilinear motion of its average position

$$\langle \vec{r} \rangle = (\vec{v}_\parallel + \vec{\alpha})t + \text{constant}. \tag{38}$$

This average position is called the "guiding centre". If we think of a disk (or wheel) always perpendicular to $\vec{B}$ and rotating uniformly about the (moving) guiding centre, the trajectory (in space-time) of any point on the disk gives a possible particle trajectory. If in particular $\vec{\alpha} = 0$, i.e. $\vec{E} = 0$, then the locus of the trajectory is a helix, with axis parallel to $\vec{B}$ - unless $\vec{v}_\parallel = 0$, in which case the locus is a circle (as in any case the projection of the locus onto a plane perpendicular to $\vec{B}$ is).

If $\vec{\alpha} \neq 0$ but $\vec{v}_\parallel = 0$, the locus is a generalized cycloid (as in any case the projection of the locus is). Several possible cycloidal trajectories of positive particles are shown in Fig. 1. We take $\vec{B}$ as pointing out of the page towards the viewer and $\vec{E}$ as pointing downward. To understand the motion we introduce the electric potential φ, satisfying

$$\vec{E} = -\vec{\nabla}\varphi\,, \tag{39}$$

and take the equipotentials to be horizontal lines in the figure. Particle (a) starts from rest and so is subject to no $\vec{v} \times \vec{B}$ force initially; it starts to

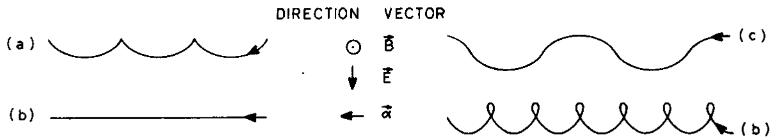

Fig.1

Possible cycloidal trajectories of positive particles

"fall" in the potential field and so picks up kinetic energy at the expense of potential energy (since the total energy

$$\mathcal{E} = \tfrac{1}{2}\,mv^2 + q\varphi = q(\tfrac{1}{2}\,\epsilon\, v^2 + \varphi) \tag{40}$$

is conserved). As v increases, the $\vec{v} \times \vec{B}$ force term becomes more important and increasingly curves the trajectory toward the left, until the particle starts to rise, slowing up and moving straighter and eventually coming momentarily to rest, before beginning another cycloidal arc. Particle (b) has initially just the right velocity ($\vec{\alpha}$) so that $\vec{E}$ and $\vec{v} \times \vec{B}$ cancel; thus it neither falls nor rises, but moves uniformly leftward (the direction of $\vec{\alpha}$) on a horizontal straight line. Particle (c) has initially a leftward velocity intermediate between zero and $\vec{\alpha}$ and alternately falls and rises back to its initial level, while always moving leftward. Particle (d) has initially a leftward velocity numerically greater than $\vec{\alpha}$, and alternately rises and falls back to its initial level, actually going rightward on the higher levels of its orbit, though drifting systematically leftward and forming a string of loops. A still faster particle would form overlapping loops, while a particle moving initially rightward would be just like (d), but starting from a highest point of the orbit instead of a lowest point.

(d) Non-electromagnetic force

If some additional previously unmentioned force $\vec{F}$ (for instance a gravitational force) acts on the particle, we must modify (10) by including the additional term $q^{-1}\,\vec{F}$ on the right. So long as this additional force is curl-free it can be absorbed into $\vec{E}$ without affecting (6). So far as this section B. 1 is concerned, $\vec{F}$ is uniform (and constant), hence curl-free, and so produces the drift velocity

$$\vec{F} \times \vec{B}/q\,B^2 \tag{41}$$

corresponding to (34).

2. Slightly varying fields

The importance of understanding the motion in constant, uniform fields (section B. 1) is that in a very wide variety of cases of interest the fields are approximately constant and uniform, at least on the distance and time scales seen by the particle during one gyration. This is so not only for many laboratory plasmas, including almost all those of relevance to the problem of controlled thermonuclear reactions, but for a great number of astrophysical and space-physical applications. In this section (B. 2) we will discuss the elementary picture of some of the effects of slight variations of the fields. The systematic formal theory of the gyrating particle will be given later.

(a) Time-varying electric field

We assume first that $\vec{B}$ is constant and uniform, while $\vec{E}$ is (spatially) uniform but varies in time (slowly compared to Ω). The parallel motion gives nothing of special interest - we again have (30), but there is nothing much more to be said.

For the perpendicular motion we again have (31). Introducing $\vec{\alpha}$ by (34), instead of (35) we now obtain (35) with an additional term,

$$\epsilon\,(\vec{v}_\perp - \vec{\alpha})^{\cdot} + \epsilon\,\dot{\vec{\alpha}} = (\vec{v}_\perp - \vec{\alpha}) \times \vec{B}, \tag{42}$$

since $\vec{E}$ (hence $\vec{\alpha}$) is no longer constant. Absorbing the new term $\epsilon\dot{\vec{\alpha}}$ into the cross-product with $\vec{B}$, as we did with the $\vec{E}$ term before, we have

$$\epsilon\,(\vec{v}_\perp - \vec{\alpha})^{\cdot} = (\vec{v}_\perp - \vec{\alpha} + \epsilon\,\frac{\dot{\vec{\alpha}} \times \vec{B}}{B^2}) \times \vec{B}, \tag{43}$$

since $\overline{\vec{B} \cdot \vec{\alpha}} = 0$. To exhibit the gyration we write this as

$$\begin{aligned} \epsilon\,(\vec{v}_\perp - \vec{\alpha} + \epsilon\,\frac{\dot{\vec{\alpha}} \times \vec{B}}{B^2})^{\cdot} - \epsilon^2\,\frac{\ddot{\vec{\alpha}} \times \vec{B}}{B^2} \\ = (\vec{v}_\perp - \vec{\alpha} + \epsilon\,\frac{\dot{\vec{\alpha}} \times \vec{B}}{B^2}) \times \vec{B}. \end{aligned} \tag{44}$$

To say that $\vec{E}$ varies slowly is to imply that its higher time derivatives are of successively smaller orders of magnitude, so we may neglect the $\ddot{\vec{\alpha}}$ term and conclude that the expression in parentheses gyrates in the familiar way. Averaging over a period gives

$$\langle \vec{v}_\perp \rangle - \vec{\alpha} = -\frac{\epsilon}{B^2}\,\dot{\vec{\alpha}} \times \vec{B} = -\frac{\epsilon}{B^2}\,(\frac{\dot{\vec{E}} \times \vec{B}}{B^2}) \times \vec{B} = \frac{\epsilon}{B^2}\,\dot{\vec{E}}_\perp \tag{45}$$

as a small drift velocity in addition to the main drift $\vec{\alpha}$ and parallel velocity $\vec{v}_\parallel$.

To understand this drift velocity in physical terms, let $\vec{E}$ average to zero over one gyration period, so $\vec{E}$ points in one direction at the start of the period and in the opposite direction at the finish, vanishing halfway along. In Fig. 2(a), the positive particle starts at the bottom of its orbit, is decelerated as it rises in the first half of its gyration period because $\vec{E}$ points downward, and is also decelerated as it falls in the second half of the period

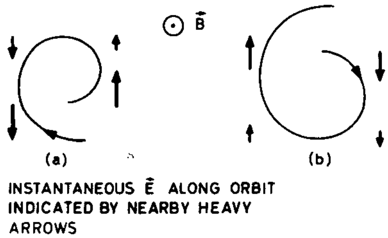

Fig. 2

Drift velocity of a positive particle due to $\vec{E}$

because $\vec{E}$ then points upward; evidently the guiding centre is moving upward, which is the direction of $\dot{\vec{E}}$. In Fig. 2(b), a positive particle starts at the top of its orbit instead and so is accelerated throughout, but again finishes higher than it started.

(b) Space-varying electric field and time-varying magnetic field

Spatial variation of $\vec{E}$ is associated with temporal variation of $\dot{\vec{B}}$ by (6). If $\vec{E}$ has a curl it tends to act on the particle the same way all around its orbit and systematically accelerate or decelerate it, changing the gyrational energy, and this is the effect we wish to calculate.

In constant fields the velocity is the sum of three terms,

$$\vec{v} = \vec{v}_{\parallel} + \vec{\alpha} + \vec{v}_{gyr}, \tag{46}$$

where $\vec{v}_{gyr}$ is the part of $\vec{v}$ which gyrates in a circle around the origin. The kinetic energy of the particle has the average value

$$\left\langle \frac{m}{2}\vec{v}^2 \right\rangle = \frac{m}{2}\left(v_{\parallel}^2 + \alpha^2 + v_{gyr}^2\right) = W_{\parallel} + W_{drift} + W_{gyr} , \tag{47}$$

where each cross term obtained in squaring (46) drops out either by orthogonality or upon averaging; here v_{gyr} is what was called a in sub-section B.1(b). We denote W_{drift} and W_{gyr} taken together as $W_{\perp}$,

$$W_{\perp} = W_{drift} + W_{gyr} = \frac{m}{2}\langle v_{\perp}^2 \rangle, \tag{48}$$

but note that in the literature $W_{\perp}$ usually denotes just W_{gyr}.

Let $\vec{E}$ be constant in time and $\vec{B}$ uniform in space, and let $\vec{E}_{\parallel} = 0$. Let $\vec{E}_{\perp} = 0$ at the guiding centre, so $\vec{\alpha} = 0$. From (31) we obtain

$$\left(\frac{\epsilon}{2} v_{\perp}^2\right)^{\cdot} = \epsilon\, \dot{\vec{v}}_{\perp} \cdot \vec{v}_{\perp} = \vec{E} \cdot \vec{v}_{\perp} . \tag{49}$$

Therefore the change in $W_{gyr} = W_{\perp}$ in one period is

$$\delta W_\perp = \delta\ (\tfrac{1}{2} m v_\perp^2) = q \oint \vec{E} \cdot \vec{v}_\perp\, dt = q \oint \vec{E} \cdot d\vec{r}, \tag{50}$$

where the loop integral is taken around the gyration orbit. Using Stoke's theorem, with $\vec{ds}$ the vector element of area, this can be written as a surface integral over the circular area bounded by the orbit:

$$\begin{aligned} \delta W_\perp &= q \iint (\vec{\nabla} \times \vec{E}) \cdot \vec{ds} = - q \iint \vec{B}_t \cdot \vec{ds} \\ &= - q\, \vec{B}_t \cdot \iint \vec{ds}, \end{aligned} \tag{51}$$

where we have used (6) and then used the fact that $\vec{B}_t$ is uniform. The magnitude of the area integral is πR_L^2 and its direction is opposite to $\vec{B}$ (by the right-hand rule for Stoke's theorem and the left-hand rule for gyration, see sub-section B. 1(b)), so

$$\begin{aligned} \delta W_\perp &= - q\, \vec{B}_t \cdot\ (-\vec{n}\, \pi R_L^2) = \pi q B_t R_L^2 \\ &= \pi B_t \frac{m^2 v_\perp^2}{q B^2} = \frac{2\pi m}{q B^2} B_t\ W_\perp\ , \end{aligned} \tag{52}$$

where we have used (29). Now the change in B over one period is just

$$\delta B = B_t\, T = B_t \frac{2\pi m}{qB} \tag{53}$$

by (26), so we have

$$\delta W_\perp = \frac{W_\perp}{B}\ \delta B. \tag{54}$$

This states that $W_\perp$ varies proportionally to B,

$$\delta\, (W_\perp / B) = 0\ , \tag{55}$$

$$\mu \equiv \frac{W_{gyr}}{B} = \text{constant}. \tag{56}$$

This quantity μ which is conserved as B changes is actually the magnetic moment of the electric current constituted by the gyrating charged particle. For the magnetic moment of a planar current loop is the product of the current by the area enclosed, while the current itself is the quotient of the charge by the period. The (scalar) magnetic moment is therefore given by

$$\frac{|q|}{T}\pi R_L^2 = \frac{q^2 B}{2\pi m}\pi\frac{m^2 v_\perp^2}{q^2 B^2} = \frac{m v_\perp^2}{2B} = \frac{W_\perp}{B} = \mu . \tag{57}$$

There is also a simple relationship between μ and the magnetic flux encircled by the gyration orbit. This (scalar) flux is the product of the field strength by the area,

$$B\pi R_L^2 = \frac{2\pi m}{q^2}\mu . \tag{58}$$

Thus, as B increases, say, and the lines of force crowd closer together, the orbit shrinks so as to enclose always the "same number" of lines of force.

(c) Space-varying magnetic field

Let $\vec{E} = 0$ and let $\vec{B}$ be in the z direction at the guiding centre of the particle. The spatial variation of $\vec{B}$ locally is given by the dyadic $\nabla\vec{B}$, i.e. the matrix

$$\begin{array}{c}
\begin{array}{ccc} \phantom{\dfrac{\partial B_x}{\partial x}} & \text{twist} & \text{curvature} \end{array} \\
\left|\begin{array}{ccc}
\dfrac{\partial B_x}{\partial x} & \dfrac{\partial B_x}{\partial y} & \dfrac{\partial B_x}{\partial z} \\[2ex]
\dfrac{\partial B_y}{\partial x} & \dfrac{\partial B_y}{\partial y} & \dfrac{\partial B_y}{\partial z} \\[2ex]
\dfrac{\partial B_z}{\partial x} & \dfrac{\partial B_z}{\partial y} & \dfrac{\partial B_z}{\partial z}
\end{array}\right| \\
\begin{array}{cc} \text{gradient B} & \text{divergence} \end{array}
\end{array} \tag{59}$$

The terms split into four kinds as indicated. We will interpret the physical significance of the terms by examining the corresponding behaviours of the lines of magnetic force. Actually all the terms are present at once, in general, and their effects add.

A magnetic line of force, by definition, is a line (i.e. curve) everywhere tangent to the magnetic field there. If $d\vec{r}$ is the differential of position vector along a magnetic line, then $d\vec{r}$ must be parallel to $\vec{B}$,

$$\vec{dr} \times \vec{B} = 0 , \tag{60}$$

so that $\vec{dr}$ is an (infinitesimal) scalar multiple of $\vec{B}$, i. e.

$$\frac{dx}{B_x} = \frac{dy}{B_y} = \frac{dz}{B_z} . \tag{61}$$

Since $\vec{B}$ is mainly in the z direction it is most convenient to describe a line of force by giving x and y as functions of z: the equations these functions satisfy are

$$\frac{dx}{dz} = \frac{B_x}{B_z}, \quad \frac{dy}{dz} = \frac{B_y}{B_z}. \tag{62}$$

Divergence terms. It is the diagonal terms of (59) which contribute to the divergence of $\vec{B}$, and we have

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \vec{\nabla} \cdot \vec{B} = 0 . \tag{63}$$

To see the significance of these terms for the magnetic line geometry we assume that they are the only non-vanishing terms and write $B_x = (\partial B_x/\partial x)\, x$, etc., by a Taylor expansion, remembering that $B_x = 0$ at the origin (the guiding centre) and keeping only the next (first-order) term. The line of force crossing the z = 0 plane at x_0 and y_0 approximately satisfies

$$\frac{dx}{dz} = \frac{1}{B_z} \frac{\partial B_x}{\partial x} x_0, \quad \frac{dy}{dz} = \frac{1}{B_z} \frac{\partial B_y}{\partial y} y_0 , \tag{64}$$

$$x = x_0 + \frac{1}{B_z} \frac{\partial B_x}{\partial x} x_0 z, \quad y = y_0 + \frac{1}{B_z} \frac{\partial B_y}{\partial y} y_0 z . \tag{65}$$

Thus the lines diverge (or converge) in the xz plane as in Fig. 3, and similarly in the yz plane.

Taking the z (i. e. parallel) component of (10) gives

$$\begin{aligned} \epsilon \dot{v}_z &= v_x B_y - v_y B_x = v_x \frac{\partial B_y}{\partial y} y - v_y \frac{\partial B_x}{\partial x} x \\ &= (a \cos) \frac{\partial B_y}{\partial y} (\epsilon \frac{v_\perp}{B} \cos) - (- a \sin) \frac{\partial B_x}{\partial x} (\frac{\epsilon v_\perp}{B} \sin), \end{aligned} \tag{66}$$

where we have used (21), (23), (27), and (28). Averaging over a period and using (24) and (63) gives

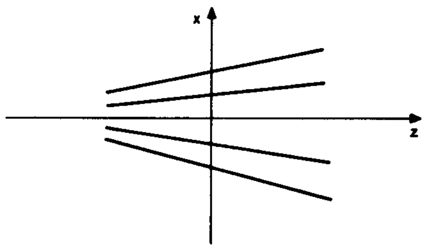

Fig. 3

Magnetic line geometry corresponding to divergence terms of matrix (59)

$$\epsilon\langle\dot{v}_z\rangle = \tfrac{1}{2}\frac{\epsilon v_\perp^2}{B}\frac{\partial B_y}{\partial y} + \tfrac{1}{2}\frac{\epsilon v_\perp^2}{B}\frac{\partial B_x}{\partial x} = -\frac{\epsilon v_\perp^2}{2B}\frac{\partial B_z}{\partial z}$$
$$= -\frac{W_\perp}{qB}\frac{\partial B_z}{\partial z} \qquad (67)$$

Since there is no electric field, the total (kinetic) energy of the particle is constant (because the $\vec{v}\times\vec{B}$ always acts perpendicular to the velocity and hence does no work on the particle), so

$$\dot{W}_\perp = -\dot{W}_\parallel = -\langle\frac{m}{2}v_z^2\rangle^{\cdot} = -mv_z\langle\dot{v}_z\rangle$$
$$= \frac{W_\perp}{B}\frac{\partial B_z}{\partial z}v_z = \frac{W_\perp}{B}\frac{\partial B_z}{\partial z}\frac{dz}{dt} = \frac{W_\perp}{B}\frac{dB}{dt}. \qquad (68)$$

Thus we find that

$$\frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0, \qquad (69)$$

so that the magnetic moment is again a constant.

Generalizing (67) from the particular choice of co-ordinate system, we obtain

$$m\dot{v}_\parallel = -\mu\vec{n}\cdot\vec{\nabla}\vec{B} = -\vec{n}\cdot\vec{\nabla}(\mu B). \qquad (70)$$

Since $\vec{n}\cdot\vec{\nabla}$ is the spatial derivative along a line of force, the particle, in moving as if it were constrained along the line, feels μB as if it were a potential. This is the basis of the well known "magnetic mirror" effect.

Curvature terms. The terms $\partial B_x/\partial z$ and $\partial B_y/\partial z$ have similar effects. Taking only the former we have, from (62),

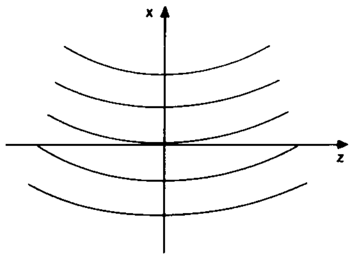

Fig. 4

Curvature of lines of force corresponding to curvature terms in matrix (59)

$$\frac{dx}{dz} = \frac{1}{B_z} \frac{\partial B_x}{\partial z} z , \tag{71}$$

$$x = x_0 + \frac{1}{B_z} \frac{\partial B_x}{\partial z} \frac{1}{2} z^2, \tag{72}$$

so that the lines of force curve in the xz plane as shown in Fig. 4. Along the line of force through the origin, we see from similarity of triangles (Fig. 5) that

$$\frac{B_x}{B} = \frac{z}{R_c}, \tag{73}$$

where R_c is the radius of curvature; from differential geometry the vector radius of curvature $\vec{R}_c$ is determined by

$$\frac{1}{R_c^2} \vec{R}_c = - \vec{n} \cdot \vec{\nabla} \vec{n} . \tag{74}$$

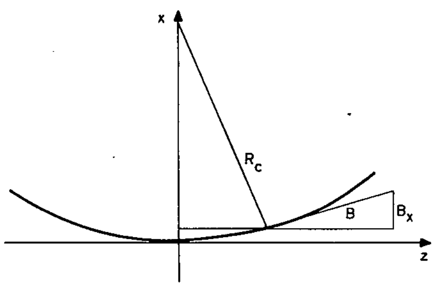

Fig. 5

Showing relation of radius of curvature R_c to B_x and B_z

The guiding centre of the particle moves along the line of force through the origin and so feels a centrifugal force

$$\vec{F} = m v_{\parallel}^2 \frac{1}{R_c^2} \vec{R}_c. \tag{75}$$

This force is perpendicular to $\vec{B}$ and can be thought of as an extraneous force as discussed in sub-section B. 1(d). By (41) we expect it to produce the drift velocity

$$\frac{\vec{F} \times \vec{B}}{qB^2} = \frac{m v_{\parallel}^2}{qB^2 R_c^2} \vec{R}_c \times \vec{B}. \tag{76}$$

Note that this drift depends on q, so that positive and negative particles drift oppositely, making an electric current.

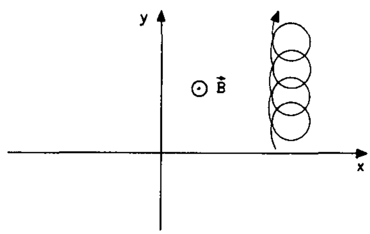

Fig. 6

Gyration of particle corresponding to $\frac{\partial B_z}{\partial x}$ term of matrix (59)

Gradient of field strength terms. The terms $\partial B_z/\partial x$ and $\partial B_z/\partial y$ have similar effects, and we take only the former. The field B has only a z component, which varies with x. If $\partial B_z/\partial x > 0$, the particle gyrates in a tighter orbit on the right than on the left (Fig. 6) and so drifts upward, i.e. in the positive y direction. It is evident that the orbit is symmetric about a point of vertical velocity ($v_x = 0$), hence there is no drift in the x direction - the x motion is periodic, the y motion periodic and uniform rectilinear superimposed. Denoting change over one gyration period by δ, we have

$$\begin{aligned} 0 &= \delta v_x = \oint dv_x = \oint \dot{v}_x \, dt \\ &= \frac{1}{\epsilon} \oint (\vec{v} \times \vec{B})_x \, dt = \frac{1}{\epsilon} \oint v_y B_z \, dt = \frac{1}{\epsilon} \oint B_z \, dy, \end{aligned} \tag{77}$$

and a Taylor expansion gives

$$B_z = B + \frac{\partial B_z}{\partial x} x, \tag{78}$$

where B means the (constant) value of B at the guiding centre. Thus we have

$$0 = \oint (B + \frac{\partial B_z}{\partial x} x)\, dy$$

$$= B \oint dy + \frac{\partial B_z}{\partial x} \oint x\, dy \tag{79}$$

$$= B\ \delta y - \frac{\partial B_z}{\partial x} \pi R_L^2,$$

since xdy is the negative of the element of area. We can now use (26) to compute the drift velocity

$$\frac{\delta y}{T} = \frac{1}{B} \frac{\partial B_z}{\partial x} \pi R_L^2 \frac{qB}{2\pi m} = \frac{m v_\perp^2}{2q B^2} \frac{\partial B_z}{\partial x}$$
$$= \frac{\mu}{qB} \frac{\partial B}{\partial x}. \tag{80}$$

Generalizing to vector form we find the drift velocity

$$\frac{\mu}{qB^2} \vec{B} \times \vec{\nabla} B. \tag{81}$$

This can be interpreted in accordance with (41) as the drift velocity resulting from a force

$$\vec{F} = - \mu \vec{\nabla} B = - \vec{\nabla}(\mu B), \tag{82}$$

so that again we find μB acting as a kind of potential.

Field twist terms. The remaining terms $\partial B_x/\partial y$ and $\partial B_y/\partial x$ enter into the z component of $\vec{\nabla} \times \vec{B}$, i.e. into $\vec{B} \cdot (\vec{\nabla} \times \vec{B})$. These terms represent twisting of the lines of force about each other. They have no particular interest for particle motion.

C. MOTION OF LINES OF MAGNETIC FORCE

It is often said that particles drift in such a way as to stay on a line of magnetic force. What amounts to the same thing but from a different point of view, it is said that lines of magnetic force are "frozen into" an ideal plasma, being carried along by the fluid motion. These motions were clarified by W. Newcomb in a basic work.

In classical electromagnetism lines of force are defined instant by instant, as discussed in sub-section B · 2 (c). No meaning is attached to the identity of a line of force as persisting throughout an interval of time, nor can be in any natural way. Any velocity field $\vec{v}(\vec{r}, t)$ such that points on a common line of force flow always into points which are still on a common line of force may be called "line-preserving", and it is permissible to think of each line of force as a curve with an identity in time, each point of the curve moving with velocity $\vec{v}$. This is even more the case if the velocity field is "flux-preserving" as well, i.e., if the "density" of lines of force is correctly represented by the flow, so that we do not have to think of lines of force as being created or destroyed during the motion.

The main theorem is the following: any velocity field $\vec{v}$ satisfying

$$\vec{E} + \vec{v} \times \vec{B} = 0 \tag{83}$$

is both line-preserving and flux-preserving. This equation is the well known "Ohm's Law with infinite conductivity" of ideal magnetohydrodynamics.

The theorem may equivalently be stated: if

$$\vec{E} \cdot \vec{B} = 0, \tag{84}$$

then the velocity field

$$\vec{v} \equiv \vec{E} \times \vec{B}/B^2 \tag{85}$$

(previously called $\vec{\alpha}$), indeed any velocity field whose perpendicular part is given by (85) and whose parallel part is arbitrary, is both line-preserving and flux-preserving. Condition (84) is generally satisfied to good approximation for hydromagnetic flows because any parallel electric field which tends to develop is neutralized by parallel flow of the electrons, which are relatively mobile compared to the heavier ions which carry the main mass.

We will prove this theorem in several different ways.

1. First proof of line preservation

To prove that lines of force are preserved, let f be any function such that

$$\vec{B} \cdot \vec{\nabla} f = 0 \tag{86}$$

at some initial time; this means that f is constant on each line of force, since $\vec{B} \cdot \vec{\nabla}$ is a tangential differential operator. Let f be "carried along" by the velocity field $\vec{v}$, i.e. let f be constant on any trajectory moving with velocity $\vec{v}$; this is expressed mathematically by

$$df/dt = 0 , \tag{87}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \ . \tag{88}$$

We wish to prove that (86) remains valid.

We start by obtaining the commutation law for the operators d/dt and $\vec{\nabla}$ as applied to any function f, not necessarily satisfying (86) and (87). Since t and $\vec{r}$ are independent variables, $\partial/\partial t$ and $\vec{\nabla}$ commute, so

$$\begin{aligned}\vec{\nabla}\frac{df}{dt} &= \vec{\nabla}\left(\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}f\right) \\ &= \frac{\partial}{\partial t}\vec{\nabla}f + \vec{\nabla}\vec{v}\cdot\vec{\nabla}f + \vec{v}\cdot\vec{\nabla}\vec{\nabla}f \qquad (89) \\ &= \frac{d}{dt}\vec{\nabla}f + \vec{\nabla}\vec{v}\cdot\vec{\nabla}f,\end{aligned}$$

$$\vec{\nabla}\frac{d}{dt} - \frac{d}{dt}\vec{\nabla} = \vec{\nabla}\vec{v}\cdot\vec{\nabla}. \qquad (90)$$

For f satisfying (87), (89) becomes

$$\frac{d}{dt}\vec{\nabla}f = -\vec{\nabla}\vec{v}\cdot\vec{\nabla}f. \qquad (91)$$

We next compute the derivative of $\vec{B}$, using (6) and (83):

$$\begin{aligned}\frac{\partial}{\partial t}\vec{B} &= -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{v}\times\vec{B}) \\ &= \vec{B}\cdot\vec{\nabla}\vec{v} + (\vec{\nabla}\cdot\vec{B})\vec{v} - \vec{v}\cdot\vec{\nabla}\vec{B} - (\vec{\nabla}\cdot\vec{v})\vec{B},\end{aligned} \qquad (92)$$

where the expansion of the triple cross-product gives two terms, each of which itself consists of two terms because $\vec{\nabla}$ acts on $\vec{v}$ and $\vec{B}$ in its capacity as a differential operator. Since $\vec{\nabla}\cdot\vec{B} = 0$ we obtain

$$\frac{d}{dt}\vec{B} = \frac{\partial}{\partial t}\vec{B} + \vec{v}\cdot\vec{\nabla}\vec{B} = \vec{B}\cdot\vec{\nabla}\vec{v} - \vec{B}\vec{\nabla}\cdot\vec{v}. \qquad (93)$$

Together (91) and (93) give

$$\begin{aligned}\frac{d}{dt}(\vec{B}\cdot\vec{\nabla}f) &= (\vec{B}\cdot\vec{\nabla}\vec{v} - \vec{B}\vec{\nabla}\cdot\vec{v})\cdot\vec{\nabla}f - \vec{B}\cdot\vec{\nabla}\vec{v}\cdot\vec{\nabla}f \\ &= -(\vec{\nabla}\cdot\vec{v})\vec{B}\cdot\vec{\nabla}f.\end{aligned} \qquad (94)$$

Thus if $\vec{B}\cdot\vec{\nabla}f$ vanishes at one time, it also vanishes a small time Δt later, at least to first order in Δt. From this it follows that (86) remains exactly valid for all time. In fact, (94) is a regular linear homogeneous first order differential equation for $\vec{B}\cdot\vec{\nabla}f$ and has therefore a unique solution for any given initial condition, and if $\vec{B}\cdot\vec{\nabla}f$ vanishes initially, that unique solution is obviously (86) for all t.

Since any function f constant on a line of force remains constant on a line of force as it is convected with the flow $\vec{v}$, it follows that the lines of force are preserved under the flow.

2. Second proof of line preservation

Let δ be a differential operator representing differencing of neighbouring points. Then the condition that neighbouring points lie on a common line of force is that $\delta\vec{r}$ be parallel to $\vec{B}$,

$$\delta\vec{r}\times\vec{B} = 0. \tag{95}$$

Since δ commutes with d/dt, we have

$$\frac{d}{dt}\,\delta\vec{r} = \delta\frac{d\vec{r}}{dt} = \delta\vec{v} = \delta\vec{r}\cdot\vec{\nabla}\vec{v}\,, \tag{96}$$

where the last step is an application of the chain rule of differentiation. By (93) we have

$$\frac{d}{dt}(\delta\vec{r}\times\vec{B}) = \delta\vec{r}\cdot\vec{\nabla}\vec{v}\times\vec{B} - \vec{B}\cdot\vec{\nabla}\vec{v}\times\delta\vec{r} - \delta\vec{r}\times\vec{B}\,\vec{\nabla}\cdot\vec{v}. \tag{97}$$

If (95) holds at one time, then the last term vanishes and also $\delta\vec{r} = a\vec{B}$, where a is an infinitesimal scalar, so that the first two terms on the right cancel. Thus (95) continues to hold if it holds initially.

We can actually put (97) into the form of a regular linear homogeneous equation for $\delta\vec{r}\times\vec{B}$, as (94) was for $\vec{B}\cdot\vec{\nabla}f$. For this we need an identity which we will have further use for later, and which it is perhaps simpler to write down directly and verify rather than to obtain by successively transforming one side till it becomes the other. The identity is

$$\vec{A}\cdot\vec{\mathbb{D}}\times\vec{B} - \vec{B}\cdot\vec{\mathbb{D}}\times\vec{A} = \vec{\mathbb{D}}:\vec{\mathbb{I}}\,\vec{A}\times\vec{B} - \vec{\mathbb{D}}\cdot(\vec{A}\times\vec{B}), \tag{98}$$

where $\vec{A}$ and $\vec{B}$ are arbitrary vectors, $\vec{\mathbb{D}}$ is an arbitrary dyadic, and $\vec{\mathbb{I}}$ is the unit dyadic; here $\vec{\mathbb{D}}:\vec{\mathbb{I}}$ is just the trace of $\vec{\mathbb{D}}$. To prove (98) it suffices to prove that its inner product with each of $\vec{A}$, $\vec{B}$ and $\vec{A}\times\vec{B}$ is an identity, for these three vectors are independent (unless $\vec{A}\times\vec{B} = 0$, in which case the right side vanishes, and so does the left side because $\vec{A}$ or $\vec{B}$ is a scalar multiple of the other).

Dotting (98) with $\vec{A}$ or $\vec{B}$ gives an obvious identity. To prove the third identity, we start from the dyadic formula.

$$\vec{\mathbb{I}}\,(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\vec{B}\times\vec{C} + \vec{B}\vec{C}\times\vec{A} + \vec{C}\vec{A}\times\vec{B}\,, \tag{99}$$

which is immediately verified by dotting on the right by $\vec{A}$, $\vec{B}$, and $\vec{C}$ in turn. Taking the double inner product of (99) with $\vec{\mathbb{D}}$ gives

$$\vec{\mathbb{D}}:\mathbb{I}\,(\vec{A}\times\vec{B})\cdot\vec{C} = \vec{A}\cdot(\vec{\mathbb{D}}\times\vec{B})\cdot\vec{C} - \vec{B}\cdot(\vec{\mathbb{D}}\times\vec{A})\cdot\vec{C} + \vec{C}\cdot\vec{\mathbb{D}}\cdot(\vec{A}\times\vec{B}), \tag{100}$$

and now taking $\vec{C} = \vec{A}\times\vec{B}$ gives a trivial rearrangement of the formula obtained by dotting (98) with $\vec{A}\times\vec{B}$.

Having proved (98) in general, we now specialize it by taking $\vec{\mathbb{D}}$ to be first $\vec{\nabla}\vec{v}$ and second the transpose of $\vec{\nabla}\vec{v}$ to obtain the two formulas

$$\vec{A}\cdot\vec{\nabla}\vec{v}\times\vec{B} - \vec{B}\cdot\vec{\nabla}\vec{v}\times\vec{A} = \vec{\nabla}\cdot\vec{v}\,\vec{A}\times\vec{B} - \vec{\nabla}\vec{v}\cdot(\vec{A}\times\vec{B}), \tag{101}$$

$$\vec{A}\times\vec{\nabla}\vec{v}\cdot\vec{B} - \vec{B}\times\vec{\nabla}\vec{v}\cdot\vec{A} = \vec{\nabla}\cdot\vec{v}\,\vec{A}\times\vec{B} - (\vec{A}\times\vec{B})\cdot\vec{\nabla}\vec{v}. \tag{102}$$

The latter will be used later; taking $\vec{A} = \delta\vec{r}$ in the former enables us to write (97) as

$$\frac{d}{dt}(\delta\vec{r}\times\vec{B}) = -\vec{\nabla}\vec{v}\cdot(\delta\vec{r}\times\vec{B}). \tag{103}$$

This is the desired linear differential equation for $\delta\vec{r}\times\vec{B}$, showing that (95) remains valid if valid initially. Therefore two neighbouring points on a line of force stay on the same line of force in the course of time. Integrating along the line, the same is true also for non-neighbouring points.

3. *Proof of flux conservation*

Let $\vec{\sigma}$ be an element of surface, so that the corresponding magnetic flux is $\vec{B}\cdot\vec{\sigma}$. The logarithmic rate of change of the volume element $\delta\vec{r}\cdot\vec{\sigma}$ is given by the divergence of the velocity field,

$$\frac{d}{dt}(\delta\vec{r}\cdot\vec{\sigma}) = \delta\vec{r}\cdot\vec{\sigma}\,\vec{\nabla}\cdot\vec{v}; \tag{104}$$

this is what the divergence means. (The formula follows immediately from the possibly more familiar equation of continuity of a field, $d\rho/dt = -\rho\,\vec{\nabla}\cdot\vec{v}$, where ρ is the mass density, together with the law of conservation of mass, $d/dt\,(\rho\,\delta\vec{r}\cdot\vec{\sigma}) = 0$). Expanding and using (96) gives

$$\delta\vec{r}\cdot\vec{\nabla}\vec{v}\cdot\vec{\sigma} + \delta\vec{r}\cdot\frac{d\vec{\sigma}}{dt} = \delta\vec{r}\cdot\vec{\sigma}\,\vec{\nabla}\cdot\vec{v}. \tag{105}$$

Since $\delta\vec{r}$ is arbitrary we have

$$\frac{d\vec{\sigma}}{dt} = \vec{\sigma}\,\vec{\nabla}\cdot\vec{v} - \vec{\nabla}\vec{v}\cdot\vec{\sigma}. \tag{106}$$

It is now simple to compute that the rate of change of flux is

$$\frac{d}{dt}(\vec{B}\cdot\vec{\sigma}) = (\vec{B}\cdot\vec{\nabla}\vec{v} - \vec{B}\,\vec{\nabla}\cdot\vec{v})\cdot\vec{\sigma} + \vec{B}\cdot(\vec{\sigma}\,\vec{\nabla}\cdot\vec{v} - \vec{\nabla}\vec{v}\cdot\vec{\sigma}) = 0, \qquad (107)$$

which shows that flux is conserved.

4. Simultaneous proof of line preservation and flux conservation

A frequently useful representation of a magnetic field $\vec{B}$ is

$$\vec{B} = \vec{\nabla} f \times \vec{\nabla} g. \qquad (108)$$

One immediate advantage of this form is that it implies

$$\vec{\nabla}\cdot\vec{B} = (\vec{\nabla}\times\vec{\nabla} f)\cdot\vec{\nabla} g - \nabla f\cdot(\vec{\nabla}\times\vec{\nabla} g) = 0, \qquad (109)$$

since the divergence of a curl vanishes; the divergence-free character of $\vec{B}$ is implicit in (108). Further, we have

$$\vec{B}\cdot\vec{\nabla} f = 0, \quad \vec{B}\cdot\vec{\nabla} g = 0, \qquad (110)$$

which state that f and g are constant along lines of force, or, in other words, that a line of force lies in a surface of constant f as well as in one of constant g, hence is the intersection of such surfaces. Evidently a line of force is specified by giving a pair of values for f and g, so that f and g serve as co-ordinates for the family of lines of force.

In fact f and g are a pair of flux functions. By this we mean that the magnetic flux of the tube of lines with f between f_0 and f_1, and g between g_0 and g_1, is $(f_1 - f_0)(g_1 - g_0)$; the magnetic flux of a tube is the flux through any cross section of the tube, all cross sections having the same flux through them because $\vec{\nabla}\cdot\vec{B} = 0$. To show that f and g are a pair of flux functions it suffices to show that the flux of an elementary tube df dg (i.e. of lines between f and f + df and between g and g + dg) is just df dg itself. To see this we introduce an arbitrary function h (not constant on lines) to serve as a third co-ordinate in space. We first observe that the volume element in physical space is

$$\frac{df\,dg\,dh}{(\vec{\nabla} f\times\vec{\nabla} g)\cdot\vec{\nabla} h}, \qquad (111)$$

where the denominator is just the Jacobian of f, g, h. Now the orthogonal distance between two neighbouring surfaces of constant h is $dh/|\vec{\nabla} h|$, so if $\vec{\sigma}$ is the element of area on such a surface, then the volume element is

$$\sigma\,\frac{dh}{|\vec{\nabla} h|}. \qquad (112)$$

Equating this to (111) and using (108) gives

$$\frac{\sigma}{|\vec{\nabla} h|} \vec{B} \cdot \vec{\nabla} h = df\, dg. \tag{113}$$

Since $\vec{\nabla}h / |\vec{\nabla}h|$ is the unit normal to the surface of constant h, we have

$$\vec{\sigma} = \sigma \frac{\vec{\nabla}h}{|\vec{\nabla}h|}, \tag{114}$$

$$\vec{B} \cdot \vec{\sigma} = df\, dg, \tag{115}$$

whence df dg is indeed the element of flux.

It is instructive to see how to construct flux functions for a given field $\vec{B}$, and how much freedom is involved. First take an arbitrary surface cutting across the lines of force (which could be thought of as a surface of constant h, though we do not use h any more). On this surface take an arbitrary (smooth) function f (see Fig. 7). Draw a single arbitrary curve cutting across the curves of constant f, and on that curve assign g arbitrarily, say g = 0. Along each curve of constant f, assign g so that gdf is the flux through an infinitesimal strip df from the g = 0 curve to the point of assignment. It is clear that df dg is the element of flux on the surface chosen. Finally, extend f and g throughout space as constants on each line of force in accord with (110).

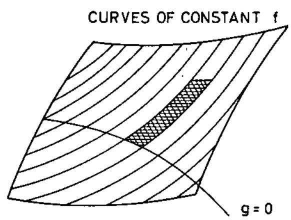

Fig. 7

Flux element g df

To show that f and g as so constructed do satisfy (108), we observe that $\vec{B}$ is orthogonal to both $\vec{\nabla}f$ and $\vec{\nabla}g$, by (110), hence parallel to $\vec{\nabla}f \times \vec{\nabla}g$, so that

$$\vec{\nabla}f \times \vec{\nabla}g = a\, \vec{B}, \tag{116}$$

where a is a scalar function. Then

$$0 = \vec{\nabla}\cdot(\vec{\nabla} f \times \vec{\nabla} g) = \vec{\nabla}\cdot(a\vec{B})$$
$$= \vec{B}\cdot\vec{\nabla} a + a\vec{\nabla}\cdot\vec{B} = \vec{B}\cdot\vec{\nabla} a, \tag{117}$$

whence a is constant on lines of force. But a = 1 on the chosen surface, since f and g were chosen to be a pair of flux functions on that surface, i.e. to satisfy (115) there. It follows that a = 1 everywhere, and so (116) reduces to (108).

We now prove that if (108) is satisfied at one instant of time and if f and g are carried along by $\vec{v}$, then it remains satisfied for all time. By (91) and the same equation for g,

$$\begin{aligned}\frac{d}{dt}(\vec{\nabla} f \times \vec{\nabla} g) &= -(\vec{\nabla}\vec{v}\cdot\vec{\nabla} f)\times\vec{\nabla} g - \vec{\nabla} f \times(\vec{\nabla}\vec{v}\cdot\vec{\nabla} g)\\ &= \vec{\nabla} g \times \vec{\nabla}\vec{v}\cdot\vec{\nabla} f - \vec{\nabla} f \times \vec{\nabla}\vec{v}\cdot\vec{\nabla} g\\ &= \vec{\nabla}\cdot\vec{v}\,\vec{\nabla} g\times\vec{\nabla} f - (\vec{\nabla} g\times\vec{\nabla} f)\cdot\vec{\nabla}\vec{v}\\ &= (\vec{\nabla} f\times\vec{\nabla} g)\cdot\vec{\nabla}\vec{v} - \vec{\nabla} f\times\vec{\nabla} g\,\vec{\nabla}\cdot\vec{v},\end{aligned} \tag{118}$$

where we have applied (102) with $\vec{A} = \vec{\nabla} g$ and $\vec{B} = \vec{\nabla} f$ (here $\vec{B}$ is not to be confused with the magnetic field). We observe that (118) is precisely the same first order differential equation for $\vec{\nabla} f \times \vec{\nabla} g$ as (93) is for the magnetic field $\vec{B}$. Therefore if $\vec{B}$ and $\vec{\nabla} f \times \vec{\nabla} g$ are equal initially, they remain equal.

D. MOTION OF PARTICLES WITH LINES OF FORCE

To lowest order the velocity of the guiding centre of a particle is $\vec{v} = \vec{v}_{\parallel} + \vec{\alpha}$. Since this velocity satisfies (83), we may ascribe it as the velocity of lines of force consistently with line preservation and flux conservation. Now $\vec{\alpha}$ is given, by (34), in terms of the electromagnetic fields alone, but $\vec{v}_{\parallel}$ varies from particle to particle. However, in thinking of the motion of a geometric entity such as a line (of force), only the perpendicular velocity is significant; the parallel velocity component moves the line into itself, i.e., does not change the line at all. This is because we have assigned an identity to a line of force as a whole, but not to points on a line. Any particular $\vec{v}$ satisfying (83) does of course lead to an assignment of identity to points, but many different velocity fields are consistent with (83) for a given electromagnetic field, and they identify points differently, but lines in the same way.

Since the perpendicular velocity of the guiding centre is $\vec{\alpha}$ for all particles whose guiding centres are at the same place, and $\vec{\alpha}$ is a preserving and conserving velocity field, we always choose $\vec{\alpha}$ to be the (perpendicular) velocity of lines of force (so long as $\vec{E}\cdot\vec{B}$ vanishes, at least approximately) and have the result that (the guiding centres of) all particles move (approximately) with the lines of force, i.e. in such a way as to stay always on the same line of force.

This statement has implications that do not involve the arbitrary and unobservable assignment of identity of lines. In particular, it implies that any two particles on a common line of force at one time are also on a common line of force at another time, even if we do not choose to think of it as the "same" line. But in fact we do so choose.

It should be noted that this "locking-together" of particles and lines is valid only in the leading approximation. The small (next order) drifts in general depend both on the particle parameter ϵ and on the velocity of the particle, so that no assignment of motion to lines could follow all particles at once.

This state of affairs is in contrast to that for the adiabatic invariant μ, which is constant not only to dominant order but, as we shall see, to all orders in the appropriate expansion. Even so, μ is not exactly constant, since the constancy is to all orders of what is not in general a convergent series, but rather an asymptotic series.

ADVANCED THEORY OF GYRATING PARTICLES *

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Let a magnetic field $\vec{B}$ and an electric field $\vec{E}$ be given as functions of position $\vec{r}$ and time t. We are interested in the non-relativistic motion of a charged "test" particle in the familiar "guiding-centre" approximation [1]. This applies when the motion of the particle may be treated approximately as a gyration (uniform circular motion) orthogonal to $\vec{B}$, of small radius but finite velocity (hence large frequency), superimposed on the finite velocity motion of a so-called guiding-centre. The electromagnetic fields at the position of the particle change only a little during one gyration period.

Let the mass of the particle be m and its charge e. The guiding-centre approximation may be formalized by considering the ratio $\epsilon = m/e$ to be numerically small, but the velocity of the particle and the electromagnetic fields to be finite. We are, therefore, interested in analysing the solution of the equation of motion of the particle, namely

$$\epsilon\, \ddot{\vec{r}} = \vec{E}(\vec{r}, t) + \dot{\vec{r}} \times \vec{B}(\vec{r}, t), \tag{119}†$$

in the limit as ϵ approaches zero.

Because the order of (119) is reduced by formally going to the limit $\epsilon = 0$, it is clear that simply a representation of $\vec{r}$ by a power series in ϵ is inappropriate. Instead, an asymptotic analysis is called for. It turns out to be appropriate to write

$$\vec{r} = \sum_{n=-\infty}^{\infty} \epsilon^{|n|}\, \vec{R}_n(t)\, \exp[n\, C(t)/\epsilon], \tag{120}$$

where each vector function $\vec{R}_n(t)$ is itself a power series in ϵ, starting with a term of zero'th order. The characteristic function C(t) can be chosen independent of ϵ, but it may (as we shall see) be convenient to choose it to be a power series in ϵ. The functions $\vec{R}_n$ and C may be complex but $\vec{r}$ must be real.

The conditions on the $\vec{R}_n$ and C are found by substituting (120) and its derivatives into (119), expanding $\vec{E}$ and $\vec{B}$ in Taylor series around $\vec{R}_0$ (note that $\vec{r} - \vec{R}_0 = O(\epsilon)$), and separately equating the corresponding coefficients of each exponential $\exp[nC/\epsilon]$. These coefficients are themselves power series in ϵ, of course.

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* Based on KRUSKAL, M., "The gyration of a charged particle", PM-S-33, NYO-7903 (1958). The Appendix, which is new, serves to fill several gaps in the treatment.

† Numbering continued from KRUSKAL, M., Elementary orbit and drift theory, these Proceedings.

For n = 0 this leads to

$$\epsilon \ddot{\vec{R}}_0 = \vec{E} + \dot{\vec{R}}_0 \times \vec{B} + \epsilon \dot{C}[\vec{R}_1 \times (\vec{R}_{-1} : \nabla\vec{B}) - \vec{R}_{-1} \times (\vec{R}_1 : \nabla\vec{B})] + O(\epsilon^2) \qquad (121)$$

where here and later through (142) $\vec{E}$ and $\vec{B}$ and their derivatives are understood to have the arguments $\vec{R}_0$, t. The physical content of (121) to lowest order is that the velocity of the guiding centre $\dot{\vec{R}}_0$ orthogonal to $\vec{B}$ is $(\vec{E} \times \vec{B})/\vec{B}^2$, which is well known.

From Eq. (121) trivially follows

$$\vec{E} \cdot \vec{B} = O(\epsilon) \qquad (122)$$

which is a condition not so much on the trajectory as on the given fields. Since physically $\vec{E}$ and $\vec{B}$ seem independent of ϵ, this would appear to require $\vec{E} \cdot \vec{B} = 0$. However, we certainly do not wish to be so severely restricted. The way out is to let the original fields $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ be given as power series in ϵ. Mathematically this is an entirely natural and almost trivial generalization, but it permits us to treat physical situations where the field vectors are not strictly orthogonal (though they must be so approximately; this will often be the case in plasmas of high electrical conductivity, for instance). We assume that $\vec{B} \neq O(\epsilon)$ at any point on the trajectory.

The physical significance of (122) is that $\vec{E}$ can have no component along $\vec{B}$ to lowest order, because if it did, it would give the particle a large acceleration and hence a large velocity (of order $1/\epsilon$). This would contradict the condition that the fields seen by the particle be nearly constant during one gyration period.

Only the components of $\dot{\vec{R}}_0$ orthogonal to $\vec{B}$ are determined directly from (121); information on the parallel component is hidden. The information may be uncovered by dotting with $\vec{B}$ and dividing through by ϵ, which yield

$$\ddot{\vec{R}}_0 \cdot \vec{B} = \epsilon^{-1} \vec{E} \cdot \vec{B} + \dot{C}[\vec{R}_1 \times (\vec{R}_{-1} : \vec{\nabla}\vec{B}) - \vec{R}_{-1} \times (\vec{R}_1 \cdot \vec{\nabla}\vec{B})] \cdot \vec{B} + O(\epsilon). \qquad (123)$$

This is the desired condition. It determines the parallel component of the acceleration, rather than of the velocity.

Returning to the substitution of (120) into (119), from the terms involving exp $[C/\epsilon]$ (i.e. n = 1) we obtain

$$\dot{C}^2 \vec{R}_1 = \dot{C} \vec{R}_1 \times \vec{B} + O(\epsilon). \qquad (124)$$

Now we do not permit $\dot{C}$ to be $O(\epsilon)$, as this would vitiate the whole point of the representation (120). Hence we may divide through by $\dot{C}$ or by $\dot{C}^2$.

Taken to lowest order, (124) may be construed as a homogeneous linear equation for $\vec{R}_1$. Indeed, we are presented with an eigen-value problem, for $\vec{R}_1$ is not generally $O(\epsilon)$. Since

$$\vec{B} \cdot \vec{R}_1 = O(\epsilon) \qquad (125)$$

(as seen by dotting (124) with $\vec{B}$), by using (124) twice we obtain

$$\begin{aligned}\dot{C}^2\vec{R}_1 &= (\vec{R}_1\times\vec{B})\times\vec{B}+O(\epsilon)\\ &= -\vec{B}^2\vec{R}_1+O(\epsilon).\end{aligned} \tag{126}$$

Thus we have determined the eigen-values $\dot{C} = \pm\, i|\vec{B}|+O(\epsilon)$, and if $\dot{C}$ were not permitted to depend on ϵ this would determine it completely (except for the trivial choice of sign). It greatly simplifies the formalism, however, to choose

$$\dot{C} = i\,|\vec{B}|, \tag{127}$$

which is a series in ϵ not only because $\vec{B}(\vec{r},t)$ is, but also because so is $\vec{R}_0$, the argument of $\vec{B}$ in (127).

We pause to point out that although from (124), by dotting with $\vec{R}_1$, we deduce that $\vec{R}_1^2 = O(\epsilon)$, this does not imply $\vec{R}_1 = O(\epsilon)$. For $\vec{R}_1$ may be (indeed must be) a complex vector.

A further point to notice is that since $\dot{C}$ is purely imaginary, the condition we must impose to insure that $\vec{r}$ as given by (120) be real is

$$\vec{R}_{-n} = \vec{R}_n^*, \tag{128}$$

where the asterisk indicates the complex conjugate. We may, therefore, restrict ourselves to the determination of $\vec{R}_n$ for $n \geq 0$.

Returning to (124), it is easily seen that the eigen-vector $\vec{R}_1$ to lowest order is uniquely determined up to an arbitrary multiplicative complex scalar. For by (125) its component along $\vec{B}$ vanishes, while its components in any two mutually orthogonal directions both orthogonal to $\vec{B}$ are determinable from each other by (124) itself.

Let ϵ F denote the $O(\epsilon)$ terms in (124), so that

$$\dot{C}^2\vec{R}_1 - \dot{C}\,\vec{R}_1\times\vec{B} = \epsilon\,\vec{F}. \tag{129}$$

Viewing this as a linear system of algebraic equations for $\vec{R}_1$, we have determined $\dot{C}$ so as to make the system degenerate (which ensures that the corresponding homogeneous system has a non-trivial solution). It follows that there is a linear condition on the non-homogeneous terms $\epsilon\,\vec{F}$ for (129) to have a solution $\vec{R}_1$. It is easily verified that the necessary and sufficient condition is

$$\dot{C}^2\vec{F}+\dot{C}\,\vec{F}\times\vec{B}+\vec{F}\cdot\vec{B}\,\vec{B} = 0. \tag{130}$$

Carrying out the derivation of (124) to the next order gives

$$\vec{F} = \vec{R}_1\cdot\vec{\nabla}\vec{E}+\dot{\vec{R}}_1\times\vec{B}+\dot{\vec{R}}_0\times(\vec{R}_1\cdot\vec{\nabla}\vec{B}) - \ddot{C}\,\vec{R}_1 - 2\,\dot{C}\,\dot{\vec{R}}_1+O(\epsilon). \tag{131}$$

This together with (130) gives to lowest order a first order differential equation for $\vec{R}_1$. Since $\vec{R}_1$ has already been determined up to a complex multiplier, this amounts to a first order differential equation for the dependence of the multiplier on t.

Returning once again to the substitution of (120) into (119), from the terms involving $\exp[n\,C/\epsilon]$ $(n \geq 2)$ we obtain

$$n^2\dot{C}^2\,\vec{R}_n = n\,\dot{C}\,\vec{R}_n \times \vec{B} + \vec{G}_n, \tag{132}$$

where to lowest order (zero'th) $\vec{G}_n$ is a polynomial in the $\vec{R}_p$ $(1 \leq p \leq n-1)$ with space derivatives of $\vec{B}$ as coefficients. Since $\dot{C}$ as previously determined is not an eigen-value of the homogeneous equation for $\vec{R}_n$ obtained from (132) by deleting $\vec{G}_n$, it follows that $\vec{R}_n$ is determined algebraically by (132).

We now have a complete set of recursion relations determining the $\vec{R}_n$ to every order (with respect to expansion in powers of ϵ) in terms of the lower order $\vec{R}_n$. The original exact equation (119), formally an ordinary vector differential equation of second order, constitutes a sixth order system of ordinary scalar differential equations. In our recursion relations these orders re-appear as follows: Two orders from the two components of $\vec{R}_0$ (which is real) orthogonal to $\vec{B}$ as determined by (121). Another two orders from the single component of $\vec{R}_0$ in the direction of $\vec{B}$ as determined by (123). None from (124) which determines $\vec{R}_1$ up to a complex scalar multiplicative function. Finally, two more from the first order complex scalar differential equation for that function.

Although our procedure has been entirely formal, it has been proved by BERKOWITZ and GARDNER [2] that the series (120) which we finally obtain really is an asymptotic series to all orders in ϵ for the exact solution of (119).

Incidentally, it turns out that by using (124) and (128) it is possible to write (123) in the form (see Appendix)

$$\ddot{\vec{R}}_0 \cdot \vec{B} = \epsilon^{-1}\,\vec{E}\cdot\vec{B} - \frac{1}{2}\,|\vec{R}_1|^2\,\vec{B}\cdot\vec{\nabla}\vec{B}^2 + O(\epsilon). \tag{133}$$

Since the only way $\vec{R}_1$ (with two degrees of freedom) enters this equation is through its absolute value (with one degree of freedom), it is suggested that we examine how $|\vec{R}_1|$ varies. From our differential equation for $\vec{R}_1$, i.e. (130) with (131), we can compute $|\vec{R}_1|^{\cdot}$. With the help of the Maxwell equations

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0,\ \vec{\nabla}\cdot\vec{B} = 0, \tag{134}$$

we can reduce the resulting expression to

$$|\vec{R}_1|^{\cdot} = -\,|\vec{R}_1|\,\frac{|\vec{B}|^{\cdot}}{2|B|} + O(\epsilon). \tag{135}$$

In view of this, $\vec{R}_0$ and $|\vec{R}_1|$ satisfy at each step of the recursion a system of differential equations of fifth order. The one remaining degree of freedom in $\vec{R}_1$ then satisfies a first order differential equation involving $\vec{R}_0$ and $|\vec{R}_1|$. (See Appendix for relation between $|\vec{R}_1|$ and the gyration radius.)

It may be noted that (135) can be integrated explicitly to lowest order, yielding

$$|\vec{R}_1|^2 \; |\vec{B}| = k + O(\epsilon), \tag{136}$$

where k is a constant. This exhibits the well-known lowest order constancy of the "magnetic moment" of the particle [1], although in the form written it more obviously exhibits the equivalent lowest order constancy of the magnetic flux enclosed by the gyration of the particle, as viewed from the Galilean frame of reference in which the guiding centre is instantaneously at rest.

Having completed the expansion of the equations of motion, we now claim that there is an adiabatic invariant which is constant to all orders in ϵ, and which is given by (136) to lowest order. (This has already been shown to the next order by HELLWIG [3]). It must be emphasized that this result is only asymptotic, i.e. that constancy to all orders does not mean exact constancy, but merely that the deviation from constancy goes to zero faster than any power of ϵ. The adiabatic invariant, furthermore, is in general not just $|\vec{R}_1|^2 \; |\vec{B}|$, but an infinite series whose leading term is $|\vec{R}_1|^2 \; |\vec{B}|$. However, if the particle is in a region of space-time where the electromagnetic field is constant (both spatially and temporally), then the higher order terms vanish and the invariant is just $|\vec{R}_1|^2 \; |\vec{B}|$.

The method of proof was suggested by KULSRUD's proof of an analogous statement for the harmonic oscillator [4]. If in (120) we replace the bracketed exponent by $n i \theta$ and specify that

$$\dot{\theta} = |\vec{B}| / \epsilon, \tag{137}$$

we have in view of (127) an equivalent representation, in which $\vec{r}$ appears as a series of (non-negative) powers of ϵ with coefficients which are functions of θ and t, periodic in θ with period 2π. From here on the argument proceeds quite generally for any system describable by a Hamiltonian; it is well known that a particle moving in a given electromagnetic field constitutes such a system (see Appendix).

Let $\vec{q}$ and $\vec{p}$ be canonical variables and $H(\vec{q}, \vec{p}, t)$ the Hamiltonian. Suppose $\vec{q}$ and $\vec{p}$ can be written as functions of two independent variables t and θ, periodic in θ with period 2π, in such a way that there is a similar function α of t and θ with the property that if θ is made any function of t satisfying $\dot{\theta} = \alpha$ (independently of the initial value of θ), then $\vec{q}$ and $\vec{p}$ constitute a trajectory. (In our particular application here α is a function of t only, given by (137).) Using subscripts to denote partial derivatives, we have Hamilton's equations of motion

$$\begin{aligned} H_{\vec{p}} &= d\vec{q}/dt = \vec{q}_t + \vec{q}_\theta \alpha, \\ -H_{\vec{q}} &= d\vec{p}/dt = \vec{p}_t + \vec{p}_\theta \alpha. \end{aligned} \tag{138}$$

Taking the inner product of these with $\vec{p}_\theta$ and $\vec{q}_\theta$ respectively and subtracting gives

$$\vec{p}_\theta \cdot \vec{q}_t - \vec{q}_\theta \cdot \vec{p}_t = H_{\vec{p}} \cdot \vec{p}_\theta + H_{\vec{q}} \cdot \vec{q}_\theta. \tag{139}$$

Remembering that θ and t are independent variables, we integrate with respect to θ from 0 to 2π. The right-hand side then obviously vanishes, and if the first term on the left-hand side is integrated by parts the result can be written

$$-\int_0^{2\pi} (\vec{p} \cdot \vec{q}_{\theta t} + \vec{q}_\theta \cdot \vec{p}_t)\, d\theta = 0. \tag{140}$$

But this is a perfect derivative with respect to t, whence

$$K = \int_0^{2\pi} \vec{p} \cdot \vec{q}_\theta \, d\theta = \oint \vec{p} \cdot d\vec{q}. \tag{141}$$

Thus the usual "adiabatic invariant" or action integral $\oint \vec{p} \cdot d\vec{q}$ is indeed a constant K whenever and to whatever extent one can introduce an auxiliary variable θ with the properties specified.

In the application to the gyrating particle, it may be noted that in regions where the fields $\vec{E}$ and $\vec{B}$ are constant (spatially and temporally) and are orthogonal to all orders, it can be easily proved that $\vec{R}_0$ is a linear function of t, $\vec{R}_1$ is independent of t, and $\vec{R}_2$, $\vec{R}_3$, ... all vanish. When $\oint \vec{p} \cdot d\vec{q}$ is evaluated in such a region, it turns out that K is given there (except for a trivial numerical factor) by $|\vec{R}_1|^2 \, |\vec{B}|$. To compute the higher order terms at a general point is in principle straight-forward but quickly becomes rather lengthy. To the next order above the lowest, however, it is not yet by any means prohibitive. (See Appendix.)

Actually, it is more useful to write the invariant in terms of familiar dynamical variables, here $\vec{r}$ and $\vec{v}$, where

$$\vec{v} \equiv \dot{\vec{r}} = \sum_{n=-\infty}^{\infty} \epsilon^{|n|} (n \dot{C} \vec{R}_n/\epsilon + \dot{\vec{R}}_n) \exp(n C/\epsilon). \tag{142}$$

It is convenient to introduce

$$\vec{v}_\perp \equiv \vec{v} - (\vec{B} \cdot \vec{v} \vec{B} + \vec{E} \times \vec{B})/\vec{B}^2; \tag{143}$$

in (143) and the following equations $\vec{E}$ and $\vec{B}$ and their derivatives are to be evaluated at $\vec{r}$, and not at $\vec{R}_0$ as in the preceding formulas. Note that $\vec{v}_\perp$ is the component of the velocity $\vec{v}$ perpendicular to $\vec{B}$ in a frame of reference in which $\vec{E}$ vanishes instantaneously at $\vec{r}$.

Without giving any details or indications of the method of calculation, we merely state that the constant of motion written explicitly to the first two orders turns out to be

$$\begin{aligned}
&|\vec{B}|^{-1}\vec{v}_{\perp}^{\,2}+\epsilon|\vec{B}|^{-5}\{-\tfrac{1}{2}\vec{v}_{\perp}^{\,2}\vec{v}_{\perp}\cdot(\vec{B}\times\vec{\nabla}\vec{B}^{2})\\
&+(\tfrac{1}{2}\vec{B}^{2}\vec{v}_{\perp}\vec{v}_{\perp}+\tfrac{3}{2}\vec{v}_{\perp}\times\vec{B}\vec{v}_{\perp}\times\vec{B}):(\vec{\nabla}\vec{E}-\vec{\nabla}\vec{B}\times\dot{\vec{R}}_{0})\\
&-2\vec{B}^{2}\vec{v}_{\perp}\cdot[\vec{E}_{t}+\dot{\vec{R}}_{0}\cdot\vec{\nabla}\vec{E}+\dot{\vec{R}}_{0}\times(\vec{B}_{t}+\dot{\vec{R}}_{0}\cdot\vec{\nabla}\vec{B})]\}+O(\epsilon^{2});
\end{aligned}\tag{144}$$

here

$$\dot{\vec{R}}_{0}=\vec{v}-\vec{v}_{\perp}+O(\epsilon)=(\vec{B}\cdot\vec{v}\vec{B}+\vec{E}\times\vec{B})/\vec{B}^{2}+O(\epsilon).\tag{145}$$

The first term of (144) is the well-known lowest order invariant [1]. If $\vec{E}$ vanishes identically the constant of motion can be simplified considerably and written

$$\begin{aligned}
&|\vec{B}|^{-1}\vec{v}_{\perp}^{\,2}-\epsilon|\vec{B}|^{-5}[(\vec{v}\times\vec{B})\cdot(\vec{\nabla}\vec{B})\cdot(\vec{v}^{2}\vec{B}+\vec{B}\cdot\vec{v}\vec{v})\\
&+\vec{B}\cdot\vec{v}(\vec{\nabla}\times\vec{B})\cdot(\tfrac{1}{2}\vec{v}_{\perp}^{\,2}\vec{B}+2\vec{B}\cdot\vec{v}\vec{v}_{\perp})]+O(\epsilon^{2}).
\end{aligned}\tag{146}$$

Similar methods can be used to prove that the adiabatic invariant for the anharmonic oscillator is constant to all orders. The same result holds for a large number of systems (at least when only one frequency is involved). (See end of Appendix for reference and for explanation of how the general theory applies to the gyrating particle.)

APPENDIX

We solve Eqs.(9) and (10) systematically to all orders in the limit $\epsilon\to 0$, keeping $\vec{B}$, $\vec{E}$, and $\vec{v}$ finite, and the characteristic distances and times of change of $\vec{B}$ and $\vec{E}$ also finite. The limit $\epsilon\to 0$ is appropriate because, by Eqs.(26) and (29), the period and radius of gyration are proportional to ϵ, so that the fields experienced by the particle are in fact then nearly constant during one gyration. [Eqs. (9, 10, 26 and 29) are on pp. 68, 70 and 71].

The derivation of (133) from (123) is based on (101). The expression in square brackets in (123) equals

$$\vec{R}_{1}\cdot\vec{\nabla}\vec{B}\times\vec{R}_{-1}-\vec{R}_{-1}\cdot\vec{\nabla}\vec{B}\times\vec{R}_{1}=\vec{\nabla}\cdot\vec{B}\vec{R}_{1}\times\vec{R}_{-1}-\vec{\nabla}\vec{B}\cdot(\vec{R}_{1}\times\vec{R}_{-1}),\tag{A.1}$$

and of course $\nabla\cdot B=0$. By (124) we have

$$\dot{C}\vec{R}_{1}\times\vec{R}_{-1}\approx(\vec{R}_{1}\times\vec{B})\times\vec{R}_{-1}=\vec{B}\vec{R}_{1}\cdot\vec{R}_{-1}-\vec{R}_{1}\vec{B}\cdot\vec{R}_{-1}\approx|\vec{R}_{1}|^{2}|\vec{B}|,\tag{A.2}$$

in view of (128) and (the complex conjugate) of (125). Since $\vec{\nabla}\vec{B}\cdot\vec{B}=\frac{1}{2}\vec{\nabla}B^{2}$, we immediately obtain (133). More generally, we can write (121) as

$$\epsilon\ddot{\vec{R}}_{0}=\vec{E}+\dot{\vec{R}}_{0}\times\vec{B}-\epsilon|\vec{R}_{1}|^{2}B\vec{\nabla}B+O(\epsilon).\tag{A.3}$$

In order to interpret $|\vec{R}_1|^2$ we compute the gyration radius. From (120),

$$\vec{r} = \vec{R}_0 + \epsilon\, \vec{R}_1 e^{C/\epsilon} + \epsilon\, \vec{R}_{-1}\, e^{-C/\epsilon} + O(\epsilon^2), \tag{A.4}$$

so the square of the gyration radius is

$$R_L^2 \equiv (\vec{r} - \vec{R}_0)^2 = 2\epsilon^2 \vec{R}_1 \cdot \vec{R}_{-1} + O(\epsilon^3) = 2\epsilon^2 |\vec{R}_1|^2 + O(\epsilon^3), \tag{A.5}$$

since $\vec{R}_1^2 = O(\epsilon)$ and $\vec{R}_{-1}^2 = O(\epsilon)$, as pointed out in the paragraph between (127) and (128). Thus $|\vec{R}_1|$ is approximately the gyration radius up to a simple factor. Note that we must write $|\vec{R}_1|$ rather than R_1 because $\vec{R}_1$ is a complex vector, so that $R_1 \equiv (\vec{R}_1^2)^{\frac{1}{2}}$ could be zero (as it is, approximately) or even negative, whereas $|\vec{R}_1| \equiv (\vec{R}_1 \cdot \vec{R}_1^*)^{\frac{1}{2}} = (\vec{R}_1 \cdot \vec{R}_{-1})^{\frac{1}{2}}$ is necessarily non-negative and vanishes only if $\vec{R}_1 = 0$.

The velocity of the particle is

$$\vec{v} = \dot{\vec{r}} = \dot{\vec{R}}_0 + \dot{C}\vec{R}_1 e^{C/\epsilon} - \dot{C}\vec{R}_{-1} e^{-C/\epsilon} + O(\epsilon). \tag{A.6}$$

The guiding-centre velocity $\dot{\vec{R}}_0$ consists of what we previously called $v_\parallel + \alpha$, and the gyration is in the exponential terms. Evidently

$$v_{gyr}^2 = (\vec{v} - \dot{\vec{R}}_0)^2 \approx -2\dot{C}^2 \vec{R}_1 \cdot \vec{R}_{-1} = 2B^2 |\vec{R}_1|^2. \tag{A.7}$$

The construction of the adiabatic invariant is based on the Hamiltonian character of the equations of motion of the particle. Let $\vec{A}$ and φ be the electromagnetic vector and scalar potentials, so that

$$\vec{\nabla} \times \vec{A} = \vec{B}, \tag{A.8}$$

$$-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi = \vec{E}. \tag{A.9}$$

The Hamiltonian of a particle of mass m and charge q is

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\varphi, \tag{A.10}$$

where $\vec{p}$ is the canonical momentum conjugate to $\vec{r}$ (the argument of $\vec{A}$ and φ). To see this we write Hamilton's equations

$$\vec{v} \equiv \dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{1}{m}(\vec{p} - q\vec{A}), \tag{A.11}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} \equiv -\vec{\nabla} H = \frac{q}{m}\vec{\nabla}\vec{A} \cdot (\vec{p} - q\vec{A}) - q\vec{\nabla}\varphi = q(\vec{\nabla}\vec{A} \cdot \vec{v} - \vec{\nabla}\varphi). \tag{A.12}$$

Eliminating $\vec{p}$ in favour of $\vec{v}$ by (A.11) we have

$$\dot{\vec{p}} = m\dot{\vec{v}} + q\dot{\vec{A}} = m\dot{\vec{v}} + q\left(\frac{\partial \vec{A}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{A}\right), \tag{A.13}$$

so (A.12) becomes

$$\begin{aligned} \frac{m}{q}\dot{\vec{v}} &= \nabla\vec{A}\cdot\vec{v} - \vec{v}\cdot\vec{\nabla}\vec{A} - \frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\varphi \\ &= \vec{v}\times(\vec{\nabla}\times\vec{A}) + \vec{E} = \vec{v}\times\vec{B} + \vec{E} \end{aligned} \tag{A.14}$$

which is indeed just the equation of motion of a particle we have been discussing throughout.

The series (120) represents the position of the particle to all orders in ϵ, assuming the fields are differentiable to all orders. (Higher and higher derivatives of the fields occur in working to higher and higher terms in the series for each $\vec{R}_m$.) In (120) we replace C by $\epsilon\, i\,\theta$. In the series for $\vec{v}$ obtained by differentiating (120), we first replace $\dot{C}$ by $i\,B$ in accord with (127), and then replace C (now appearing only in the exponents) by $\epsilon\, i\,\theta$. We intend to treat θ as an independent variable, but to obtain the correct motion of the particle θ cannot be independent but must vary according to (137). At any time the values of $\vec{r}$ for different values of θ are the positions of particles with entirely similar behaviour (to all orders), whose guiding centres are all at the same place and move together, and which differ from each other only in gyration phase.

The integration over θ in obtaining the adiabatically invariant action integral may be thought of as an integration over a "ring" of such associated particles. The invariant obtained is nevertheless an invariant of each particle separately, because the ring of associated particles is determined when the position and velocity of any one of them is given.

To compute the constant of motion (adiabatic invariant) K given by (141), we write

$$K \equiv \oint \vec{p}\cdot d\vec{r} = q\oint(\epsilon\,\vec{v} + \vec{A})\cdot d\vec{r} \tag{A.15}$$

by (A.11). By (A.6) and (A.4) the first term gives

$$\begin{aligned} \oint \epsilon\,\vec{v}\cdot d\vec{r} &= \epsilon \int_0^{2\pi} \vec{v}\cdot r_\theta\, d\theta \\ &\approx \epsilon \int_0^{2\pi} (\dot{R}_0 + \dot{C}\vec{R}_1 e^{i\theta} - \dot{C}\vec{R}_{-1}\, e^{-i\theta}) : (i\,\epsilon\,\vec{R}_1 e^{i\theta} - i\,\epsilon\,\vec{R}_{-1} e^{-i\theta})\, d\theta \\ &= -4\pi\, i\,\epsilon^2\, \dot{C}|\vec{R}_1|^2 = 4\pi\,\epsilon^2 B|\vec{R}_1|^2 \approx 2\pi B R_L^2, \end{aligned} \tag{A.16}$$

where we have used (127) and (A.5). By Stoke's theorem the second term gives

$$\oint \vec{A}\cdot d\vec{r} = \iint (\vec{\nabla}\times\vec{A})\cdot d\vec{s} = \iint \vec{B}\cdot d\vec{s}, \tag{A.17}$$

where the surface integral is taken over a circular disk spanning the "gyration circle", i.e. spanning the ring of associated particle positions. Actually (A.17) is just the flux through the ring, the field times the area, so,

$$\oint \vec{A}\cdot d\vec{r} \approx - B\,\pi\, R_L^2, \tag{A.18}$$

the minus sign arising as discussed after (51). Thus the second term is half the first and of opposite sign, so

$$K \approx \pi\, q\, B\, R_L^2; \tag{A.19}$$

the conserved quantity is approximately the flux through the gyration ring. Using (A.5), (A.7) and (56), we can also write

$$K \approx \pi\, q\, B\, 2\epsilon^2 |\vec{R}_1|^2 \approx \pi\, q\, \epsilon^2 B^{-1} v_{gyr}^2 \approx 2\,\pi\,\epsilon\, B^{-1}\, W_{gyr} = 2\pi\epsilon\,\mu. \tag{A.20}$$

In carrying out the evaluation of K to higher orders, no difficulty arises with the first term on the right in (A.15); evidently we need only retain the higher order terms neglected in (A.16). With the second term on the right in (A.15), however, our evaluation (A.18) was conceptual rather than formal, hence not easily extended to higher order. An obvious formal procedure we could use is to Taylor-expand $\vec{A}(\vec{r})$, in the closed line integral, around its value at the guiding centre,

$$\vec{A}(\vec{r}) = \vec{A} + (\vec{r}-\vec{R}_0)\cdot\vec{\nabla}\vec{A} + \tfrac{1}{2}(\vec{r}-\vec{R}_0)(\vec{r}-\vec{R}_0) : \vec{\nabla}\vec{\nabla}\vec{A} + \ldots, \tag{A.21}$$

where $\vec{A}$ and its derivatives, on the right side, are evaluated at $\vec{R}_0$. The integral $\int_0^{2\pi} \vec{A}(\vec{r})\cdot\vec{r}_\theta\, d\theta$ is then easily evaluated. The result would come out in terms of derivatives of $\vec{A}$, however, and $\vec{A}$ is not a local physical quantity. The result would have to be expressible in terms of $\vec{B}$ and its derivatives, but the task of so expressing it would involve somewhat awkward vector and dyadic manipulations.

It seems preferable to work from the surface integral on the right of (A.17), in which $\vec{A}$ no longer appears. We need to choose some (any) particular surface spanning the gyration ring. A natural choice is to replace ϵ by φ in our formula for $\vec{r}$,

$$\vec{r} = \vec{R}_0 + \varphi(\vec{R}_1 e^{i\theta} + \vec{R}_{-1} e^{-i\theta}) + \varphi^2(\vec{R}_2 e^{2i\theta} + \vec{R}_{-2} e^{-2i\theta}) + \ldots, \tag{A.22}$$

and view this as the parametric formula of a surface, with the parameters φ and θ satisfying

$$0 \leqslant \varphi < \epsilon, \quad 0 \leqslant \theta < 2\pi. \tag{A.23}$$

The boundary of this disk-like surface, $\varphi = \epsilon$, is indeed just the gyration ring. We now expand $\vec{B}(\vec{r})$, for $\vec{r}$ on the surface (A.22), around its value at $\vec{R}_0$, much as we did with $\vec{A}$ in (A.21). The vector surface element of the parametric surface $\vec{r}(\varphi, \theta)$ is

$$d\vec{s} = \vec{r}_\varphi \times \vec{r}_\theta \, d\phi \, d\theta, \tag{A.24}$$

so we obtain

$$\begin{aligned} \iint \vec{B}\cdot d\vec{s} = \int_0^\epsilon d\varphi \int_0^{2\pi} d\theta [\vec{B} + \varphi(\vec{R}_1 e^{i\theta} + \vec{R}_{-1} e^{-i\theta})\cdot\nabla\vec{B} + O(\varphi^2)] \\ \cdot\{[\vec{R}_1 e^{i\theta} + \vec{R}_{-1} e^{-i\theta} + O(\varphi)] \times [\varphi(i\vec{R}_1 e^{i\theta} - i\vec{R}_{-1} e^{-i\theta}) + O(\varphi^2)]\}. \end{aligned} \tag{A.25}$$

It is trivial to carry out both integrations explicitly to however high an order we care to write out the expansions, and although vector manipulations are needed to simplify the result, they are much easier than those needed in the other procedure to make $\vec{A}$ appear only in the combination $\vec{\nabla}\times\vec{A}$. To dominant order, for instance, (A.25) gives

$$\begin{aligned} \iint \vec{B}\cdot d\vec{s} &\approx \int_0^\epsilon d\varphi \; 2\pi\vec{B}\cdot(-i\varphi\vec{R}_1\times\vec{R}_{-1} + i\varphi\vec{R}_{-1}\times\vec{R}_1) \\ &\approx 2\pi(\tfrac{1}{2}\epsilon^2)(-2i)\vec{B}\cdot(\vec{R}_1\times\vec{R}_{-1}) \\ &\approx 2\pi i\epsilon^2(\vec{R}_1\times\vec{B})\cdot\vec{R}_{-1} \approx 2\pi i\epsilon^2(\dot{C}\vec{R}_1)\cdot\vec{R}_{-1} \\ &\approx -2\pi\epsilon^2 B|\vec{R}_1|^2, \end{aligned} \tag{A.26}$$

of course agreeing with (A.18).

It may be mentioned that a vast generalization of the gyrating particle theory is presented in my paper "Asymptotic Theory of Hamiltonian and Other Systems with All Solutions Nearly Periodic" (J. Math Phys. 3 (1962) 806). What is studied there is an abstract system of ordinary differential equations

$$\vec{x}_s = \vec{f}(\vec{x}, \epsilon), \tag{A.27}$$

where $\vec{x}$ is a vector of any number of dimensions and $\vec{f}$ a corresponding vector function of $\vec{x}$, independent of s but depending on a small parameter ϵ in such a way that the solutions of the lowest order system,

$$\vec{x}_s = \vec{f}(\vec{x}, 0), \tag{A.28}$$

are all periodic in s no matter what initial value for $\vec{x}$ is chosen. In application to the gyrating particle $\vec{x}$ is the seven-dimensional phase-space vector

$$\vec{x} \equiv (\vec{r}, \vec{v}, t) \tag{A.29}$$

and the time-like variable s is not actually the time, but rather

$$s \equiv \epsilon^{-1} t. \tag{A.30}$$

The equations of motion of the particle may be written

$$\begin{aligned} \vec{r}_s &= \dot{\vec{r}} t_s = \epsilon \vec{v}, \\ \vec{v}_s &= \dot{\vec{v}} t_s = \epsilon \dot{\vec{v}} = \vec{E}(\vec{r}, t) + \vec{v} \times \vec{B}(\vec{r}, t), \\ t_s &= \epsilon, \end{aligned} \tag{A.31}$$

which give (A.27) with

$$\vec{f}(\vec{r}, \vec{v}, t, \epsilon) = (\epsilon \vec{v}, \vec{E} + \vec{v} \times \vec{B}, \epsilon). \tag{A.32}$$

To lowest order (A.31) implies

$$\vec{r}_s = 0, \; t_s = 0, \tag{A.33}$$

whence $\vec{r}$ and t are constant, so that the fields in the $\vec{v}_s$ equation are constant and $\vec{v}$ undergoes the usual circular motion (as always $\vec{E} \cdot \vec{B} = 0$ to lowest order is required). Any solution of the lowest order equation is therefore periodic. Note that the significant gyration is not in physical space but rather velocity space.

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DERIVATION OF MACROSCOPIC EQUATIONS

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Since we shall be concerned with collections of large numbers of particles interacting with their self-created and/or externally imposed electromagnetic fields and since it is in general a problem of prohibitive difficulty to follow the detailed motion of all the particles we must rest content to describe the behaviour of the plasma system in some average or statistical sense.

It is often desirable to be content to describe the plasma in terms of the crudest of statistical theories, essentially a hydrodynamic description in terms of mean number density, mean velocity, pressure, etc., modified to include electromagnetic effects.

The value of working at this low level of description is that one can quickly get an insight into the behaviour of the plasma and obtain a large body of qualitative results which are, in general, only somewhat modified by more complete (and thus complex) statistical theories.

We shall now suppose the totally ionized plasma to consist of electrons and positive ions (of one species), masses m^- and m^+, and charges $e^-(=-e)$ and e^+, respectively. The generalization to more complex systems with ions in various stages of ionization is straightforward as long as the internal dynamics of an ion is negligible.

For each particle of our system we have an equation of motion

$$m_n \dot{\vec{v}}_n = e_n \left[\vec{E}(\vec{r}_n, t) + \frac{\vec{v}_n}{c} \times \vec{B}(\vec{r}_n, t) \right], \tag{1a}$$

$$\dot{\vec{r}}_n = \vec{v}_n \,. \tag{1b}$$

(We have written down the non-relativistic equation of motion and indeed for most cases of interest this description is adequate. In some problems, however, e.g. synchrotron radiation, the relativistic description must be invoked and we shall write down for completeness the equation of motion of a single particle in relativistic form:

$$\frac{d}{dt}(\gamma m\vec{v}) = e\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right), \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and m is the rest mass, but we shall not pursue this equation further at this time.)

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The equations of motion for the electromagnetic field, Maxwell's equations, are

$$\frac{\partial \vec{B}}{\partial t}(\vec{r}, t) = -c\vec{\nabla} \times \vec{E}(\vec{r}, t), \tag{3}$$

$$\frac{\partial \vec{E}}{\partial t} = c\vec{\nabla} \times \vec{B} - 4\pi\vec{J}(\vec{r}, t) , \tag{4}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{5}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi\sigma(\vec{r}, t). \tag{6}$$

These equations relate the development of the electromagnetic field intensities $\vec{E}$, $\vec{B}$ in terms of the sources, charge density σ and current density $\vec{J}$. Notice that we have made no distinction between $\vec{E}$ and $\vec{D}$, $\vec{B}$ and $\vec{H}$ since we shall keep full track of our internal sources. It is only when one abandons some of the sources appearing in Eqs. (4) and (6) that the distinction must arise.

We now must relate the sources σ, $\vec{J}$ to the dynamical motion of the plasma. Now instead of relating the electromagnetic fields to the exact dynamical motion of the particles,

$$\sigma(\vec{r}, t) = \sum_n e_n \delta\,[\vec{r} - \vec{r}_n(t)], \tag{7}$$

$$\vec{J}(\vec{r}, t) = \sum_n e_n \vec{v}_n \delta[\vec{r} - \vec{r}_n(t)] \tag{8}$$

we shall embark on the statistical description.

We presuppose the existence of a distribution function for each species $f^+(\vec{r}, \vec{v}, t)$ and $f^-(\vec{r}, \vec{v}, t)$. The distribution function has the meaning that $f(\vec{r}, \vec{v}, t)d^3r\, d^3v$ ($d^3r \equiv dx\,dy\,dz$ and $d^3v = dv_x\, dv_y\, dv_z$) represents the probable number of particles of each type in the volume element $d^3r\, d^3v$ at the point $(\vec{r}, \vec{v})$ in the six-dimensional phase space (μ-space). The particle mean number density and mean velocity of each type are defined by the zero'th and first moment of these distribution functions with respect to velocity:

$$n^{\pm}(\vec{r}, t) = \int f^{\pm}(\vec{r}, \vec{v}, t) d^3v, \tag{9}$$

$$\vec{u}^{\pm}(\vec{r}, t) = \frac{1}{n^{\pm}(\vec{r}, t)} \int \vec{v} f^{\pm}\, d^3v. \tag{10}$$

We now take the sources σ, $\vec{J}$ to be given by

$$\sigma(\vec{r}, t) = \sum_{+,-} en, \tag{11}$$

$$\vec{J} = \sum_{+,-} en\vec{u}\ . \tag{12}$$

The fields $\vec{E}$ and $\vec{B}$ computed from these averages σ, $\vec{J}$ are called the self-consistent, or average internal fields. Any sample particle is subject not only to the self-consistent and/or external fields but also to rapidly fluctuating micro-fields, i.e. forces due to encounters with neighbouring particles in the configuration space. Let us for the time being ignore these micro-fields and determine the laws of motion for $f^{\pm}(\vec{r}, \vec{v}, t)$.

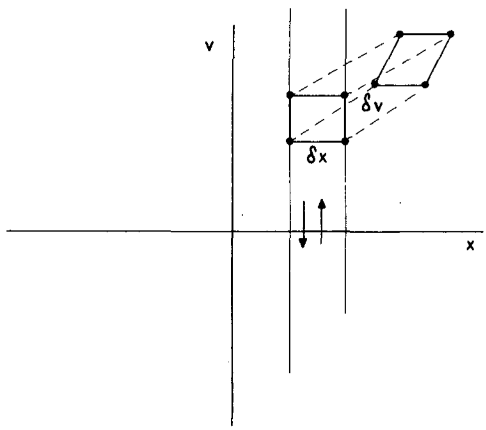

Fig. 1

Motion of volume element in phase space

Consider a small element of volume $\delta\Omega = d^3r\,d^3v$ in the phase space (Fig. 1). The number of phase points associated with particles of either species in this element is $f^{\pm}(\vec{r}, \vec{v}, t)\delta\Omega$. During the course of time the particles corresponding to these phase points move over the phase space but at all times keeping the same number of particles in $\delta\Omega$, in virtue of the fact that neighbouring particles execute neighbouring motions. (The external and/or self-consistent fields are "smooth" over $\delta\Omega$.) Thus,

$$\frac{d}{dt}\,[f^{\pm}(\vec{r}, \vec{v}, t)\delta\Omega] = 0, \tag{13}$$

or

$$\frac{df}{dt}\delta\Omega + f\frac{d(\delta\Omega)}{dt} = 0. \tag{13a}$$

We shall now show $d/dt(\delta\Omega) = 0$.

We shall give this derivation of the law of conservation of extension in phase in more general terms, applicable also to the Liouville equation.

Consider the laws of motion for a system of n degrees of freedom in their first order form

$$\dot{z}_i = g_i(z_1, z_2, \ldots, z_n),$$

where z_i may be a co-ordinate or a velocity. Consider the extension in phase

$$\delta\Omega = \prod_i \delta z_i .$$

Then

$$(\delta\Omega)^{\cdot}/\delta\Omega = \sum_i \frac{\delta \dot{z}_i}{\delta z_i}$$

$$= \sum_i [g_i(z_1, \ldots, z_i + \delta z_i, \ldots, z_n) - g_i(\ldots, z_i, \ldots)] / \delta z_i$$

$$= \sum_i \frac{\partial g_i}{\partial z_i} + \text{higher order terms}$$

$$\equiv \vec{\nabla}_n \cdot \vec{g} + \text{higher order terms.}$$

For Hamiltonian systems $\vec{\nabla}_n \cdot \vec{g} = 0$.

In our present case, since $\dot{\vec{r}} = \vec{v}$, $\dot{\vec{v}} = \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right)$,

$$\frac{\partial}{\partial \vec{r}} \cdot \vec{v} = 0; \quad \frac{\partial}{\partial \vec{v}} \cdot \left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right] = \vec{\nabla}_v \cdot \vec{E} + \frac{1}{c}\vec{B} \cdot (\vec{\nabla}_v \times \vec{v}) - \frac{1}{c}\vec{v} \cdot \vec{\nabla}_v \times \vec{B} = 0,$$

where the three terms of the last expression are equal to zero. We thus have

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{r}} \cdot \frac{\partial f}{\partial \vec{r}} + \dot{\vec{v}} \cdot \frac{\partial f}{\partial \vec{v}} = 0, \tag{14}$$

or

$$\frac{\partial f^{\pm}}{\partial t} + \vec{v} \cdot \frac{\partial f^{\pm}}{\partial \vec{r}} + \frac{e^{\pm}}{m^{\pm}}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right) \cdot \frac{\partial f^{\pm}}{\partial \vec{v}} = 0. \tag{15}$$

If we now reconsider the fluctuating micro-fields, i.e. collisions between particles, then particles are continually transferred from one element $\delta x\, \delta v$ to another element in the same strip δx, so that a term must be added to the right-hand side of Eq. (15) to record the balance between particles entering and leaving a given volume element of the phase space because of collisions. We then write our kinetic equation for either species as

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial \vec{r}}+\frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}}=\left(\frac{\delta f}{\delta t}\right)_c . \tag{16}$$

The detailed structure and properties of the collision term is discussed in other lectures and we shall merely posit at this time any properties of this term that we shall need.

Let us now take moments of this equation with respect to velocity, i.e. multiply by $1, \vec{v}, \vec{v}\vec{v}, \vec{v}\vec{v}\vec{v}$, etc., and integrate over velocity space. For the zero'th moment

$$\int d^3v \frac{\partial f}{\partial t}(\vec{x}, \vec{v}, t) = \frac{\partial}{\partial t}\int d^3v f = \frac{\partial n}{\partial t}(\vec{r}, t), \tag{17}$$

$$\int d^3v\, \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} = \frac{\partial}{\partial \vec{r}}\cdot\int d^3v\, \vec{v} f = \vec{\nabla}\cdot(n\vec{u}), \tag{18}$$

$$\int d^3v \frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial \vec{v}} = \frac{e}{m}\int d^3v \vec{\nabla}_{\vec{v}}\cdot\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right) = 0, \tag{19}$$

where we have assumed f vanishes sufficiently strongly for large velocities so that all surface integrals in velocity space vanish. Since individual collisions conserve number, the collision term has no zero'th moment.

We thus arrive at the equation of number continuity for each species

$$\frac{\partial n^{\pm}}{\partial t} + \vec{\nabla}\cdot(n^{\pm}\vec{u}^{\pm}) = 0 . \tag{20}$$

The equation of charge continuity,

$$\frac{\partial \sigma}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \tag{21}$$

and mass continuity,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{U}_0) = 0 \tag{22}$$

follow from Eq. (20) by multiplying by the charge of each species and summing over species, and by multiplying by the mass of each species and summing over species, respectively. Here

$$\rho = \sum_{+,-} mn \tag{23}$$

$$\vec{U}_0 = \frac{1}{\rho} \sum_{+,-} mn\vec{u} \tag{24}$$

are the mass density and velocity of centre of mass, respectively.

To arrive at the macroscopic equations of motion we take the first moment, i. e. multiply by $m\vec{v}$ and integrate over velocities:

$$m\int d^3v\,\vec{v}\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}(mn\vec{u}), \tag{25}$$

$$m\int d^3v\,\vec{v}\vec{v}\cdot\frac{\partial f}{\partial \vec{r}} = \vec{\nabla}\cdot(m\int d^3v\,\vec{v}\vec{v}f), \tag{26}$$

$$\begin{aligned} m\int d^3v\,\vec{v}\left[\frac{e}{m}\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)\cdot\frac{\partial f}{\partial\vec{v}}\right] &= -e\int d^3v\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)f \\ &= -en\left(\vec{E}+\frac{\vec{u}}{c}\times\vec{B}\right) \end{aligned} \tag{27}$$

$$m\int d^3v\,\vec{v}\left(\frac{\delta f}{\delta t}\right)_c = p_{ss'}, \quad \text{momentum transferred per unit time by collisions with opposite species.} \tag{28}$$

(Like-like collisions produce no net momentum change in virtue of Newton's third law.)

If we now define the stress dyadic (or tensor) for each species as

$$\begin{aligned} \vec{P}^{\pm} &= m^{\pm}\int d^3v(\vec{v}-\vec{U}_0)(\vec{v}-\vec{U}_0)f^{\pm} \\ &= m^{\pm}\int d^3v\,\vec{v}\vec{v}f^{\pm} - m^{\pm}n^{\pm}\vec{u}^{\pm}\vec{U}_0 - m^{\pm}n^{\pm}\vec{U}_0\vec{u}^{\pm} + m^{\pm}n^{\pm}\vec{U}_0\vec{U}_0 \end{aligned} \tag{29}$$

we then have

$$\frac{\partial}{\partial t}(mn\vec{u}) + \vec{\nabla}\cdot[m^{\pm}n^{\pm}(\vec{u}^{\pm}\vec{U}_0 + \vec{U}_0\vec{u}^{\pm} - \vec{U}_0\vec{U}_0)] + \vec{\nabla}\cdot\vec{P}^{\pm} = e^{\pm}n^{\pm}\left(\vec{E}+\frac{\vec{u}^{\pm}}{c}\times\vec{B}\right). \tag{30}$$

If we sum over both species we have, using Eq. (24), and with $\vec{P} \equiv \vec{P}^{+} + \vec{P}^{-}$,

$$\frac{\partial}{\partial t}(\rho\vec{U}_0) + \vec{\nabla}\cdot\vec{P} + \vec{\nabla}\cdot(\rho\vec{U}_0\vec{U}_0) = \sigma\vec{E} + \frac{\vec{J}}{c}\times\vec{B}, \tag{31}$$

where again we have used the third law, $p_{ss'} = -p_{s's}$. If we note

$$\vec{\nabla}\cdot(\rho\vec{U}_0\vec{U}_0) = \vec{U}_0\vec{\nabla}\cdot(\rho\vec{U}_0) + \rho\vec{U}_0\cdot\vec{\nabla}\vec{U}_0 \tag{32}$$

and employ the equation of mass continuity (22), we have

$$\rho \frac{d\vec{U}_0}{dt} = -\vec{\nabla} \cdot \vec{P} + \sigma \vec{E} + \frac{\vec{J}}{c} \times \vec{B}, \tag{33}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{U}_0 \cdot \vec{\nabla}. \tag{34}$$

If there are external body forces, or other average internal forces (e.g. gravity), they appear naturally on the right-hand side of Eq. (33).

(The definition (29) of the stress tensor is not universal, but is appropriate when both species are very closely Maxwellian relative to the centre of mass velocity and at the same temperature.) Often in plasmas macroscopic phenomena take place on a time scale fast compared to that on which any significant alteration of the distribution function due to collisions can take place and another definition of pressure is often appropriate.

$$\vec{P}^{\pm} = m^{\pm} \int d^3v (\vec{v} - \vec{u})(\vec{v} - \vec{u}) f. \tag{35}$$

The analogue of Eq. (30) with this definition of the stress tensor is

$$m^{\pm} n^{\pm} \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right) = e^{\pm} n^{\pm} \left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \right) - \vec{\nabla} \cdot \vec{P}^{\pm} + p_{ss'}. \tag{36}$$

We now notice that

$$\tfrac{1}{2} \vec{P}^{\pm} : \vec{I} = \tfrac{1}{2} \mathrm{Trace}\, \vec{P}^{\pm} = \tfrac{1}{2} m \int d^3v (\vec{v} - \vec{U}_0)(\vec{v} - \vec{U}_0) f^{\pm} \tag{37}$$

represents the mean kinetic energy of the system relative to the centre of mass and may be referred to as the internal energy of the system or the thermal energy of the gas. We may define a generalized temperature through

$$\tfrac{1}{2} \mathrm{Trace}\, \vec{P}^{\pm} = \tfrac{3}{2} N^{\pm} \Theta, \tag{38}$$

where $\Theta = kT$ (k is here Boltzmann's constant). Again, it is possible and sometimes desirable to use two separate temperatures for the separate species.

There are two important points to notice at this stage. Equations (22) and (33) together with Maxwell's Eqs. (3)–(6), although in some sense exact, do not form a closed set of macroscopic equations. The Maxwell equations governing the time evolution of the electromagnetic fields involve the charge and current densities. To find the time evolution of these quantities we could use Eqs. (20) and (30) for each species but Eq. (30) involves the knowledge of $p_{ss'}$, i.e. properties of the collision term. Equations (21), (22), and (33) are not enough since we are still one vector equation short. We could write an equation for $\partial \vec{J}/\partial t$ (the so-called generalized Ohm's law) but this involves the $p_{ss'}$ again.

The second point is that we still have no equations for $\vec{P}$. If we compute the equation of motion for $\vec{P}$ by ascending the moment ladder further we find for each species

$$\frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{U}_0\vec{P}) + \vec{P}\cdot\vec{\nabla}\vec{U}_0 + (\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T + mn\frac{d\vec{U}_0}{dt}\vec{U} + mn\vec{U}\frac{d\vec{U}_0}{dt}$$

$$+\frac{e}{c}(\vec{B}\times\vec{P} - \vec{P}\times\vec{B}) - en\vec{U}\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right) - e\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right)n\vec{U} = \left(\frac{\delta\vec{P}}{\delta t}\right)_c, \quad (39)$$

where

$$\vec{Q} = m\int d^3v(\vec{v} - \vec{U}_0)(\vec{v} - \vec{U}_0)(\vec{v} - \vec{U}_0)_f \quad (40)$$

is the heat flow triadic, and

$$\vec{U} = \frac{1}{n}\int d^3v(\vec{v} - \vec{U}_0)f \quad (41)$$

is the mean velocity of each species relative to the centre of mass, and

$$(\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T$$

is the transpose of the dyadic $(\vec{P}\cdot\vec{\nabla}\vec{U}_0)$. In this last equation we have omitted the species label + or − . If we now sum over species we have

$$\frac{\partial \vec{P}}{\partial t} + \vec{\nabla}\cdot(\vec{Q} + \vec{U}_0\vec{P}) + \vec{P}\cdot\vec{\nabla}\vec{U}_0 + (\vec{P}\cdot\vec{\nabla}\vec{U}_0)^T$$

$$+\sum_{+,-}\frac{e}{mc}(\vec{B}\times\vec{P} - \vec{P}\times\vec{B}) + (\vec{J} - \sigma\vec{U}_0)\left(\vec{E} + \frac{\vec{U}_0}{c}\times\vec{B}\right), \quad (42)$$

where we have made use of the fact that

$$\sum mn\vec{U} = 0 \quad (43)$$

and

$$\sum en\vec{U} = \vec{J} - \sigma\vec{U}_0. \quad (43a)$$

By taking one-half the trace of Eq. (42) we obtain the equation of energy balance

$$\frac{\partial}{\partial t}\left(\frac{3}{2}n\Theta\right) + \vec{\nabla}\cdot\left(\vec{q} + \frac{3}{2}n\Theta\vec{U}_0\right) + \vec{P}:\vec{\nabla}\vec{U}_0 - (\vec{J} - \sigma\vec{U}_0)\left(E + \frac{\vec{U}_0}{c}\times\vec{B}\right) = 0. \quad (44)$$

Here

$$q = \frac{1}{2} \sum_{+,-} m \int d^3v (\vec{v} - \vec{U}_0)^2 (\vec{v} - \vec{U}_0) f \tag{45}$$

is the heat flow vector and represents the flow of internal energy relative to the centre of mass. The term involving $(\vec{B} \times \vec{P} - \vec{P} \times \vec{B})$ has zero trace. Here n is the sum of the number densities. The collision term gives no contribution because collisions preserve energy.

We see from Eq. (42) that the equation governing the time rate of change involves the heat flow tensor. There is no rigorous way to close the moment hierarchy.

There are two important limiting situations where the moment scheme may be closed:

(a) The first is where collisions dominate. This situation is most often realized in weakly ionized gases, where collisions with neutrals dominate, but sometimes even in totally ionized gas this situation obtains. Here one can make a development along the lines of the Chapman-Enskog theory. One arrives at a set of transport coefficients, resistivity coefficients, coefficient of thermal conductivity, coefficient of viscosity, etc., which relate the fluxes such as current field, thermal gradients and mass velocity gradients. This will be discussed in other papers.

(b) In these situations where to lowest approximation collisions are negligible (see chart), and if the characteristic frequencies are high and/or wave numbers are small such that, crudely, $L/T \approx \omega/k \gg v_{th}$ (the so-called "Low Temperature Approximation"), then to lowest approximation the pressure may be dropped from the equation of motion. The next approximation taking into account thermal corrections consists in dropping the heat flow term from the equation for the pressure development (Eq. (42)) which leads in interesting situations to certain adiabatic laws for the pressure development.

There is one additional comment preparatory to the mutilation of these equations as we begin our study of plasma properties, that is their non-linearity. A few idealized situations have been studied which capture this feature. The usual procedure is to examine small departures from some known equilibrium or steady flow. Unfortunately, far too often these situations prove to be unstable, and to examine their fate (turbulence of "hash") the non-linearity must be invoked. Only very recently have we come to get even the slightest grip on these problems, and finding suitable techniques for handling them is one of the outstanding current problems in plasma physics and will be discussed later.

We have just outlined the two essential difficulties in closing the macroscopic moment equations:

(a) If we think in terms of the two macroscopic velocities $\vec{u}^+$, $\vec{u}^-$ or equivalently in terms of $\vec{J}$, $\vec{U}_0$, there remains something to do with the term $p_{ss'}$, the momentum transferred per unit time by collisions with opposite species.

(b) How shall the pressure be determined, when, perhaps, neither the "Low-Temperature" nor collision-dominated situation prevails?

Let us return to Eq. (36) for each species, divide by $m^{\pm}$, multiply by $e^{\pm}$, and sum over species. With $e^{+} = Ze$, $e^{-} = -e$, we then have, utilizing Eqs. (11), (12), (23), and (24),

$$\frac{\partial \vec{J}}{\partial t} + \sum_s [\vec{u}_s \vec{\nabla} \cdot (n_s e_s \vec{u}_s) + e_s n_s \vec{u}_s \cdot \vec{\nabla} \vec{u}_s] = e^2 \left(\frac{Z^2 n^+}{m^+} + \frac{n^-}{m^-} \right) \vec{E}$$

$$+ \frac{e^2}{c(m^+ + Zm^-)} \rho \left(\frac{Z^2}{m^+} + \frac{Z}{m^-} \right) \vec{U}_0 \times \vec{B} + \frac{e}{c(m^+ + Zm^-)} \left(Z^2 \frac{m^-}{m^+} - \frac{m^+}{m^-} \right) \vec{J} \times \vec{B}$$

$$- e \left(\frac{Z}{m^+} \vec{\nabla} \cdot \vec{P}'^{+} - \frac{1}{m^-} \vec{\nabla} \cdot \vec{P}'^{-} \right) + e \left(\frac{Z}{m^+} + \frac{1}{m^-} \right) p_{+-}. \tag{46}$$

Now, under the assumptions (1) $m^- \ll m^+$, (2) all terms quadratic in the $\vec{u}_s$ and their derivatives may be neglected (generally valid if $|\vec{u}_s| \ll (p/\rho)^{1/2}$, $(B^2/4\pi\rho)^{1/2}$, i. e. if macroscopic velocities are $\ll$ sound speed or hydromagnetic speed), (3) $n^+ \approx n^-/Z$, and (4) $P'^- \sim P'^+$, then this equation reduces to

$$\frac{m^-}{e^2 n^-} \frac{\partial \vec{J}}{\partial t} = \vec{E} + \frac{\vec{U}_0}{c} \times \vec{B} - \frac{\vec{J} \times \vec{B}}{cn^-} + \frac{1}{en^-} \vec{\nabla} \cdot \vec{P}'^- - \frac{1}{en^-} p_{+-}. \tag{47}$$

The real difficulties are now concentrated on the last two terms on the right-hand side of Eq. (47). The term involving p_{+-}, if the deformations of the distribution function from local Maxwellian are small, should be proportional to the relative velocity of the two types of particles. We shall take this term equal to $-\eta J$ where η is defined by

$$\eta = |p_{+-}| / en^- |J|. \tag{48}$$

Actually η is frequency and magnetic field dependent, and not even scalar, effects which will be discussed in other papers.

The term involving the stress tensor is troublesome. We may take this scalar and isotropic only in a collision dominated theory, where there are many collisions during a characteristic time, in which case

$$\vec{P} = p\vec{I} \tag{49}$$

and

$$\frac{d}{dt}(p\rho^{-5/3}) = 0. \tag{50}$$

For rapid changes in which the internal kinetic energy changes in only one or two directions the appropriate γ is 3 or 2, respectively. Actually in the presence of thermal gradients heat flow terms appear in Eq. (47) but this point will not be discussed further at this time.

The equation of motion under these approximations is

$$\rho \frac{d\vec{U}_0}{dt} = \frac{\vec{J}}{c} \times \vec{B} - \vec{\nabla} \cdot \vec{P}. \tag{51}$$

(The charge neutrality condition $n_- \approx Zn_+$ has been invoked to throw out the term $\sigma\vec{E}$ that should appear in Eq. (51).)

Let us now try to estimate the size of the terms appearing in the generalized Ohm's law, to see under what situation certain terms or another might be omitted from these equations.

If we let L and T measure typical spatial and temporal variations, then

$$\frac{4\pi\vec{J}}{\omega_p T} \approx \vec{E} + \frac{\vec{U}_0}{c} \times \vec{B} + \frac{\Theta}{eL}\vec{1}_L - \frac{\vec{J}\times\vec{B}}{cn^-} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}. \tag{52}$$

We can simplify this equation further by eliminating the term involving $\vec{J}\times\vec{B}$ in Eq. (47) via the equation of motion. This yields

$$\frac{4\pi\vec{J}}{\omega_{pe}^2 T} \approx \vec{E} + \frac{\vec{U}_0}{c} \times \vec{B} - \frac{\Theta}{eL}\vec{1}_L - \frac{m^+}{e}\frac{\vec{U}_0}{T} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}. \tag{53}$$

In terms of characteristic frequencies and speeds this becomes

$$\frac{4\pi\vec{J}}{\omega_{pe}^2 T} \approx \vec{E} + \frac{\vec{U}_0}{c} \times \vec{B} + \frac{c_s a_{ci}}{cL}\vec{B} - \frac{\vec{U}_0}{c}\times\frac{\vec{B}}{\omega_{ci}T} - \frac{4\pi\nu_c}{\omega_p^2}\vec{J}, \tag{54}$$

where

$$c_s = \text{sound speed} \sim (\Theta/m^+)^{1/2}, \tag{55}$$

$$\omega_{ci} = \text{ion gyro-frequency} = eB/m^+c, \tag{56}$$

$$a_{ci} = \text{ion gyro-radius} = (\Theta/m^+)^{1/2}/\omega_{ci}. \tag{57}$$

In Eq. (54) we have performed a very dangerous simplification in annihilating the vectorial nature of the equation. It so often happens, especially in the presence of a strong magnetic field, that terms which are large in one direction may vanish in another, so that for every particular problem one must respect the vectorial character in performing estimates on the size of the various terms.

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EQUILIBRIUM OF A MAGNETICALLY CONFINED PLASMA IN A TOROID*

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A. INTRODUCTION

A static equilibrium of plasma (or of conducting fluid) with scalar pressure p and magnetic field $\vec{B}$ is often described by the so-called "magnetostatic equations",

$$\vec{\nabla} p = \vec{j} \times \vec{B}, \tag{1}$$

$$\vec{\nabla} \times \vec{B} = \vec{j}, \tag{2}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{3}$$

where $\vec{j}$ is the electric current density. In particular, these equations apply to many proposed controlled thermonuclear reactors and their prototypes, especially the stellarator [1] and the recently much discussed stabilized pinch effect. In sections B, C, and D are derived a variety of properties possessed by solutions of Eq. (1-3).

One of these properties is that if p is constant on the boundary of its region of definition, then under some mild additional assumptions that boundary must be topologically toroidal. However, prescribing such a boundary surface and the value of p on it by no means determines a unique solution, even though there are as many equations as unknowns (two vector and one scalar). One of our objects is to establish the additional conditions which together with the magnetostatic equations (and the boundary prescription) do determine a unique solution. This is achieved in several different ways, the additional conditions always amounting to the specification of two numbers for each surface of constant p.

An experiment is imagined (section E) in which an ideal viscous hydromagnetic fluid exhibits a damped motion until coming to rest in an equilibrium configuration. A number of invariants with respect to any such motion are described in section F. These lead to constraints on the admissible trial states in a variational principle (sections G, H, and J) suggested by the experiment. The quantity varied is the potential energy, the sum of the magnetic and the internal fluid energies. The variational principle provides a potentially powerful tool for proving the existence of solutions of the magnetostatic equations and for obtaining them numerically. It also provides a characterization of solutions by their values of the invariants.

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* This is based on the article by M. D. Kruskal and R. M. Kulsrud in The Physics of Fluids 1 4 (1958) 265. The work was supported by the USAEC under contract AT (30-1) - 1238 with Princeton University. The sections and equations have been relabelled, and the last section is new.

In section K equations governing a steady state of magnetic field and slowly diffusing plasma are introduced. These amount to the magnetostatic equations together with two auxiliary conditions (73, 77) for each surface of constant pressure (section L). The system of equations obtained is expected to have a solution which is unique, and this is verified in section M for the limiting case of low pressure*.

B. MAGNETIC SURFACES

The magnetostatic Eqs. (1-3) have the simple consequences

$$\vec{\nabla}\cdot\vec{j} = 0, \tag{4}$$

$$\vec{B}\cdot\vec{\nabla}p = 0, \tag{5}$$

$$\vec{j}\cdot\vec{\nabla}p = 0, \tag{6}$$

$$\vec{B}\cdot\vec{\nabla}\vec{B} = \vec{\nabla}(p + \tfrac{1}{2}B^2), \tag{7}$$

$$\vec{\nabla}\cdot(\vec{B}\times\vec{\nabla}p) = 0, \tag{8}$$

$$\vec{B}\cdot\vec{\nabla}\vec{j} = \vec{j}\cdot\vec{\nabla}\vec{B}, \tag{9}$$

$$\vec{B}\cdot\vec{\nabla}(\vec{B}\cdot\vec{j}) = \vec{j}\cdot\vec{\nabla}\vec{B}^2. \tag{10}$$

Here (4) follows from (2), (5, 6) from (1), (7) from (1, 2), (8) from (2) and (6), (9) from the curl of (1) in view of (3) and (4), and finally (10) from (1) and (8) in view of (3) and (4).

If p is reasonably smooth and not constant in any (small) region, the equation $p = P$ determines a family of surfaces characterized by their values of the parameter P. By (5) they are "magnetic surfaces", in the sense that they are made up of lines of magnetic force, and similarly by (6) they are "current surfaces". If such a surface lies in a bounded volume of space and has no edges (because of not intersecting the edge of the region of definition of p), and if either $\vec{B}$ or $\vec{j}$ nowhere vanishes on it, then by a well-known theorem [2] it must be either a toroid (by which we mean a topological torus) or a Klein bottle. The latter, however, is not realizable in physical space.

Under normal circumstances each surface $p = P$ (excepting a set of values of P of measure zero) is traversed ergodically and consequently

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* Sections K-M form a somewhat revised version of a previous work: M. D. Kruskal, U.S. Atomic Energy Commission Report No. NYO-7307, (PM-S-17), 1955. Except for the two auxiliary conditions, which were new, this was largely a more mathematical version of an earlier theory: L. Spitzer, Jr., U.S. Atomic Energy Commission Report No. NYO-997 (PM-S-4), 1952.

determined by any line of force contained in it. Even when this is not so, however, we shall call it a magnetic surface.

As suggested by the foregoing discussion, we now explicitly assume that the magnetic surfaces form a family of nested toroids. The innermost toroid is degenerate, consisting of a single closed curve called the (magnetic) axis. We shall usually also assume that p increases monotonically going inward (as is proved for steady diffusing plasmas in section L) and indeed that $\vec{\nabla} p \neq 0$ except on the axis.

C. SURFACE QUANTITIES

We now introduce two co-ordinate functions η and θ. Each is to be multiple-valued, its values at any point differing by integers. The function η is to be continuous everywhere and to increase by unity during one traversal of the magnetic axis. The function θ is to be continuous everywhere except at the axis and is to increase by unity during one small loop around the axis. Finally, a pair of values of η and θ is to determine a unique point on each magnetic surface. For definiteness we assume that η, θ, p form a left-handed co-ordinate system.

For each particular magnetic surface we now define

$$V \equiv \int d\tau \, , \tag{11}$$

$$U \equiv \int d\tau \vec{B}^2, \tag{12}$$

$$K \equiv \int d\tau \vec{B} \cdot \vec{j} \, , \tag{13}$$

$$\psi \equiv \int d\tau \vec{B} \cdot \vec{\nabla} \eta, \tag{14}$$

$$\chi \equiv \int d\tau \vec{B} \cdot \vec{\nabla} \theta, \tag{15}$$

$$I \equiv \int d\tau \vec{j} \cdot \vec{\nabla} \eta \, , \tag{16}$$

$$J \equiv \int d\tau \vec{j} \cdot \vec{\nabla} \theta \, , \tag{17}$$

where $d\tau$ is the volume element and the region of integration is always the interior of the particular surface. (The integrals are well defined, since $\vec{\nabla}\eta$ and $\vec{\nabla}\theta$ are single-valued). We note that V is the enclosed volume and U is twice the enclosed magnetic energy. There seems to be no simple physical interpretation of K, but its vanishing will be seen to be significant (section L). The integrands of (14-17) can be written as divergences by (3) and (4), so we may apply Gauss' theorem; however, since η and θ are not single-valued, it is necessary first to cut the region of integration, say at η = 0 or at θ = 0 as appropriate. Since by (5) or (6) the boundary contribution vanishes except for the double boundary at the cut, we then obtain, for example,

$$\psi = \int_{\eta=1} dS \frac{\vec{\nabla}\eta}{|\vec{\nabla}\eta|} \cdot \vec{B}\eta - \int_{\eta=0} dS \frac{\vec{\nabla}\eta}{|\vec{\nabla}\eta|} \cdot \vec{B}\eta$$
$$= \int_{\eta=1} dS \frac{\vec{\nabla}\eta}{|\vec{\nabla}\eta|} \cdot \vec{B}, \tag{18}$$

where dS is the area element and the integrations are over those parts of the indicated surfaces of constant η which are interior to the particular magnetic surface. Thus ψ is the longitudinal magnetic flux inside the magnetic surface, i.e. the magnetic flux through any cross section of the interior. Similarly, I is the longitudinal current inside the particular surface. On the other hand, χ is what may be called the azimuthal magnetic flux inside the magnetic surface, since it is the flux through any ribbon-like surface of constant θ of which one edge is the magnetic axis and the other lies on the particular surface. Similarly, J is the azimuthal current inside the particular surface.

Functions of position which, like p, are constant on magnetic surfaces will be called surface quantities. The quantities defined by (11-17) may be interpreted as functions of position in an obvious way and are then surface quantities. Any surface quantity may be considered as a function of any other, and derivatives of one with respect to another are meaningful and are themselves surface quantities.

It may be noted that definitions (14-17) are invariant under continuous deformation of the co-ordinate functions η and θ. All functions η with the same direction of increase along the axis are deformable into each other. The analogous statement does not hold for θ, however; two functions θ are continuously deformable into one another if, and only if, their ribbons of constancy wind around the axis the same number of times. Two functions θ differ by an integral multiple of an acceptable function η, the integral multiplier being the difference of the winding numbers. If θ is increased by an integral multiple of η, then χ and J are increased by the corresponding integral multiples of ψ and I, respectively. The results of the next section are manifestly invariant under these changes.

D. RELATIONS AMONG SURFACE QUANTITIES

Let $\vec{w}$ be any single-valued vector field satisfying

$$\vec{\nabla}p \cdot (\vec{\nabla} \times \vec{w}) = 0 . \tag{19}$$

Let $\vec{z}(P)$ be a particular point on each surface p = P. For each point $\vec{x}$ in space define

$$\nu(\vec{x}) \equiv \int_{\vec{z}(p)}^{\vec{x}} d\vec{x} \cdot \vec{w} \,, \tag{20}$$

where the path of integration lies on the surface: in view of (19) it follows from Stokes' theorem that the value of $\nu(\vec{x})$ is independent of the path joining $\vec{z}$ to $\vec{x}$ for all paths continuously deformable into each other. However, not all paths are deformable into each other, so ν is multiple-valued. It can clearly be written

$$\nu = \lambda + \eta \oint_{\substack{\theta=0 \\ p=P}} d\vec{x} \cdot \vec{w} + \theta \oint_{\substack{\eta=0 \\ p=P}} d\vec{x} \cdot \vec{w}, \tag{21}$$

where λ is some single-valued function and the loop integrals are taken in the direction of increasing η and θ respectively.

If it were not for the variability of the lower limit in (20), we would have $\vec{\nabla}\nu = \vec{w}$; as it is we have

$$\vec{\nabla} p \times (\vec{\nabla}\nu - \vec{w}) = 0 \tag{22}$$

or equivalently

$$\vec{B} \cdot (\vec{\nabla}\nu - \vec{w}) = 0, \quad \vec{j} \cdot (\vec{\nabla}\nu - \vec{w}) = 0. \tag{23}$$

Now introduce two general surface quantities

$$F \equiv \int d\tau \vec{B} \cdot \vec{w}, \quad G \equiv \int d\tau \vec{j} \cdot \vec{w}. \tag{24}$$

By (23, 21) and (14-17) we then obtain

$$\begin{aligned} dF &= d \int d\tau \vec{B} \cdot \vec{\nabla}\nu \\ &= d\psi \oint_{\theta=0} d\vec{x} \cdot \vec{w} + d\chi \oint_{\eta=0} d\vec{x} \cdot \vec{w} \,, \end{aligned} \tag{25}$$

$$dG = dI \oint_{\theta=0} d\vec{x} \cdot \vec{w} + dJ \oint_{\eta=0} d\vec{x} \cdot \vec{w} \,. \tag{26}$$

We are now in a position to obtain various relations among the surface quantities by special choices of $\vec{w}$ satisfying (19). In view of (2) and (6) we are justified in choosing $\vec{w} = \vec{B}$. In this case $F = U$ and $G = K$, while by

Stokes' theorem applied to that part of the respective surface $\eta = 0$ or $\theta = 0$ which lies inside the magnetic surface we have

$$\oint_{\eta=0} d\vec{x} \cdot \vec{B} = -I, \tag{27}$$

$$\oint_{\theta=0} d\vec{x} \cdot \vec{B} = J + \oint_{\psi=0} d\vec{x} \cdot \vec{B}, \tag{28}$$

where the last integral is taken around the magnetic axis ($\psi = 0$) in the direction of increasing η. From (25, 26) we therefore obtain

$$dU = \left(J + \oint_{\psi=0} d\vec{x} \cdot \vec{B} \right) d\psi - I\, d\chi, \tag{29}$$

$$dK = \left(J + \oint_{\psi=0} d\vec{x} \cdot \vec{B} \right) dI - I\, dJ. \tag{30}$$

Another choice for $\vec{w}$ is the vector potential $\vec{A}$ (with $\vec{\nabla} \times \vec{A} = \vec{B}$), justified by (5). This leads analogously to

$$d \int d\tau \vec{B} \cdot \vec{A} = \left(\chi + \oint_{\psi=0} d\vec{x} \cdot \vec{A} \right) d\psi - \psi\, d\chi, \tag{31}$$

$$d \int d\tau \vec{j} \cdot \vec{A} = \left(\chi + \oint_{\psi=0} d\vec{x} \cdot \vec{A} \right) dI - \psi\, dJ. \tag{32}$$

Our next choice is

$$\vec{w} = (\vec{B} \times \vec{\nabla} p)/(\vec{\nabla} p)^2, \tag{33}$$

which may be justified by observing that

$$\vec{\nabla} p \times \vec{w} = \vec{B}, \tag{34}$$

in view of (5) and then using (3). In this case we have $F = 0$ and $G = V$ in view of (1). Also,

$$\oint_{\eta=0} d\vec{x}\cdot\vec{w} = \oint_{\eta=1} d\vec{x}\cdot\vec{w}\eta - \oint_{\eta=0} d\vec{x}\cdot\vec{w}\eta$$

$$= -\int_{p=P} dS \frac{\vec{\nabla}p}{|\vec{\nabla}p|}\cdot[\vec{\nabla}\times(\vec{w}\eta)]$$

$$= \int_{p=P} \frac{dS}{|\vec{\nabla}p|}\,\vec{\nabla}p\cdot(\vec{w}\times\vec{\nabla}\eta) \tag{35}$$

$$= \int_{p=P} \frac{dS}{|\vec{\nabla}p|}\,\vec{B}\cdot\vec{\nabla}\eta = -\frac{d\psi}{dp}\ ,$$

where the first step is trivial, the second follows from Stokes' theorem applied to the cut magnetic surface, the third follows from (19), the fourth from (34), and the last from (14) and the fact that $d\tau = -dS\,dp/|\vec{\nabla}p|$ (since $|dp|/|\vec{\nabla}p|$ is the distance between two neighbouring magnetic surfaces). Similarly

$$\oint_{\theta=0} d\vec{x}\cdot\vec{w} = \frac{d\chi}{dp}\ . \tag{36}$$

Thus (25) is tautological, but (26) gives

$$dp\ dV = d\chi\ dI - d\psi\, dJ. \tag{37}$$

Another possibility is to choose $\vec{w}$ to be a gradient, thus satisfying (19) trivially. Indeed, let $\vec{w} = \vec{\nabla}\vec{q}$ be a gradient of a vector field and therefore a dyadic, and note that nothing in the derivation of (25, 26) is invalidated. If $\vec{q}$ is single-valued the loop integrals in (25, 26) vanish and we may conclude that F and G themselves (now vectors) vanish, which is also obvious from Gauss' theorem. Since $\vec{\nabla}\vec{x}$ is the unit dyadic, taking $\vec{q} = \vec{x}$ gives

$$\int d\tau\vec{B} = 0, \qquad \int d\tau\vec{j} = 0. \tag{38}$$

Taking $\vec{q} = \vec{B}$ instead and using (7) gives

$$0 = \int d\tau\vec{\nabla}(p + \tfrac{1}{2}\vec{B}^2)\ ,$$

$$= -\int_{p=P} dS \frac{\vec{\nabla}p}{|\vec{\nabla}p|}(p + \tfrac{1}{2}\vec{B}^2), \tag{39}$$

and since the first term of the integrand contributes nothing (take p outside the integral and convert back to a volume integral),

$$\int_{p=P} dS \frac{\vec{\nabla} p}{|\vec{\nabla} p|} \vec{B}^2 = 0. \tag{40}$$

E. AN IMAGINED EXPERIMENT

Suppose that everywhere in a given rigid toroidal tube T with perfectly conducting walls there is a viscous perfectly conducting fluid with an adiabatic equation of state, and also a magnetic field tangential to the tube walls at the walls. Suppose that any heat generated by the viscosity is somehow magically removed, so that each element of fluid is isentropic. The system can then lose energy but not gain it, since there can be no energy flux through the walls.

Let the fluid be initially at rest. In general, it will not be in equilibrium and will start to move. As long as it moves it loses energy, so it must eventually come to rest in a state of less energy than its initial state. Clearly an initially resting state of minimum energy cannot start moving at all, and so must be in equilibrium, i. e. satisfy the magnetostatic equations.

Since we are comparing resting states, we are interested in the non-kinetic (i. e. potential) energy W,

$$W = \int_T d\tau \left(\frac{1}{2} \vec{B}^2 + \frac{p}{\gamma - 1}\right), \tag{41}$$

where the first and second terms of the integrand are the energy densities of the magnetic field and of the fluid respectively, γ being the ratio of specific heats of the fluid. (If $\gamma = 1$ the second term should be p log p).

F. INVARIANTS

It now appears that minimizing W should provide equilibrium solutions. However, we must be careful. If we minimize W outright we obtain $\vec{B} = 0$, $p = 0$, which, though certainly an equilibrium solution, is of no interest, and is clearly not a state which will be reached eventually by every initial state. We have neglected to observe that any motion of our fluid is subject to certain constraints. By a constraint here is meant a condition that some quantity be an invariant during any motion, an invariant being a constant of motion which depends only on the instantaneous state, not on the velocity. (All constraints here are holonomic). Only states with the same invariants can possibly be transformed into each other by a motion. We should therefore not minimize W among all states, but only among states with the same values of the invariants as the initial state.

We must therefore find invariants. Since the fluid is a perfect conductor, it carries lines of force with it [3]. Therefore, any topological property of the lines of force is an invariant. For instance, if in the initial state there were a line of force ergodic in T, then this would have to be carried into a similarly ergodic line in the final state. But the final state has to satisfy (5), so that p will be constant on the line and hence constant everywhere (if it is to be continuous). Such a state is not of interest. This example shows that we must choose the initial magnetic field to have precisely those topological properties possessed by equilibria of interest.

Accordingly, we choose an initial magnetic field which has a family of nested toroidal magnetic surfaces, which are, however, not necessarily surfaces of constant p. The quantities ψ and χ are then defined, and since the lines of force are carried with the fluid, ψ and χ for the magnetic surface formed by some definite set of fluid particles are invariant during a motion. The quantities V, U, K, I, and J are also defined, but need not be invariant.

The way in which a line of force intertwines with itself as it is continued around its magnetic surface many times is a topological property and therefore another invariant, but it turns out to be describable in terms of ψ and χ, and therefore does not provide an independent constraint. Indeed, this intertwining is characterized by the limit of the ratio of the number of loops around the magnetic axis to the number of traversals around the length of the toroid made by a line of force indefinitely prolonged, i. e. the limit of θ/η following the line. This limit is usually denoted by $\iota/2\pi$ and is equal to $d\chi/d\psi$. (See section N.)

Let ρ be the mass density of the fluid. Then $\rho\, d\tau$ is the mass of an element of fluid, and is therefore invariant during a motion. Furthermore, the adiabatic law assumed amounts to requiring that p/ρ^{γ} be invariant for a fluid element. We thus have two purely hydrodynamic constraints. But ρ is of no interest, since it enters neither into the magnetostatic equations nor into the potential energy W. Eliminating ρ we have only one invariant $p^{1/\gamma}\, d\tau$ for each element of fluid.

The invariants we have found (ψ and χ for magnetic surfaces, $p^{1/\gamma}\, d\tau$ for fluid elements) apply if we know which fluid element in the final state corresponds to each element in the initial state. However, there is no reference to this correspondence in (41). We wish to minimize W for all states p, $\vec{B}$ which could possibly be reached by a motion from the initial state, i. e. for which there exists some correspondence preserving the values of the invariants. The correspondence not being known ahead of time, it now makes no sense to require that ψ and χ are individually preserved. Nevertheless, the correspondence must be chosen to preserve ψ and that same correspondence must preserve χ. Then χ considered as a function of ψ (for example) must be the same in the final state as in the initial state. In short, the functional relationship between χ and ψ is an invariant which can be specified without knowing the correspondence ahead of time. What has been done, in effect, is to label the surfaces with their values of ψ, after which the only magnetic invariant left is χ, now as a function of the label ψ.

We have now eliminated consideration of the correspondence of surfaces as a whole. To eliminate consideration of the correspondence of fluid elements within a given magnetic surface, we must use the invariant $p^{1/\gamma}\, d\tau$ to form a label. Assuming that lines of force are ergodic on almost all surfaces, and choosing the correspondence for one particular fluid element as a reference point arbitrarily, we may use the integral of $p^{1/\gamma}$ $d\tau$ along a little flux tube going from some point on the surface to the reference point as a label for that point. This label is clearly invariant and hence extends the correspondence from the arbitrary reference point to the whole surface (since the line of force through the reference point is assumed ergodic on the surface and hence covers a dense set of points).

However, in establishing this labelling we have not exhausted the information available from the invariance of $p^{1/\gamma}\, d\tau$. There remains the condition that the integral of $p^{1/\gamma}$ $d\tau$ over the shell-like volume bounded by two neighbouring surfaces must obviously be invariant. Introducing the surface quantity

$$M \equiv \int d\tau p^{1/\gamma} \tag{42}$$

(the integral being taken over the interior of the surface), we may require equivalently that M be invariant. (It may be observed that M is just proportional to the mass contained within the surface if ρ happens to be such that the fluid is isentropic, i. e. if p/ρ^{γ} is the same for all fluid elements. The invariance of M then represents conservation of mass). As with χ, by considering M now to be a function of ψ we eliminate any reference to the correspondence.

G. STATEMENT OF VARIATIONAL PRINCIPLE

The preceding considerations suggest the following variational principle: A function p and a solenoidal magnetic vector field $\vec{B}$ in T, forming nested toroidal magnetic surfaces and having a fixed total longitudinal flux and no normal component at the walls, make W stationary among all such pairs with the same invariant functions $\chi(\psi)$ and $M(\psi)$ if, and only if, $\vec{\nabla} p = (\vec{\nabla} \times \vec{B}) \times \vec{B}$.

Before proving this it is desirable to reformulate it so as to include ψ explicitly in the characterization of a state, since otherwise it is difficult to tell whether a neighbouring field $\vec{B}$ has nested toroidal surfaces.. Correspondingly we have an additional constraint and an additional variational condition. Thus we propose the following variational principle: consider all triples p, $\vec{B}$, and ψ in T satisfying the constraints

(a) ψ has toroidal level surfaces, $\psi = C$ at the walls, $\min \psi = 0$, $\max \psi = C$,

(b) $\vec{\nabla} \cdot \vec{B} = 0, \qquad (43)$

(c) $\vec{B} \cdot \vec{\nabla}\psi = 0, \qquad (44)$

(d) $\int_{\psi \leq c} d\tau \vec{B} \cdot \vec{\nabla}\eta = c, \qquad (45)$

(e) $\int_{\psi \leq c} d\tau \vec{B} \cdot \vec{\nabla}\theta = \chi(c),$ (46)

(f) $\int_{\psi \leq c} d\tau p^{1/\gamma} = M(c);$ (47)

here C is a constant and $\chi(c)$ and $M(c)$ are arbitrary fixed functions defined for $0 \leq c \leq C$, which is also the range of the last three conditions. Then a particular triple makes W stationary among all such triples if, and only if, it satisfies the variational conditions

(y) p is a function of ψ alone,

(z) $\vec{\nabla}p = (\vec{\nabla} \times \vec{B}) \times \vec{B}.$ (48)

It should be noted that in non-degenerate cases, in which (almost) any magnetic line of force covers a complete magnetic surface ergodically, (z) implies (y) in view of (5) and (44). In degenerate cases (y) is (partly) independent. But in these cases ψ is not itself physically significant, only p and $\vec{B}$ which are consistent with a variety of functions ψ. Just in these degenerate cases we could have found additional constraints in our imagined experiment (the integral of $p^{1/\gamma}\, d\tau$ in each thin closed flux tube), and by omitting these from our formulation of the variational principle we force upon ψ a physical significance (the only constraint on the flow of fluid elements is that they stay on surfaces of constant ψ) which is reflected in the variational condition (y).

H. PROOF

Given a function ψ satisfying (a), it is easily seen by the method used at the beginning of section D that the most general field $\vec{B}$ satisfying (b) and (c) is given by

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\nu, \tag{49}$$

where ν is a multiple-valued function such that (49) determines $\vec{B}$ uniquely, i.e. such that on each surface of constant ψ the various branches of ν differ only by constants. Furthermore, (d) and (e) are then satisfied if and only if ν can be written

$$\nu = \lambda + \eta\chi'(\psi) - \theta , \tag{50}$$

with λ single-valued, as can be seen upon comparing (49) with (34, 22) and referring to (21, 35, 36).

Let us first assume that W is stationary and derive (y) and (z). For the moment hold ψ and λ fixed and vary only p. Then

$$
\begin{aligned}
0 = \delta W &= \frac{1}{\gamma - 1}\int_T d\tau\ \delta p \\
&= \frac{1}{\gamma - 1}\int_0^C dc \int_{\psi = c} \frac{dS}{|\vec{\nabla}\psi|}\ \delta p
\end{aligned}
\tag{51}
$$

for any perturbation δp satisfying

$$
\frac{1}{\gamma}\int_{\psi = c} \frac{dS}{|\vec{\nabla}\psi|}\ p^{1/\gamma - 1}\ \delta p = 0, \tag{52}
$$

which is obtained from (f) by varying p and also differentiating with respect to c. Picking δp so that the integrand of (52) approximates to the difference of two Dirac delta functions with peaks at two points of the same surface, we satisfy (52) and see from (51) that p must have the same value at the two points. Thus (y) is established. We now have $p = P(\psi)$, where by (f)

$$
P(c) = \left[M'(c) \Big/ \int_{\psi = c} \frac{dS}{|\vec{\nabla}\psi|} \right]^{\gamma}. \tag{53}
$$

Next we vary only λ, obtaining

$$
\begin{aligned}
0 = \delta W &= \int_T d\tau \vec{B}\cdot\delta\vec{B} = \int_T d\tau \vec{B}\cdot(\vec{\nabla}\psi \times \vec{\nabla}\delta\lambda) \\
&= -\int_T d\tau\ \delta\lambda \vec{\nabla}\cdot(\vec{B}\times\vec{\nabla}\psi),
\end{aligned}
\tag{54}
$$

$$
(\vec{\nabla}\times\vec{B})\cdot\vec{\nabla}\psi = 0. \tag{55}
$$

Finally we wish to vary only ψ. When we do so, $p = P(\psi)$ varies at a fixed point not only on account of the argument ψ but also because P(c) does. To compute the contribution of $\delta P(c)$ to δW, we note that

$$
\begin{aligned}
\delta \int_{\psi = c} \frac{dS}{|\vec{\nabla}\psi|} &= \delta \frac{d}{dc}\int_{\psi \le c} d\tau \\
&= -\frac{d}{dc}\int_{\psi = c} dS \frac{\delta\psi}{|\vec{\nabla}\psi|},
\end{aligned}
\tag{56}
$$

$$\int_T d\tau\, \delta P(\psi) = \int_T d\tau P\, \delta \log P$$

$$= \int_0^C dc \int_{\psi=c} \frac{dS}{|\vec{\nabla}\psi|} \frac{P\gamma \left(\frac{d}{dc} \int_{\psi=c} dS \frac{\delta\psi}{|\vec{\nabla}\psi|} \right)}{\int_{\psi=c} \frac{dS}{|\vec{\nabla}\psi|}}$$

$$= \gamma \int_0^C dc P(c) \frac{d}{dc} \int_{\psi=c} dS \frac{\delta\psi}{|\vec{\nabla}\psi|} \tag{57}$$

$$= -\gamma \int_0^C dc P' \int_{\psi=c} dS \frac{\delta\psi}{|\vec{\nabla}\psi|}$$

$$= -\gamma \int_T d\tau P'(\psi)\, \delta\psi,$$

in which we have used the fact that $\delta\psi = 0$ at the walls in view of (a). Accordingly we have

$$0 = \delta W = \int_T d\tau\, [\vec{B} \cdot \delta\vec{B} + (P'\, \delta\psi + \delta P)/(\gamma - 1)]$$

$$= \int_T d\tau\, [\vec{B} \cdot (\vec{\nabla}\, \delta\psi \times \vec{\nabla}\nu + \vec{\nabla}\psi \times \vec{\nabla}\, \delta\nu) - P'\, \delta\psi] \tag{58}$$

$$= \int_T d\tau\, \delta\psi [\vec{\nabla} \cdot (\vec{B} \times \vec{\nabla}\nu) - P'],$$

in which we have used (55); there is no trouble with the discontinuity of $\vec{\nabla}\theta$ at the magnetic axis because $\delta\psi$ vanishes there (since ψ and $\vec{\nabla}\psi$ both do). Thus we obtain

$$(\vec{\nabla} \times \vec{B}) \cdot \vec{\nabla}\nu = P'. \tag{59}$$

Taking the cross product of (49) with $\vec{\nabla} \times \vec{B}$ and using the variational conditions (55, 59) gives

$$(\vec{\nabla} \times \vec{B}) \times \vec{B} = P'\vec{\nabla}\psi = \vec{\nabla}p, \tag{60}$$

which establishes (z).

It is clear that all the steps can be reversed to show that (y) and (z) imply that $\delta W = 0$ for all perturbations.

J. REMARKS

There is a noteworthy modification of the variational principle. Suppose we omit condition (e). We are then free to vary $\chi(c)$, so we obtain an additional variational condition from

$$0 = \delta W = \int_T d\tau \vec{B} \cdot \delta \vec{B} = \int_0^C dc\ \delta\chi'(c) \int_{\psi=c} \frac{dS}{|\vec{\nabla}\psi|} \vec{B} \cdot (\vec{\nabla}\psi \times \vec{\nabla}\eta), \tag{61}$$

$$0 = \int_{\psi=c} dS \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|} \cdot (\vec{B} \times \vec{\nabla}\eta) = \int_{\psi \leq c} d\tau \vec{\nabla} \cdot (\vec{B} \times \vec{\nabla}\eta) = \int d\tau \vec{j} \cdot \vec{\nabla}\eta = I. \tag{62}$$

That is, we obtain just the additional condition appropriate for the steady state of a diffusing plasma (see section L).

Our variational principle characterizes equilibria as stationary states of the potential energy W, i.e. states for which the first variation of W vanishes. The stability of such equilibria has been investigated elsewhere [4] by examining the positive-definiteness of the second variation of W.

Two limiting choices of γ are particularly simple. The first is $\gamma \to \infty$ (incompressibility), for which $M \to V$, so that we prescribe the volume to be enclosed by each surface and vary the magnetic energy alone. The second is $\gamma \to 0$ (pressure completely independent of density), for which M'^{γ} approaches the maximum value of p on the surface, so that we prescribe $p(\psi)$ and vary the integral* of $\frac{1}{2}\vec{B}^2 - p$.

K. STEADY SLOWLY DIFFUSING PLASMA

We now wish to obtain a complete set of equations governing a steady state plasma slowly diffusing through a magnetic field to the containing walls of the toroidal tube T. We assume that the walls are perfect electric conductors with purely tangential magnetic field and also perfect plasma absorbers ($p = 0$ there), and that new plasma in somehow introduced or injected into T (necessary to maintain a steady state) at the source density rate Q, which may depend on position. We assume that there is no temperature gradient and ignore a variety of complicating factors, such as nuclear reactions and radiation, which might occur in applications of interest.

* A variational principle for simply connected regions based on this integral was formulated by H. Grad in a talk at Princeton, October, 1954.

Our steady state is almost static, so to zeroth order in the diffusion velocity $\vec{v}$ we have the magnetostatic equations (1-3). We also have the (first order) equation of continuity

$$\vec{\nabla} \cdot (p\vec{v}) = Q, \tag{63}$$

where we have taken the plasma density to be p in view of the assumed isothermality. In addition we have Maxwell's equation

$$\vec{\nabla} \times \vec{E} = 0 \tag{64}$$

and Ohm's law [5] which we take in the form

$$\vec{E} + \vec{v} \times \vec{B} = \frac{1}{\sigma}\vec{j} + \frac{\alpha}{p}\vec{\nabla}p, \tag{65}$$

where $\vec{E}$ is the electric field, σ the conductivity (assumed constant and scalar), and α a physical constant.

Now (64) holds everywhere in space (not just in T), so $\vec{E}$ is the gradient of a single-valued scalar. Since $\vec{\nabla}p/p$ is also such a gradient, we can introduce a single-valued scalar ϕ in T satisfying

$$\vec{E} = \vec{\nabla}\phi + \frac{\alpha}{p}\vec{\nabla}p, \tag{66}$$

so that (65) becomes

$$\vec{\nabla}\phi + \vec{v} \times \vec{B} = \frac{1}{\sigma}\vec{j}. \tag{67}$$

Let us consider (67) as an equation for $\vec{v}$. The condition that it have a solution is

$$\vec{B} \cdot \vec{\nabla}\phi = \frac{1}{\sigma}\vec{B} \cdot \vec{j}, \tag{68}$$

and if this is satisfied the general solution of (67) is

$$\vec{v} = \frac{1}{B^2}\left(\vec{\nabla}\phi - \frac{1}{\sigma}\vec{j}\right) \times \vec{B} + a\,\vec{B}, \tag{69}$$

where a is an arbitrary scalar function. Eliminating $\vec{v}$ by (69) and using (1), we have for (63)

$$\vec{\nabla} \cdot (pa\vec{B}) = Q + \vec{\nabla} \cdot \left[\frac{p}{B^2}\left(\vec{B} \times \vec{\nabla}\phi + \frac{1}{\sigma}\vec{\nabla}p\right)\right], \tag{70}$$

which may be viewed as a diffusion equation of sorts for p, the diffusion coefficient being $p/\vec{B}^2\sigma$.

L. THE TWO AUXILIARY CONDITIONS

By (3) and (5) the left-hand side of (70) may be written $p\vec{B}\cdot\vec{\nabla}a$. Thus (70) as well as (68) are what we may call "magnetic differential equations", namely equations of the type

$$\vec{B}\cdot\vec{\nabla}r = s \tag{71}$$

with scalar r and s. Viewed as an equation for r with s given, (71) determines how r varies along a magnetic line of force. In the non-degenerate case that the line covers a magnetic surface ergodically, (71) and the assignment of a value to r at one point détermine r at a set of points dense in the surface. A necessary condition that the values of r so obtained be extendable to a continuous single-valued function over the whole surface is easily derived by integrating (71) over the shell volume between two neighbouring magnetic surfaces p = P and p = P + dP. By (3), Gauss' theorem, and (5) the left-hand side then vanishes, while after dividing by dP the right-hand side becomes

$$\int_{p=P}\frac{dS}{|\vec{\nabla}p|}\, s = 0. \tag{72}$$

It is plausible to assume (and we shall) that, in the non-degenerate case, (72) is also a sufficient condition for (71) to have a continuous single-valued solution r. It is then clear that (71) determines r up to a surface quantity.

In accordance with the foregoing paragraph, the conditions that (68) permit a solution ϕ and (70) a solution a are

$$\int_{p=P}\frac{dS}{|\vec{\nabla}p|}\,\vec{B}\cdot\vec{j} = 0\,, \tag{73}$$

$$\int_{p=P}\frac{dS}{|\vec{\nabla}p|}\left\{Q + \vec{\nabla}\cdot\left[\frac{p}{\vec{B}^2}\left(\vec{B}\times\vec{\nabla}\phi + \frac{1}{\sigma}\vec{\nabla}p\right)\right]\right\} = 0. \tag{74}$$

The latter may be considerably simplified by multiplying by dP and integrating over all magnetic surfaces interior to a particular one; using Gauss' theorem then leads to

$$\int d\tau Q \mp \int dS\,\frac{\vec{\nabla}p}{|\vec{\nabla}p|}\cdot\left(\vec{B}\times\vec{\nabla}\phi + \frac{1}{\sigma}\vec{\nabla}p\right)\frac{p}{\vec{B}^2} = 0, \tag{75}$$

where the minus or plus sign is to be adopted accordingly as p decreases or increases going outward (so that $\mp \vec{\nabla}p/|\vec{\nabla}p|$ is the unit outward normal to the magnetic surface). Eliminating ∇p by (1) except in the denominator, expanding out the inner products of the cross products, and using (68) give

$$\int d\tau Q \mp \int \frac{dS}{|\vec{\nabla}p|} \left(-\vec{j}\cdot\vec{\nabla}\phi + \frac{1}{\sigma}\vec{j}^2\right) p = 0. \tag{76}$$

By (4, 6) the term involving ϕ can be written as a divergence; converting the surface integral to a shell volume integral, we see by Gauss' theorem and (6) that the contribution of that term is zero. Since Q, σ, and p are essentially positive, we must take the minus sign. Thus p decreases going outward (as assumed at the end of section B) and (76) may be written

$$\int_{p \geq P} d\tau Q - \frac{1}{\sigma} P \int_{p=P} \frac{dS}{|\vec{\nabla}p|} \vec{j}^2 = 0 . \tag{77}$$

Our system of equations now consists of the magnetostatic equations (1-3) together with the auxiliary conditions (73, 77). For any solution p, $\vec{B}$, $\vec{j}$ of this system we can find ϕ and a and therefore $\vec{E}$ and $\vec{v}$, which together with the solution represent a slowly diffusing equilibrium. (The arbitrariness of a surface quantity each in a and ϕ corresponds to a physical arbitrariness in the total fluid flow along lines of force and in the total charge on a magnetic surface respectively).

Condition (77) can be shown to be equivalent to the energy balance equation, which could have been written down a priori. Condition (73) may be written $-dK/dp = 0$ or, integrating, $K = 0$. When (73) holds, (30) can be integrated to show that I is proportional to $J + \oint d\vec{x}\cdot\vec{B}$. Since I and J, but not the loop integral, vanish at the axis, the constant of proportionality vanishes and $I = 0$. (Conversely the vanishing of I for all surfaces entails that of K).

Thus (73) is equivalent to (62), the extra variational condition obtained by minimizing W without prescribing χ. In other words, in the diffusing plasma the azimuthal magnetic flux adjusts to give the lowest energy; the lines of force associated with the longitudinal magnetic flux are permanently trapped by the perfectly conducting walls, but "untwist" themselves locally as much as possible.

It is not hard to see that the vanishing of dI at a particular magnetic surface implies that the lines of electric current there are closed curves, and indeed closed curves topologically like (deformable into) curves of constant η. The converse is even easier to see (choose η to be constant on the current lines).

M. THE LOW PRESSURE LIMIT

It is physically quite plausible to suppose that our system of equations has a solution p, $\vec{B}$, $\vec{j}$, and a unique one for any reasonable general prescription of the tube T, the source function Q, the conductivity σ, and the total trapped longitudinal magnetic flux C. This supposition is further borne out by the variational principle, which shows that two auxiliary conditions for each magnetic surface are just the appropriate number. We now proceed to prove the supposition in the limiting case of low pressure under an assumption on the geometry of T which is apparently necessary if the solution is to behave regularly in the limit. This assumption is that the unique vacuum magnetic field which is purely tangential at the walls and has a prescribed total longitudinal flux C vanishes nowhere and determines a non-degenerate family of nested toroids. (For a wide class of toroidal geometries this is assured to a high approximation by rotational transform theory [6]).

For p small (1) becomes, in the limit, $\vec{j} \times \vec{B} = 0$, or

$$\vec{j} = g\vec{B}, \tag{78}$$

with g a scalar function. From (3) and (4) we obtain the magnetic differential equation

$$\vec{B} \cdot \vec{\nabla} g = 0, \tag{79}$$

which implies that g is a surface quantity. But then using (78) in (73) and taking g out of the integral gives g = 0 and therefore $\vec{j} = 0$. Thus $\vec{j}$ must be small if p is.

Starting over now with p and $\vec{j}$ both small, we see from (2, 3) that to lowest order $\vec{B}$ must be the unique magnetic field of our assumption. Now (1) is equivalent to the pair of equations obtained by taking the inner and the cross products with $\vec{B}$, namely (5) and

$$\vec{j} = \frac{1}{\vec{B}^2} \vec{B} \times \vec{\nabla} p + h\vec{B}, \tag{80}$$

with h a scalar function. The only restriction (2) places on our remaining unknowns p and $\vec{j}$ is (4). We now adopt the point of view that (80), (4), and (73) are conditions on $\vec{j}$, while (5) and (77) are conditions on p.

Now (80) expresses $\vec{j}$ in terms of what we may consider a new scalar unknown h. But then (4) is equivalent to the magnetic differential equation

$$\vec{B} \cdot \vec{\nabla} h = -(\vec{B} \times \vec{\nabla} p) \cdot \vec{\nabla} \frac{1}{\vec{B}^2} \tag{81}$$

by (3) and (8), while (73) becomes

$$\int_{p=P} \frac{dS}{|\vec{\nabla} p|} h\vec{B}^2 = 0 \tag{82}$$

and may be viewed as determining the additive surface quantity left arbitrary in h by (81).

Of course, by (5) p is a surface quantity of the magnetic surfaces of the vacuum field and is therefore a function of ψ only. It is now obvious from (81, 82) that h is determined completely independently of p except for being proportional to $p'(\psi)$, and by (80) the same is true for $\vec{j}$. Indeed, setting

$$\vec{j} = p'\vec{y}, \quad h = p'k, \tag{83}$$

we see that $\vec{y}$ and k are uniquely determined independently of p by the three equations obtained from (80-82) by replacing p, $\vec{j}$, and h everywhere by ψ, $\vec{y}$, and k, respectively.

We can now write (77) in the form

$$\int_{\psi \le c} d\tau Q + \frac{1}{\sigma} pp' \int_{\psi = c} \frac{dS}{|\vec{\nabla}\psi|} \vec{y}^2 = 0. \tag{84}$$

(Note that Q is of the second order of smallness compared to p and $\vec{j}$). Thus $(p^2)'$ is determined, and since p vanishes at the wall p is determined, whereupon $\vec{j}$ is determined by (83). This completes the proof.

It may be noted that $(p^2)'$ remains finite at the walls. Thus p' becomes infinite there, p varying as the square root of the distance to the walls.

N. COMMENT

There is one point in section F of this paper which needs further comment, namely the remark that the rotational transform angle ι (in radians) is given by

$$\frac{\iota}{2\pi} = \frac{d\chi}{d\psi} \tag{85}$$

The transform angle ι is defined by

$$\frac{\iota}{2\pi} \equiv \lim \frac{\theta}{\eta} \tag{86}$$

where the limit is taken moving infinitely along a line of force, in other words as $\eta \to \pm\infty$. Here θ and η are to be thought of as varying continuously as one moves along the line of force, so they do not remain between 0 and 1; in fact, to say that $\eta \to \pm\infty$ is to say that one follows the line of force around and around the toroidal surface the "long way", and ι then measures a sort of average rate at which one is going around the "short way". It so happens that this limit always exists and has the same value for every line of force on a given surface of constant pressure and whichever direction one goes to infinity along it.

In order to obtain (85) we consider the η, θ plane image of the magnetic surface of interest, as shown in Fig. 1. It is a square with opposite sides

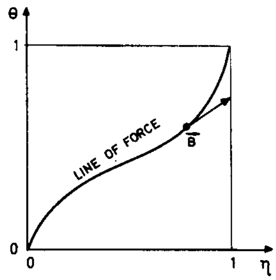

Fig. 1

η, θ plane image of the magnetic surface

identified (in physical space), therefore topologically a torus. Suppose, for example, that a line of force goes exactly once around the short way as it goes once around the long way, thus closing upon itself as pictured. Then θ increases by unity whenever η does, and $\iota/2\pi = 1$ by (86).

Consider a closed loop of ribbon-like surface one edge of which is the indicated line of force and the other edge of which is on an extremely close neighbouring magnetic surface. This other edge should run along a line of force as nearly as possible; in general it cannot do so exactly because no line of force on the neighbouring surface need close on itself after going once around the torus, but this produces a negligible (higher order) error which we may and do ignore. Thus we may think of the ribbon as made up of lines of force, and there is no magnetic flux through the ribbon.

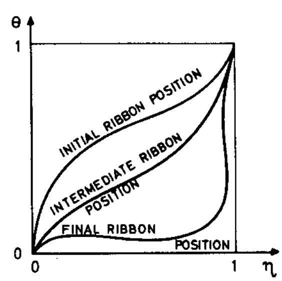

Fig. 2

Different positions of the ribbons

We now deform the ribbon continuously, moving each edge (and similar internal line) from its initial position along a line of force to a final position running first horizontally along the η axis and then vertically along the θ axis, as in Fig. 2. The flux through the ribbon remains constant during the deformation, by Gauss' theorem applied to the volume swept out by the ribbon during the deformation; we see this by integrating $\vec{\nabla} \cdot \vec{B} = 0$ over the volume swept out and noting that there is no flux through the areas swept out by the edges since they are pieces of magnetic surfaces (on which $\vec{B}$ has no normal component).

Measuring the flux through the ribbon in the upper leftward direction in the figures, we see that the flux through the part of the ribbon along the η

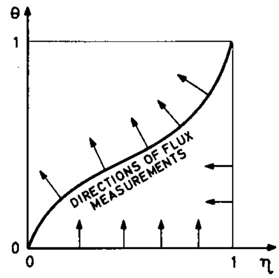

Fig. 3
Flux measurement

axis ($\theta = 0$) counts positively and that through the part along the θ axis (actually $\eta = 1$) counts negatively (Fig. 3). This latter flux would be just ψ if taken all the way down to the magnetic axis, as shown by (18); since we take it only between two neighbouring surfaces it is $d\psi$ instead. Similarly the flux through the η axis part of the ribbon is $d\chi$. We therefore have

$$d\chi - d\psi = 0, \tag{87}$$

so that

$$\frac{d\chi}{d\psi} = 1 = \frac{\iota}{2\pi} \tag{88}$$

in accord with (85).

Now suppose, more generally, that the line of force goes m times around the short way while going n times around the long way and then closes on itself. Since θ increases by m as η increases by n, we now have $\iota/2\pi = m/n$ by (86).

The flux through the ribbon of lines of force is again zero. The deformed ribbon now runs n times along the η axis ($\theta = 0$) and m times along the θ axis ($\eta = 1$), so

$$n\,d\chi - m\,d\psi = 0, \tag{89}$$

$$\frac{d\chi}{d\psi} = \frac{m}{n} = \frac{\iota}{2\pi}, \tag{90}$$

again in accord with (85).

If the line of force does not close on itself after going any finite number of times around then $\iota/2\pi$ is irrational. We do not have m and n to work with, so (90) is meaningless. But (85) is still valid, by continuity. In fact, to within any prescribed accuracy, the line approximately closes after some sufficiently large number of times around. Then (90) holds approximately, and going to the limit we obtain (85).

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HYDROMAGNETIC STABILITY THEORY

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1. IDEAL HYDROMAGNETICS

If we are concerned with phenomena for which $\omega_{ci}T$, $\omega_{pe}T \gg 1$ then, to a good approximation, the inertial terms may be neglected in the generalized Ohm's law. If, in addition, $(c_s/\bar{u})(a_{ci}/L) \ll 1$ (take $\bar{u} \sim c_H$ or c_s) then the term involving the ion pressure may be dropped. (If we are involved, as we shall be, with studying the stability of systems in which there is no mass flow in equilibrium ($\bar{u}_0 = 0$) then certainly this criterion cannot be satisfied in the equilibrium. However, it can be shown that the stability criteria are not affected by its omission.) We shall for simplicity omit body forces such as gravity from discussion but they may be readily included. We are thus left with the situation

$$\vec{E} + \frac{\vec{u}}{c} \times \vec{B} \doteq \eta \vec{J}. \tag{1}$$

If the time $\tau \approx 4\pi L^2/\eta c^2 \gg T$,

or

$$(\bar{u}^2/c^2)\, \frac{\omega_{pe}^2 T^2}{\nu_c T} \gg 1,$$

then the collisional term can be neglected. (Here ω_{ci} is the ion gyrofrequency, ω_{pe} is the electron plasma frequency, $\bar{u}$ is the average mass flow, and c_s and c are sound and hydromagnetic speed respectively. Characteristic length and time are denoted by L and T respectively.) However, one must reconcile this approximation with $\nu_c T \gg 1$, in order to maintain the pressure isotropy during the motion and thus have

$$\frac{d}{dt}(p/\rho^{5/3}) = 0. \tag{2}$$

We shall assume such conditions to obtain and shall write down the equations of ideal hydrodynamics:

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} p + \vec{J} \times \vec{B}. \tag{3}$$

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$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0. \tag{4}$$

$$\vec{E} + \vec{u} \times \vec{B} = 0. \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}\right) (p/\rho^{\gamma}) = 0. \tag{6}$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}. \tag{7}$$

$$\vec{\nabla} \times \vec{B} = \vec{J}. \tag{8}$$

Although one might argue, and quite rightfully, that this set of equations has a rather limited regime of approximate applicability for most plasma situations, their simplicity allows us to handle, say, more complicated geometrical situations. A more solid reason is that this set of equations leads to a pessimistic appraisal of stability as we shall see later.

This set of differential equations must be supplemented by a set of boundary conditions, obtained by integrating these equations across a region of sharp variation.

For a plasma-plasma interface,

$$\left[p + \frac{B^2}{2}\right] = 0. \tag{9}$$

$$n \cdot [\vec{u}] = 0. \tag{10}$$

$$\vec{n} \times [\vec{E}] = \vec{n} \cdot \vec{u} [\vec{B}]. \tag{11}$$

$$\vec{n} \cdot [\vec{B}] = 0, \quad n \times [\vec{B}] = \vec{K}. \tag{12}$$

Here $\vec{n}$ is a unit vector normal to the interface. $\vec{K}$ is any surface current (idealization) which might exist at the interface.

At a fluid vacuum interface, Eq. (10) is meaningless, but the other equations are valid.

A region of interest is obtained by considering the situation at rigid perfectly conducting walls. The appropriate boundary conditions here are

$$\vec{n} \times \vec{E} = 0. \tag{13}$$

$$\vec{n} \cdot \frac{\partial \vec{B}}{\partial t} = 0. \tag{14}$$

$$\vec{n} \cdot \vec{u} = 0. \tag{15}$$

A further condition which must be satisfied at an interface carrying a sheet current but no sheet mass is that the lines of force of the magnetic field lie in the surface. If it were not so the refraction of lines of force would yield infinite accelerations to the surface. We shall restrict ourselves to situations where the statement Eq. (14) is strengthened to $\vec{n} \cdot \vec{B} = 0$.

It can be shown that this dissipationless system has an energy integral

$$U = \int d\tau \left\{ \rho \frac{u^2}{2} + \frac{B^2}{2} + \frac{p}{\gamma - 1} \right\}. \tag{16}$$

At present we shall be concerned with the stability of those hydrodynamic equilibria in which $\vec{u}$ vanishes. The "normal mode" technique is the usual one for investigating the stability of such systems. It consists in solving the linearized equations of motion for small perturbations around the equilibrium state. The system is said to be unstable if any solution increases indefinitely in time; if no such solution exists, the system is stable.

There exists an energy principle technique [1, 2, 3], on the other hand, which depends on a variational formulation of the equation of motion. This was first formulated by Rayleigh for calculation of the normal frequencies of vibrating systems. The power of this method lies in the fact that if one is concerned with determining only if the system is stable, and not with actual frequencies or growth rates, then we need only discover whether there exists any perturbation which decreases the potential energy from its equilibrium value. This makes practicable the stability analysis of more complicated equilibria than the normal mode method.

It is convenient at this time to adopt a Lagrangian description of the "fluid" motion. Here we consider all quantities to be functions of $\vec{r}_0$, the initial location of a fluid mass element, and of time. Let the displacement vector $\vec{\xi}(\vec{r}_0, t)$ be determined by

$$\vec{r} = \vec{r}_0 + \xi(\vec{r}_0, t) \tag{17}$$

where $\vec{r}$ is the location of the element at time. Clearly $\xi(\vec{r}_0, t = 0) = 0$. The mass velocity $\vec{u}$ is given by

$$\vec{u} = \frac{d\vec{\xi}}{dt}. \tag{18}$$

If we now treat $\vec{\xi}$ as a small quantity, we shall derive the time evolution of all the field quantities in terms of $\vec{\xi}$ and then determine the law of motion for $\vec{\xi}$.

A basic relation we need is the following: By the chain rule for differentiation we have

$$\vec{\nabla} \equiv \frac{\partial}{\partial \vec{r}} = \frac{\partial \vec{r}_0}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{r}_0} \equiv \vec{\nabla}\vec{r}_0 \cdot \vec{\nabla}_0. \tag{19}$$

Now from (17)

$$\vec{r}_0 = \vec{r} - \vec{\xi}(\vec{r}_0, t) = \vec{r} - \vec{\xi}(\vec{r} - \vec{\xi}, t) \tag{20}$$

so that

$$\begin{aligned} \vec{\nabla}\vec{r}_0 &= \vec{\nabla}\vec{r} - \vec{\nabla}\vec{\xi} \\ &= \vec{\nabla}\vec{r} - \vec{\nabla}_0\vec{\xi} + O(\xi^2). \end{aligned} \tag{21}$$

Since $\vec{\nabla}\vec{r} = \vec{I}$, we have

$$\vec{\nabla} = \vec{\nabla}_0 - \vec{\nabla}_0\vec{\xi} \cdot \vec{\nabla}_0. \tag{22}$$

From the equation of continuity we have

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \vec{u} \cdot \vec{\nabla}\rho = -\rho\vec{\nabla} \cdot \vec{u}(\vec{r}, t). \tag{23}$$

Consequently

$$\frac{d}{dt} \ln (\rho/\rho_.) = -\vec{\nabla} \cdot \frac{d\vec{\xi}}{dt}(\vec{r}, t). \tag{24}$$

If we now work out an expression for the commutator of $\vec{\nabla}$ and d/dt, it is readily found that

$$\left[\vec{\nabla}, \frac{d}{dt}\right] = \vec{\nabla}\vec{u} \cdot \vec{\nabla} = \vec{\nabla}\dot{\vec{\xi}} \cdot \vec{\nabla}. \tag{25}$$

Thus, correct to first order in $\vec{\xi}$,

$$\frac{d}{dt} \ln (\rho/\rho_0) = -\frac{d}{dt}(\vec{\nabla} \cdot \vec{\xi}) \tag{26}$$

and thus

$$\rho = \rho_0 (1 - \vec{\nabla}_0 \cdot \vec{\xi}) + O(\xi^2). \tag{27}$$

It follows at once that

$$\rho_0/\rho = 1 + \vec{\nabla}_0 \cdot \vec{\xi} + \theta(\xi^2). \tag{28}$$

Now, since

$$\rho\, d\tau = \rho_0\, d\tau_0,$$

then

$$d\tau = d\tau_0[1 + \vec{\nabla}_0 \cdot \vec{\xi}] + O(\xi^2). \tag{29}$$

Likewise,

$$\frac{d}{dt}\left(\frac{\vec{B}}{\rho}\right) = (\rho\,\dot{\vec{B}} - B\dot{\rho})/\rho^2 = (\vec{B} + \vec{B}\vec{\nabla}\cdot\ \vec{u}/\rho)$$

$$= \frac{\vec{B}}{\rho} \cdot \vec{\nabla}\vec{u} = \frac{\vec{B}}{\rho} \cdot \vec{\nabla}\xi = \frac{d}{dt}\left(\frac{\vec{B}}{\rho} \cdot \vec{\nabla}\vec{\xi}\right) + O(\xi^2). \tag{30}$$

Consequently

$$\frac{\vec{B}}{\rho} - \frac{\vec{B}}{\rho} \cdot \vec{\nabla}\vec{\xi} = \frac{\vec{B}_0}{\rho_0}. \tag{31}$$

Thus

$$\begin{aligned} \vec{B} &= \frac{\rho}{\rho_0}(\vec{B}_0 + \vec{B}_0 \cdot \vec{\nabla}_0 \vec{\xi}) \\ &= B_0 + B_0 \cdot \vec{\nabla}_0 \xi - \vec{B}_0 \vec{\nabla} \cdot \vec{\xi}. \end{aligned} \tag{32}$$

The perturbations in p and $\vec{J}$ are easily found from the above equations and expanding out

$$\vec{\nabla} \times \vec{B} = J$$

$$\frac{d}{dt}(p\rho^{-\gamma}) = 0.$$

In summary with $Q_0 = \vec{\nabla}_0 \times (\xi_0 \times B_0)$ and if we keep only terms to first order in ξ, we have:

$$\rho = \rho_0 - \rho_0\, \vec{\nabla}_0 \cdot \vec{\xi}. \tag{33}$$

$$p = p_0 - \gamma p_0 \vec{\nabla} \cdot \vec{\xi}. \tag{34}$$

$$\vec{J} = \vec{J}_0 - (\vec{\nabla}_0\vec{\xi} \cdot \vec{\nabla}_0)\times\vec{B} + \vec{\nabla}_0 \times \vec{Q} + \vec{\nabla}_0 \times (\vec{\xi} \cdot \vec{\nabla}_0 \vec{B}_0) \tag{35}$$

$$B = \vec{B}_0 + \vec{Q} + \vec{\xi} \cdot \vec{\nabla}_0 \vec{B}_0. \tag{36}$$

The equation of motion for $\vec{\xi}(r_0, t)$ is found to be

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{\Gamma}[\vec{\xi}] \tag{37}$$

with

$$\vec{F}(\vec{\xi}) = \vec{\nabla}_0 [\gamma p_0 \vec{\nabla}_0 \cdot \xi + \xi \cdot \vec{\nabla} p_0] + \vec{J}_0 \times \vec{Q} - \vec{B}_0 \times \vec{\nabla} \times \vec{Q}. \tag{38}$$

Note that $\vec{F}$ depends only on ξ and not on $\dot{\xi}$. At a perfectly conducting fixed boundary the condition on ξ is

$$\vec{\xi} \cdot \vec{n} = 0. \tag{39}$$

Let us confine ourselves, for simplicity, to cases where plasma is confined by such a boundary, with no intervening vacuum region. Results for the case where there is an intervening vacuum region shall merely be stated with their derivation left as an exercise.

It is now within our power, in principle, to follow in time any small motion about an equilibrium state in which $u_0 = 0$. The central problem is to determine for a given equilibrium whether such a small motion grows in time. Again, if we confine ourselves just to a determination of stability, and do not inquire into the details of the motion, we have only to examine the sign of the change in potential energy. We shall prove that the system is unstable, if, and only if, there exists some $\vec{\xi}$ which makes this change in energy negative.

This demonstration requires the proof of self-adjointness of the operator $\vec{F}(\vec{\xi})$. That is, given any two arbitrary vector fields $\vec{\xi}(\vec{r}_0, t)$, $\vec{\eta}(\vec{r}_0, t)$, we must show

$$\int d\tau_0 \vec{\eta} \cdot \vec{F}(\vec{\xi}) = \int d\tau_0 \vec{\xi} \cdot \vec{F}(\vec{\eta}). \tag{40}$$

This self-adjointness can be proved directly, but tediously, by integrating by parts, and using the equilibrium conditions

$$\vec{\nabla}_0 p_0 = \vec{J}_0 \times \vec{B}. \tag{41}$$

$$\vec{\nabla} \times \vec{B}_0 = \vec{J}. \tag{42}$$

$$\vec{\nabla} \cdot \vec{B}_0 = 0. \tag{43}$$

We shall give here a simple direct proof, which depends only on the existence of the energy integral of the linearized system in which terms of the form of the product $\vec{\xi}$ and $\vec{\xi}$ do not appear.

$$\begin{aligned} \Delta U &= U(t) - U(0) = \text{const.} \\ &= \Delta K(\vec{\xi}, \vec{\xi}) + \Delta W(\vec{\xi}, \vec{\xi}). \end{aligned} \tag{44}$$

Here

$$\Delta K(\vec{\xi}, \vec{\xi}) = \frac{1}{2} \int d\tau_0 \rho_0 \left(\frac{\partial \xi}{\partial t}\right)^2 \tag{45}$$

and $\Delta W \{\vec{\xi}, \vec{\xi}\}$ is arrived at by inserting Eqs. (33), (34), (35), (36) into (16). Thus

$$\begin{aligned} \Delta \dot{K} &= \int d\tau_0 \, \dot{\vec{\xi}} \cdot \vec{F} [\vec{\xi}] = - \Delta \dot{W} \\ &= \Delta W \{\dot{\vec{\xi}}, \vec{\xi}\} - \Delta W \{\vec{\xi}, \dot{\vec{\xi}}\} \end{aligned} \tag{46}$$

where we have used the equation of motion Eq. (38). From Eq. (46) it follows that

$$\int d\tau_0 \dot{\vec{\xi}} \cdot \vec{F}[\vec{\xi}] = \int d\tau_0 \vec{\xi} \cdot \vec{F} [\dot{\vec{\xi}}]. \tag{47}$$

Since $\dot{\vec{\xi}}$ satisfies the same boundary as ξ and may be chosen arbitrarily initially to be $\vec{\eta}$, we have the self-adjointness. Further, from Eq. (46)

$$\Delta W \{\vec{\xi}, \vec{\xi}\} = - \frac{1}{2} \int d\tau_0 \vec{\xi} \cdot \vec{F} [\vec{\xi}]. \tag{48}$$

Since the time does not appear explicitly in Eq. (38), we may seek normal mode solutions of the form

$$\vec{\xi}_k (\vec{r}_0, t) = \eta_k(\vec{r}_0) \, e^{i\omega_k t}. \tag{49}$$

The corresponding characteristic value equation is

$$-\omega_k^2 \rho_0 \eta_k = \vec{F}\,[\eta_k]\,. \tag{50}$$

Since $\vec{F}$ is self-adjoint and real, the $\vec{\eta}_k$ can be chosen real and to satisfy the orthonormality condition

$$\frac{1}{2}\int d\tau_0 \rho_0 \vec{\eta}_\ell \cdot \vec{\eta}_m = \delta_{\ell m}. \tag{51}$$

Further,

$$\omega_k^2 \text{ is real} \tag{52}$$

and the phenomenon of "over-stability" (ω is pure real or pure imaginary) cannot occur. Any normal-mode with positive ω_k^2 is stable. Thus the necessary and sufficient condition for stability is that there must be at least one negative ω_k^2.

To show this (assuming completeness of η_k), let

$$\eta = \sum a_k \eta_k \tag{53}$$

then

$$\begin{aligned} \Delta W\,[\vec{\eta}, \vec{\eta}] &= -\frac{1}{2}\sum_n \sum_m a_n a_m \int d\tau_0 \vec{\eta}_n \cdot \vec{F}\,[\vec{\eta}_m] \\ &= \frac{1}{2}\sum_n a_n^2\, \omega_n^2\,. \end{aligned} \tag{54}$$

Thus ΔW can be negative if, and only if, there exists at least one negative ω_n^2.

These conditions are closely related to Rayleigh's principle, namely, that the Euler-Lagrange equation of the variational principle

$$\omega^2 = \frac{\Delta W\{\vec{\xi}, \vec{\xi}\}}{\Delta K\{\vec{\xi}, \vec{\xi}\}} \tag{55}$$

$$\delta\omega^2 = 0 \tag{56}$$

gives just Eq. (50).

The utility of the Rayleigh-Ritz procedure is that when the ratio Eq. (55) has a minimum, we may estimate the eigenvalues (oscillation frequencies and growth rates). Even if ω^2 is not bounded from below it can still be employed for information on the structure of the eigenmodes.

The power of the energy principle over both normal mode or, equivalently, the Rayleigh-Ritz procedure is that one can abandon the normalizing condition $\Delta K\{\vec{\xi},\vec{\xi}\} = 1$, especially if ρ_0 is complicated, and preferably use any convenient one to keep the energy inflated (bounded from below). One, of course, loses precise knowledge of the exact eigenfrequencies but often gains great analytical simplification.

Let us now illustrate these ideas with a simple example. Consider a plasma in which the magnetic field vanishes and the pressure is constant and outside which there is a vacuum region with a magnetic field. (We shall now suppress the subscript 0 on equilibrium quantities).

One has, in general, using Eqs. (38) and (48), and after integrating by parts,

$$\Delta W = \frac{1}{2}\int_p d\tau\,\{Q^2 + \vec{J}\cdot\vec{\xi}\times\vec{Q} + \gamma p\,(\vec{\nabla}\cdot\vec{\xi})^2 + \vec{\xi}\cdot\vec{\nabla}p\,\vec{\nabla}\cdot\vec{\xi}\}$$

$$+\frac{1}{2}\int_v d\tau(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}\int_I d\sigma\,(\vec{n}\cdot\vec{\xi})^2\vec{n}\cdot[\vec{\nabla}(p+B^2]\,. \qquad (57)$$

where the integrations are over the interior of the plasma, the vacuum region, and along the plasma-vacuum interface, respectively. The unit normal $\vec{n}$ points out of the plasma. At the interface the boundary condition is

$$\vec{n}\times\vec{A} = -\,(\vec{n}\cdot\vec{\xi})\,\vec{B}_{vac.} \qquad (58)$$

In our present problem we take in the plasma $\vec{B} = 0$, p = constant. Thus

$$\Delta W = \frac{1}{2}\int_v d\tau\,(\vec{\nabla}\times\vec{A})^2 + \frac{1}{2}\int_I d\sigma(\vec{n}\cdot\vec{\xi})^2\vec{n}\cdot\vec{\nabla}|B^2| + \frac{1}{2}\int_p d\tau\gamma p(\vec{\nabla}\cdot\xi)^2 \qquad (59)$$

Clearly ΔW is only lowered by choosing $\vec{\xi}$ such that $\vec{\nabla}\cdot\vec{\xi} = 0$, and we shall do so.

Denote by $\vec{R}$ the vector from a point on the line of force to the local centre of curvature. Then

$$\frac{1}{2}\vec{n}\cdot\vec{\nabla}|B|^2 = \vec{n}\cdot\vec{R}B^2/|\vec{R}|^2. \qquad (60)$$

Clearly, if $\vec{R}$ everywhere points away from the plasma ΔW is positive for all $\vec{\xi}$ and is stable. This is the principle of cusped geometries for obtaining stable MHD equilibria.

Consider situations where there is some region where $\vec{R}$ is directed into the plasma. Construct a local Cartesian co-ordinate system in a small region about such a point, with the z-axis normal to the surface and pointing into the vacuum, and the x-axis in the local direction of $\vec{B}$. Choose a displacement $\vec{\xi}$ such that

$$\xi_z(x, y, o) = \xi_0\, f(x) \sin ky$$

which falls to zero in a small distance $a \ll R$ but such that $ka^2 \gg R$. Choose for the perturbed vector potential $\vec{A}$

$$\vec{A} = f(x)\, \vec{\nabla}\left(\frac{\xi_0 B}{k} \cos ky\, e^{-kz}\right) \tag{61}$$

which satisfies the boundary condition Eq.(60). It is clear that the vacuum contribution

$$\int_V d\tau\, |\vec{\nabla} \times \vec{A}|^2 = \int_V d\tau \left\{\vec{\nabla} f \times \vec{\nabla}\left(\frac{\xi_0 B}{k} \cos ky\, e^{-kz}\right)\right\}^2$$

$$\approx \int d\tau\, |\vec{\nabla} f|^2\, \xi_0^2\, B^2\, e^{-2kz}$$

$$\simeq \xi_0^2\, B^2/2\,k \tag{62}$$

while the surface contribution,

$$\int d\sigma\, (\vec{n} \cdot \vec{\xi})\, \vec{n} \cdot \vec{R}\, \frac{|B|^2}{|R|^2} \sim \frac{1}{R}\, \xi_0^2\, B^2 a^2. \tag{63}$$

Thus the system is unstable.

These "flute instabilities" or "interchange instabilities" tend to move magnetic lines of force into a region previously occupied by matter, thus shortening them while only slightly bending them. The net result is a decrease in magnetic energy with no change in gas energy ($\vec{\nabla} \cdot \vec{\xi} = 0$).

To estimate the growth rate choose

$$\xi_x = 0, \;\; \xi_y = \xi_0 f \cos ky\, e^{-kz}, \;\; \xi_z = \xi_0\, zu\, ky\, e^{-kz}. \tag{64}$$

Then

$$\vec{\nabla} \cdot \xi = 0 + O\left(\frac{1}{ka}\right). \tag{65}$$

Then

$$\Delta K = \frac{1}{2}\int d\tau\rho\,|\xi|^2 = \rho\,\frac{\xi^2 a^2}{4k} \tag{66}$$

and thus

$$\omega^2 = \frac{\Delta W}{\Delta K} \sim -\frac{B^2}{\rho}\,\frac{k}{R}. \tag{67}$$

This is unbounded as k goes to infinity. It is true that this simple problem could be treated as well using normal modes or the Rayleigh-Ritz procedure, but had the geometry been more complicated, had there been no sharp separation, and had the equilibrium been spatially dependent, then the power of the energy principle would come to the fore.

2. DOUBLE ADIABATIC THEORY

If collisions are rare, $\nu_c T \ll 1$, then the assumption of the isotropy of the stress tensor becomes untenable and it is best to abandon the effect of collisions. In the presence of a strong magnetic field it is still possible to obtain essentially a hydrodynamic description under certain conditions.

The magnetic field serves to keep particles together, but only for the directions perpendicular to the magnetic lines. The magnetic field forces each charged particle to gyrate around a guiding centre which sticks to and moves along a line of force. As a result the particles cannot disperse in any direction perpendicular to the magnetic field but only among the field lines. In this case the pressure is no longer isotropic but different for the directions parallel and perpendicular to the magnetic field lines.

CHEW, LOW and GOLDBERGER [4] have given a fluid description in terms of a few macroscopic moments $\vec{u}$, ρ, $\vec{B}$ and the two pressures $p_\perp$ and $p_\parallel$. The difficulty in this procedure is the truncation of the moment system. If one simply throws away the heat flow term, and collisions, in the moment equation for the time development of the stress tensor, one finds

$$\vec{P} = p_\perp(\vec{I} - \vec{e}\vec{e}) + p_\parallel \vec{e}\vec{e} \tag{68}$$

where $\vec{e}$ is a unit vector along $\vec{B}$.
(This form for the stress tensor can be arrived at directly as a property of the distribution function in the large magnetic field limit or by taking our

moment equation (without collisions), and letting $\vec{B}$ be large forces

$$\vec{P} \times \vec{B} - \vec{B} \times \vec{P} = 0 \tag{69}$$

Eq. (68) then follows.)

There are two situations when this neglect of heat flow along the lines is a meaningful approximation: (a) when the longitudinal invariant exists, and (b) when the "low-temperature" approximation is valid. It is instructive to derive these laws from rather elementary thermodynamic arguments as follows:

The internal energy per unit volume is given by half the trace of the stress tensor

$$E = E_{\parallel} + E_{\perp} \tag{70}$$

with

$$E_{\parallel} = p_{\parallel}/2\rho$$

$$E_{\perp} = p_{\perp}/2\rho \tag{71}$$

In the absence of collisions $E_{\parallel}$ and $E_{\perp}$ are independent, except for the magnetic field constraint. Consider an element of volume $d\tau = dl ds$ where dl is an element of length along $\vec{B}$ and ds an element of area orthogonal to $\vec{B}$.

Under a $\vec{\xi}$ displacement, you have seen that

$$\frac{\delta dl}{dl} = (\vec{e} \cdot \vec{\nabla}\vec{\xi}) \cdot \vec{e} \tag{72}$$

and since

$$\frac{\delta dl}{dL} + \frac{\delta ds}{ds} = \frac{\delta d\tau}{d\tau} = \vec{\nabla} \cdot \vec{\xi} \tag{73}$$

then

$$\frac{\delta ds}{ds} = \vec{\nabla} \cdot \vec{\xi} - (\vec{e} \cdot \vec{\nabla}\vec{\xi}) \cdot \vec{e}. \tag{74}$$

Thus from the First Law of Thermodynamics

$$\delta(E_{\parallel} \rho d\tau) = \delta\left(\frac{1}{2} p_{\parallel} d\tau\right) = - p_{\parallel} ds \, \delta dl \tag{75}$$

and

$$\delta\,(E_{\perp}\rho d\tau) = \delta\,(p_{\perp} d\tau) = -\; p_{\perp} dl \delta ds. \tag{76}$$

It then follows at once that

$$\frac{dp_{\parallel}}{dt} = -\; p_{\parallel} \vec{\nabla}\cdot\vec{u} - 2p_{\parallel}(\vec{e}\cdot\vec{\nabla}\vec{u})\cdot e \tag{77}$$

$$\frac{dp_{\perp}}{dt} = -\; 2p_{\perp}\, \vec{\nabla}\cdot\vec{u} + p_{\perp}(\vec{e}\cdot\vec{\nabla}\vec{u})\cdot\vec{e}. \tag{78}$$

These may be written in a suggestive form

$$\frac{d}{dt}\left(\frac{p_{\parallel}B^2}{\rho^3}\right) = 0 \tag{79}$$

$$\frac{d}{dt}\left(\frac{p_{\perp}}{\rho B}\right) = 0\,. \tag{80}$$

This latter formula is equivalent to the constancy of the magnetic moment and these two latter laws replace the single adiabatic law

$$\frac{d}{dt}\left(p\rho^{-\gamma}\right) = 0. \tag{81}$$

Because of the holonomic nature of these equations we can repeat the arguments which lead to the previous energy principle and arrive at the form which will be written down in the following section.

3. ADIABATIC (PARTICLE) THEORY

3.1. Introduction

Our purpose now is to derive, from the kinetic equation in the small m/e limit, criteria useful in the discussion of stability of plasmas in static equilibrium [5]. At first we ignore collisions but later show how their effects may be taken into account. Our approach yields a generalization of the aforementioned energy principles for investigating the stability of hydromagnetic systems to situations where the effect of heat flow along magnetic lines of force is not negligible, and hence to situations where the strictly hydrodynamic approach is inapplicable.

In section 3.2. we characterize our general method of approach. In section 3.3. we delineate the properties of the small m/e limit of which we make use in the present problem, the constants of the motion, and the condition for static equilibrium. (We omit, for simplicity, the possibility of an electric field along the lines of force (but O (m/e) !) in the equilibrium we consider, but this is readily included. See [6]). In sections 3.4. and 3.5. we calculate the first and second variations of the energy and conclude with a statement of the general stability criterion.

In section 3.6. we state several theorems which relate the stability criterion to those of ordinary hydromagnetic theory, while in section 3.7. we show how to incorporate (at least a particular model of) collisions into the theory. Finally, we touch upon the problem of incorporating the charge neutrality condition into the theory.

3.2. General method

Our method consists of writing down the energy of the system to second order in the perturbation fields of f and $\vec{B}$, where f is the distribution function in $\vec{x}$, $\vec{v}$ space and $\vec{B}$ is the magnetic field intensity. We eliminate the terms involving the second-order perturbation f^{**} by employing certain constraints, namely that certain constants of the motion have their equilibrium values. (This method was suggested by a technique used by W. A. Newcomb to show stability in the much simpler case of a plasma with a Maxwellian equilibrium distribution.) The constants of the motion we employ are time-independent and are functionals of f and $\vec{B}$ which are regular at (permit expansion about) their equilibrium values.

The resulting expression for the energy is a quadratic form in the first-order perturbation f^* and $\vec{\xi}$ jointly (we have seen that ξ describes the displacement of magnetic lines of force away from their equilibrium positions) whose positive-definiteness provides a sufficient condition for stability. We rid ourselves of the dependence on f^* by minimizing the energy with respect to it, subject to the constraint that all constants of the aforementioned type have their equilibrium values. We then have a sufficient condition for stability involving $\vec{\xi}$ alone.

Generally, the constants of the motion of the type we employ do not specify the motion completely, so that there exist many motions evolving from the same equilibrium (at $t = -\infty$). By restricting these constants of motion to their equilibrium values, the only possible motions other than the equilibrium are instabilities (the pure modes of which have an exponential time behaviour and hence vanish at $t = -\infty$).

To illustrate the method we consider the simplest of examples, the equation of motion

$$\ddot{x} = -\lambda x . \tag{82}$$

This system has one time-independent constant, the energy

$$\mathscr{E} = \frac{\dot{x}^2}{2} + \lambda \frac{x^2}{2}. \tag{83}$$

There exists another constant (the initial phase) of more complicated behaviour and involving the time explicitly. We do not employ this latter constant.

The first-order variation of the energy vanishes, since in static equilibrium x(t) vanishes. The second-order variation leads to the form

$$0 = \mathscr{E}^{**} = \left(\frac{dx^*}{dt}\right)^2/2 + \lambda (x^*)^2/2 \tag{84}$$

for the perturbation. Clearly if $\lambda > 0$ then the energy is a positive-definite form, and the system is stable, for there exist no motions away from equilibrium. The stable (oscillatory) motions must necessarily increase the energy from its equilibrium value and hence are disregarded. If $\lambda < 0$, however, the form is indefinite, Eq. (84) can be satisfied non-trivially, and there exist (exponential) motions away from equilibrium.

3.3. Small m/e limit

In the present investigation we examine the second-order variation of the energy

$$\mathscr{E} = \int d^3x \ \frac{1}{2} (\vec{B}^2 + \vec{E}^2)$$

$$+ \sum \int \int \int (\beta/q) d\nu d\epsilon d^3x \ [mf(q^2/2 + \vec{\alpha}^2/2 + \nu\beta)] \tag{85}$$

from its equilibrium value. Here $\vec{E}$ and $\vec{B}$ are the electromagnetic field intensities, f is the distribution function in $\vec{x}$, $\vec{v}$ space of a particular species of charged particles, the summation is over all species, and

$$\vec{v} = \vec{\alpha} + \vec{v}_\perp + q\vec{n}. \tag{86}$$

$$\vec{B}(\vec{x}, t) = |\vec{B}(\vec{x}, t)| \vec{n}(\vec{x}, t) \equiv \beta\vec{n}. \tag{87}$$

$$\alpha = \vec{E} \times \vec{B}/B^2. \tag{88}$$

$$q = \vec{v} \cdot \vec{n}. \tag{89}$$

$$\nu = v_\perp / 2\beta. \tag{90}$$

$$\epsilon = q^2/2 + \nu\beta. \tag{91}$$

The quantity $(\beta/q)d\nu d\epsilon$ represents the volume element in velocity space. We assume for simplicity that any boundaries present are such as to present no complications, e.g., rigid and perfectly conducting walls with entirely tangential. The properties of the small m/e limit we employ are

(a) ν is constant following a particle motion.

(b) f is rotationally symmetric in velocity space about a line parallel to $\vec{B}$ and passing through the point $\vec{\alpha}$.

(c) $\vec{\alpha}$ is the common perpendicular drift velocity of all particles. This last fact, as is well known, is what permits the introduction into the formalism of a displacement vector $\vec{\xi}(\vec{x}, t)$ which governs not only the development of the field quantities but also the transverse motion of the particles. There is an additional property, charge neutrality (see section 3.8.), which we do not employ, but we remark at this time that in contradistinction to the Chew, Goldberger and Low (C.G.L.) theory where one particle species is taken of much smaller mass than another in order to satisfy the condition that $\vec{E} \cdot \vec{n}$ vanish, we treat all particle species on an equal footing, regard $\vec{E}$ and $\vec{B}$ as participants in the m/e expansion, and find that $\vec{E} \cdot \vec{n}$ is indeed zero to lowest order in m/e, which is all that is necessary for the evolution of the expansion. It can be shown that the stability criterion to be obtained is not affected by not employing this property in many cases, including that of isotropic equilibrium distribution functions.

The equilibrium distribution function we denote by $g(\nu, \epsilon, L)$ where L labels the line of force passing through a point in space. We take g monotonic in ϵ with

$$g_\epsilon < 0 \tag{92}$$

for reasons we shall see later. The equilibrium condition is

$$\begin{aligned} 0 &= -\vec{\nabla} \cdot \vec{P} + (\vec{\nabla} \times \vec{B}) \times \vec{B} \\ &= -\vec{\nabla} \cdot (p_+ \vec{I} + p_- \vec{n}\vec{n}) - \frac{1}{2} \vec{\nabla} \vec{B}^2 + \vec{B} \cdot \vec{\nabla} \vec{B} \end{aligned} \tag{93}$$

where $\vec{p}$ is the stress dyadic, $\vec{I}$ is the unit dyadic,

$$p_+ \equiv p_\perp, \quad p_- \equiv p_\parallel - p_\perp \tag{94}$$

with $p_\perp$ and $p_\parallel$ given by

$$p_\parallel = m \int\int (\beta/q)\, d\nu\, d\epsilon\, q^2 g \tag{95}$$

and

$$p_\perp = m \int\int (\beta/q)\, d\nu\, d\epsilon\, \nu \beta g \tag{96}$$

3.4. First-order variation of energy

The first-order change in $\mathscr{E}$ is given by

$$\mathscr{E}^* = \int d^3 x \left(\frac{1}{2}\vec{B}^2 \vec{\nabla}\cdot\vec{\xi} + \vec{B}\cdot\vec{B}^*\right) - m\iiint d\nu\, d\epsilon\, d^3 x\; [(g + \epsilon g_e)(\beta q \vec{\nabla}\cdot\vec{\xi}$$

$$+ (\beta q)^* + \beta q\,(f^* + \epsilon f_\epsilon^*\;)] . \tag{97}$$

Here we take for convenience the change following the displacement rather than at a fixed point, and we now take equilibrium quantities without special designation and denote perturbation quantities with starts. (We shall frequently perform an integration by parts with respect to ϵ, as we have in arriving at Eq.(97), by making use of the fact that $1/q$ equals q_ϵ, which follows immediately from Eq.(88). We do this in order to avoid the appearance of non-integrable integrands like $1/q^3$). The perturbed magnetic field intensity at the new position $\vec{x}+\vec{\xi}$ is given up to second order in $\vec{\xi}$ by

$$\vec{B}(\vec{x}) + \vec{B}^*(x) + \vec{B}^{**}(\vec{x}) = \vec{B} + [\vec{B}\cdot\vec{\nabla}\vec{\xi} - \vec{B}\,\vec{\nabla}\cdot\vec{\xi}]$$

$$+ \frac{1}{2}\left[\vec{B}\left((\vec{\nabla}\cdot\vec{\xi})^2 + \vec{\nabla}\vec{\xi} : \vec{\nabla}\vec{\xi}\right) - 2\,\vec{\nabla}\cdot\vec{\xi}\,\vec{B}\cdot\vec{\nabla}\vec{\xi}\right] \tag{98}$$

and the perturbed volume element at the new position is given by

$$d^3 x \left[1 + \vec{\nabla}\cdot\vec{\xi} + \frac{1}{2}\,[(\vec{\nabla}\cdot\vec{\xi})^2 - \vec{\nabla}\vec{\xi} : \vec{\nabla}\vec{\xi}]\right]. \tag{99}$$

However, we find that $\mathscr{E}^*$ vanishes trivially when we make use of the general constraint condition that all constants of the motion have their equilibrium values, as we shall now see.

The general constant of motion of our system constructed from individual particle constants may be written

$$\iiint (\beta/q)\, d\nu\, d\epsilon\, d^3 x\, G\,(f, \nu, L) \tag{100}$$

since the volume element in phase space is constant and so is G, which is an arbitrary function of the distribution function f (remember $\dot{f} = 0$), ν, and L where again L labels a line of force passing through a point in space. That *magnetic lines of force maintain their identity during a displacement* is a consequence of the fact that $\vec{E}\cdot\vec{n}$ is zero (to lowest order in m/e); that particles stick to magnetic lines of force is a consequence of (c).

The condition that Eq. (100) vanish to first order yields

$$0 = -\iiint d\nu\, d\epsilon\, d^3x\, G_f(g(\nu,\epsilon,L),\nu,L)[\beta q\,\vec{\nabla}\cdot\vec{\xi}\,g_\epsilon + g_\epsilon(\beta q)^* - \frac{\beta}{q}\,f^*]\,. \tag{101}$$

We may now regard G_f, as an arbitrary function of ϵ, ν, L because, by Eq. (92), g is monotonic in ϵ. Accordingly we may strip Eq. (101) to the basic constraint condition

$$0 = \int_T d^3x\,[\beta q\,\vec{\nabla}\cdot\vec{\xi}\,g_\epsilon + (\beta/q)(-\,\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n}:\vec{\nabla}\vec{\xi})(q^2 - \nu\beta)g_\epsilon - (\beta/q)f^*] \tag{102}$$

where the integration is over a thin tube of force T. (We may, in general, translate integrals over thin tubes of force of flux $d\varphi$ to those along lines of force according to the prescription

$$\int_T d^3x\ \beta A(\vec{x}) = d\varphi \int_L dl A(\vec{x}) \tag{103}$$

for arbitrary A.)

We now make the particular choice

$$G_f(g(\nu,\epsilon,L),\nu,L) = -\,m\,\epsilon \tag{104}$$

for G_f, in Eq. (101), add the resulting expression to Eq. (97) in order to eliminate f^*, and obtain

$$\mathscr{E}^* = \int d^3x\left(\frac{1}{2}\,B^2\vec{\nabla}\cdot\vec{\xi} + \vec{B}\cdot\vec{B}^*\right) - m\iiint d\nu\, d\epsilon\, d d^3x \times g\left[\beta q\,\vec{\nabla}\cdot\vec{\xi} + (\beta q)^*\right]. \tag{105}$$

The right-hand side of Eq. (105) now vanishes identically as stated when use is made of Eqs. (91), (98), (93), (95) and (96).

3.5. Second-order variation of energy

The second-order change in energy is given by

$$
\begin{aligned}
\mathscr{E}^{**} = \int d^3x \Big\{ & \frac{1}{4}\,[\vec{\nabla}\cdot\vec{\xi})^2 - \vec{\nabla}\vec{\xi} : \vec{\nabla}\vec{\xi}\,]\,\vec{B}^2 + (\vec{\nabla}\cdot\vec{\xi})\vec{B}\cdot\vec{B}^* + \frac{1}{2}\vec{B}^{*2} \\
& + \vec{B}\cdot\vec{B}^{**} + \frac{1}{2}E^{*2} + \frac{1}{2}\rho\,\vec{\alpha}^2\Big\} \\
& + \frac{m}{3}\iiint d\nu\, d\epsilon\, d^3x \Big\{\Big(\frac{1}{2}\,[(\vec{\nabla}\cdot\vec{\xi})^2 - \vec{\nabla}\vec{\xi} : \vec{\nabla}\vec{\xi}\,]\beta q^3 \\
& + \vec{\nabla}\cdot\vec{\xi}\,(\beta q^3)^* + (\beta q^3)^{**}\Big)(\epsilon g_{\epsilon\epsilon} + 2g_\epsilon) \\
& + \Big((\beta q^3)^* + \vec{\nabla}\cdot\vec{\xi}\,\rho q^3\Big)(\epsilon f^*_{\epsilon\epsilon} + 2f^*_\epsilon) \\
& + \beta q^3(\epsilon f^{**}_{\epsilon\epsilon} + 2f^{**}_\epsilon)\Big\}.
\end{aligned} \tag{106}
$$

Here ρ is the mass density. If we write the constraint condition Eq. (100) to second order,

$$
\begin{aligned}
0 &= \Big\{\frac{1}{3}\iiint \beta\, q^3\, d\nu\, d\epsilon\, d^3x\,\Big(G'' f_\epsilon^2 + G' f_{\epsilon\epsilon}\Big)\Big\}^{**} \\
&= \frac{1}{3}\iiint d\nu\, d\epsilon\, d^3x \Big\{G' f^{**}(\beta q^3)_{\epsilon\epsilon} + \frac{1}{2}G' f^{*2}(\beta q^3)_{\epsilon\epsilon} \\
&\quad + (G' f^*)_{\epsilon\epsilon}(\beta q^3)^* + G' f^*\,\vec{\nabla}\cdot\vec{\xi}\,(\beta q^3)_{\epsilon\epsilon} \\
&\quad + \Big(\frac{1}{2}\Big[(\vec{\nabla}\cdot\vec{\xi})^2 - \vec{\nabla}\vec{\xi} : \vec{\nabla}\vec{\xi}\Big]\beta q^3 + (\vec{\nabla}\cdot\vec{\xi})(\beta q^3)^* \\
&\quad + (\rho q^3)^{**}\Big)\Big(G'' g_\epsilon^2 + G' g_{\epsilon\epsilon}\Big)\Big\}
\end{aligned} \tag{107}
$$

and make the same choice Eq. (104) for G_f ($= G'$), we can eliminate f^{**} in the same way that f^* was eliminated in first order, and $\mathscr{E}^{**}$ then becomes a quadratic form in $\vec{\xi}$ and f^* jointly. (We have not explicitly introduced the next-order correction to the displacement $\vec{\xi}$ since its contribution to $\mathscr{E}^{**}$

vanishes in this second order, as the contribution of $\vec{\xi}$ to $\mathscr{E}^*$ vanished in the first order.) We now have

$$\mathscr{E}^{**} = \frac{1}{2}\int d^3x\,(\vec{E}^{*2} + \rho\vec{\alpha}^2) + \delta W \tag{108}$$

with δW defined by

$$\delta W = \frac{1}{2}\int d^3x\,\left\{\vec{Q}^2 + \vec{\nabla}\times\vec{B}\cdot\vec{\xi}\times\vec{Q} + \vec{\xi}\cdot\vec{\nabla}\vec{\xi}\cdot\,(\vec{\nabla}\cdot\vec{P})\right\}$$

$$-(m/2)\iiint d\nu\, d\epsilon\, d^3x\,(\beta/q)f^{*2}/g_\epsilon$$

$$+(m/3)\iiint d\nu\, d\epsilon\, d^3x\, g_\epsilon\left\{\frac{1}{2}\left[(\vec{\nabla}\cdot\vec{\xi})^2 - \vec{\nabla}\vec{\xi}:\vec{\nabla}\vec{\xi}\right]\beta q^3\right.$$

$$\left. + (\vec{\nabla}\cdot\vec{\xi})(\rho q^3)^* + (\beta q^3)^{**}\right\} \tag{109}$$

and

$$\vec{Q} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}). \tag{110}$$

We find after using Eqs.(91), (98), (95) and (96) that Eq.(109) becomes

$$\delta W = \frac{1}{2}\int d^3x\,\left\{\vec{Q}^2 + (\vec{\nabla}\times\vec{B})\cdot\vec{\xi}\times\vec{Q} + (\vec{\nabla}\cdot\vec{\xi})\vec{\xi}\cdot\vec{\nabla}p_+\right.$$

$$+ 2p_+\,[\vec{\nabla}\cdot\vec{\xi} - \vec{n}\vec{n}:\vec{\nabla}\vec{\xi}]^2$$

$$+ p_-\,[-\vec{n}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{n} - \vec{n}\vec{n}\vec{\xi}\cdot\vec{\nabla}\vec{\nabla}\vec{\xi}$$

$$\left. - (\vec{n}\cdot\vec{\nabla}\vec{\xi})^2 + (\vec{n}\vec{n}:\vec{\nabla}\vec{\xi})^2]\right\}$$

$$-(m/2)\iiint(\beta/q)\,d\nu\, d\epsilon\, d^3x\,\left\{f^{*2}/g_\epsilon\right.$$

$$\left. - g_\epsilon\nu^2\beta^2(-\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n}:\vec{\nabla}\vec{\xi})^2\right\} \tag{111}$$

We are now prepared to state our stability criterion: δW is a quadratic form in $\vec{\xi}$ and f^* jointly, otherwise depending only on equilibrium quantities. If this form is positive-definite (i.e. positive for all non-trivial permissible $\vec{\xi}$ and f^*), then our system is stable. Indeed, the only $\vec{\xi}$ and f^* for which $\mathscr{E}^{**}$ can vanish (as it must since $\mathscr{E}$ is a constant of the motion) are the trivial ones and hence no instability can develop.

Let us now hold $\vec{\xi}$ fixed and minimize this expression with respect to f^* (find the worst f^* from the point of view of stability), subject to the general constraint condition Eq.(102). To do this we multiply Eq.(102) by the Lagrange multiplier $\lambda(\epsilon, \nu, L)$, integrate over ν and ϵ, and then integrate (sum) over tubes of force to obtain

$$0 = \iiint d\nu\, d\epsilon\, d^3x\ \lambda(\nu, \epsilon, L)\Big[\beta q g_\epsilon \vec{\nabla}\cdot\vec{\xi} + (\beta/q) g_\epsilon (-\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n}:\vec{\nabla}\vec{\xi})(q^2 - \nu\beta) - (\beta/q) f^*\Big]. \tag{112}$$

We now add this expression to Eq.(111) and then vary with respect to f^*, obtaining the Euler equation

$$-f^*/g_\epsilon + \lambda = 0. \tag{113}$$

If we now use this to eliminate f^* in Eq.(102), we find

$$\lambda = \left\{\int_T d^3x\ (\beta/q)\Big[q^2\vec{\nabla}\cdot\vec{\xi} + (-\vec{\nabla}\cdot\vec{\xi} + n\,n:\vec{\nabla}\vec{\xi})(q^2 - \nu\beta)\Big]\right\} \Big/ \int_T d^3x\,(\beta/q). \tag{114}$$

These give the minimizing f^* in terms of $\vec{\xi}$, so Eq.(111) now represents a quadratic form on ξ alone, otherwise involving only equilibrium quantities. In the hydromagnetic fluid theory where the pressure develops according to the adiabatic laws

$$\frac{d}{dt}\left(p_\parallel p^2/\rho^3\right) = 0,\quad \frac{d}{dt}\left(p_\perp/\rho\beta\right) = 0 \tag{115}$$

the corresponding expression δW_D is

$$\begin{aligned}\delta W_D = \frac{1}{2}\int d^3x \Big\{&\vec{Q}^2 + (\vec{\nabla}\times\vec{B})\cdot\vec{\xi}\times\vec{Q} + \frac{5}{3}p_+(\vec{\nabla}\cdot\vec{\xi})^2 + (\xi\cdot\vec{\nabla}p_+)\vec{\nabla}\cdot\vec{\xi}\ .\\ &+ p_+\Big[\frac{1}{3}(\vec{\nabla}\cdot\vec{\xi})^2 - 2(\vec{\nabla}\cdot\vec{\xi})\vec{n}\vec{n}:\vec{\nabla}\vec{\xi} + 3(\vec{n}\vec{n}:\vec{\nabla}\vec{\xi})^2\Big]\\ &+ p_-[-\vec{\xi}\cdot\vec{\nabla}(\vec{n}\vec{n}:\vec{\nabla}\vec{\xi}) + 4(\vec{n}\vec{n}:\vec{\nabla}\xi)^2 - \vec{n}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{n}\\ &+ \vec{\xi}\cdot\vec{\nabla}\vec{n}\cdot\vec{\nabla}\vec{\xi}\cdot\vec{n} - (\vec{n}\cdot\vec{\nabla}\vec{\xi})^2 + \vec{n}\cdot\vec{\nabla}\vec{\xi}\cdot(\vec{\xi}\cdot\vec{\nabla}\vec{n})]\Big\}\end{aligned} \tag{116}$$

We can now write Eq. (111) as

$$\delta W = \delta W_D + I - \int d^3x\ \vec{n}\vec{n} : \vec{\nabla}\vec{\xi} \left\{ p_+ (2\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n} : \vec{\nabla}\vec{\xi}) + 3p_- \vec{n}\vec{n} : \vec{\nabla}\vec{\xi} \right\} \quad (117)$$

where

$$I = -(m/2) \iiint (\beta/q)\, d\nu\, d\epsilon\, d^3x\ g_\epsilon \left\{ \lambda^2 - \nu^2\beta^2(-\vec{\nabla}\cdot\vec{\xi} + nn : \vec{\nabla}\xi)^2 \right\} \quad (118)$$

(This expression for δW can be shown to be independent of the component of $\vec{\xi}$ parallel to $\vec{B}$, as it should be on physical grounds.)

3.6. Comparison theorems

The condition $\delta W > 0$ can be shown to be a necessary as well as sufficient condition for stability once self-adjointness has been established. The necessity of the principle was first shown directly by Newcomb (unpublished) and, with a technique that employed a shorter proof along the lines indicated in Part I, Ideal Hydromagnetics, was given by KULSRUD [6]. This proof of self-adjointness is intimately tied to the monotonicity condition $g_\epsilon < 0$. This condition eliminates the possibility of over stable low frequency oscillations along the lines of force. Overstability (complex eigenfrequencies) implies lack of self-adjointness. Nevertheless there is strong indication that it is possible to form an energy principle even if $g_\epsilon < 0$, by using only first-order perturbations and not relying on the technique employed in section 3.5. [7].

We shall now show that stability under the presently considered adiabatic (particle) theory implies stability under the Double Adiabatic Theory. For, by means of Schwarz's inequality

$$\begin{aligned} \lambda^2 \le \int_T d^3x\ (\beta/q)\, [\, & \nu^2\rho^2(\vec{\nabla}\cdot\vec{\xi} - \vec{n}\vec{n} : \vec{\nabla}\vec{\xi})^2 \\ & + 2\nu\beta q^2 (\vec{n}\vec{n} : \vec{\nabla}\vec{\xi})(\vec{\nabla}\cdot\vec{\xi} - \vec{n}\vec{n} : \vec{\nabla}\vec{\xi}) \\ & + (\vec{n}\vec{n} : \vec{\nabla}\vec{\xi})^2 q^4] \Big/ \int_T (\beta/q) d^3x \end{aligned} \quad (119)$$

If we now insert this inequality into Eq. (118) we find

$$\begin{aligned} I \le (m/2) \iiint (\beta/q) d\nu\, d\epsilon\, d^3x\ g[& 2\nu\beta(\vec{n}\vec{n} : \vec{\nabla}\vec{\xi})(\vec{\nabla}\cdot\vec{\xi} - \vec{n}\vec{n} : \vec{\nabla}\vec{\xi}) \\ & + 3\, q^2 (\vec{n}\vec{n} : \vec{\nabla}\vec{\xi})^2] . \end{aligned} \quad (120)$$

When the right-hand side of Eq. (120) is expressed in terms of p_- and p_+, it becomes precisely the last integral on the right-hand side of Eq. (117). Hence

$$\delta W \le \delta W_D . \tag{121}$$

We can obtain an important inequality in the opposite direction when the equilibrium distribution functions are isotropic ($g_\nu = 0$). In this case

$$p = p_\perp = p_\parallel = (m/3)\int_0^\infty d\epsilon\,(2\epsilon)^{3/2} g(\epsilon, L) \tag{122}$$

and we may write

$$\lambda = 2\epsilon \int_T \beta d^3 x\,(1 - y\beta)^{-\frac{1}{2}}\left[(1-\frac{3}{2}y\beta y\cdot\beta)\vec{n}\vec{n} : \vec{\nabla}\vec{\xi} + \frac{1}{2}y\beta\vec{\nabla}\cdot\vec{\xi}\right] \Big/ \int_T \beta d^3 x\,(1 - y\beta)^{-\frac{1}{2}} \tag{123}$$

where

$$y = \nu/\epsilon . \tag{124}$$

If we now take y and ϵ as variables in velocity space rather than ν and ϵ, we find we may write I in terms of the moment p after an integration by parts in ϵ and obtain

$$\begin{aligned} I = I_1 + I_2 = (15/4)\int\int_0^{1/\beta} dy d^3 x\, \beta p(1 - y\beta)^{-\frac{1}{2}} \\ \cdot\left\{\int_T \beta d^3 x\,(1 - y\beta)^{\frac{1}{2}}\left[(1-\frac{3}{2}y\beta)\vec{n}\vec{n} : \vec{\nabla}\vec{\xi} + \frac{1}{2}y\beta\vec{\nabla}\cdot\vec{\xi}\right] \Big/ \int_T \beta d^3 x\,(1 - y\beta)^{-\frac{1}{2}}\right\}^2 \\ - \frac{1}{2}\int d^3 x \left[2p(-\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n} : \vec{\nabla}\vec{\xi})^2\right]. \end{aligned} \tag{125}$$

For this isotropic case we now have

$$\delta W = \frac{1}{2}\int d^3 x \left\{\vec{Q}^2 + (\vec{\nabla}\times\vec{B})\cdot\vec{\xi}\times\vec{Q} + \vec{\xi}\cdot\vec{\nabla}p\,\vec{\nabla}\cdot\vec{\xi}\right\} + I . \tag{126}$$

This result has been also obtained independently by M. Rosenbluth using another method. Since the integrand in I_1 is positive, we may take Schwarz's inequality in the opposite direction, perform the integration, and obtain

$$\delta W \geq \delta W_H = \frac{1}{2}\int d^3x \left\{ Q^2 + (\vec{\nabla}\times\vec{B})\cdot\vec{\xi}\times\vec{Q} + \vec{\xi}\cdot\vec{\nabla}p(\vec{\nabla}\cdot\vec{\xi}) + \frac{5}{3}\,(\overline{\vec{\nabla}\cdot\vec{\xi}})^2 \right\} \quad (127)$$

where

$$\overline{\vec{\nabla}\cdot\vec{\xi}} = \int_T d^3x\ \vec{\nabla}\cdot\vec{\xi} \Big/ \int_T d^3x. \quad (128)$$

Thus if for a $\gamma = 5/3$ hydromagnetic fluid we can show stability $\delta W_H > 0$, then we may conclude that the system will indeed be stable under our more refined particle picture.

3.7. *Collisions*

In case collisions are not negligible, the situation is somewhat altered in that we lose most of the constants of the motion. Those of the type in Eq. (100), for which G is independent of f and ν, remain. However, the fact that the equilibrium distribution function is now locally Maxwellian, with modulus $\Theta(L)$, (as it must be for static equilibrium under collisions) enables us to proceed with the argument. We do not lose the property that particles stick to magnetic lines of force, however, since the size of step away from a magnetic line of force after a collision goes to zero with m/e.

The Boltzmann function per tube

$$\mathscr{H} = \int_T\!\!\int\!\!\int (\beta/q)\,d\nu\, d\epsilon\, d^3x\, \Theta(L) f \log f = \mathscr{H}_0 + \mathscr{H}_1 + \mathscr{H}_2 + \dots \quad (129)$$

has the well-known property

$$d\mathscr{H}/dt \leq 0 \quad (130)$$

because to lowest order in m/e there is no heat flow across lines of force.

We now assume that all regular, time-independent, phase functions have their equilibrium values at $t = -\infty$ and in particular obtain

$$\mathscr{H}(t = -\infty) = \mathscr{H}_0. \quad (131)$$

Now

$$d\mathscr{H}_0/dt = 0 \quad (132)$$

and, therefore,

$$d\mathscr{H}^*/dt \leq 0. \quad (133)$$

But $\mathcal{H}^*$ is linear in f^* and $\vec{\xi}$ and hence a reversal in the sign of f^* and $\vec{\xi}$ leads to $d\,\mathcal{H}^*/dt \geq 0$. We conclude, therefore, that $d\mathcal{H}^*/dt$ equals zero and so

$$\mathcal{H}^* = 0 \tag{134}$$

and finally obtain

$$\mathcal{H}^{**} \geq 0. \tag{135}$$

Now $\mathcal{E}$ is still a constant of the motion and we may use the expression for $\mathcal{H}^{**}$ to eliminate f^{**} from Eq. (106) obtaining the expression Eq. (109) for this Maxwellian case with the additional positive term $-\mathcal{H}^{**}$ on the right-hand side. We could have used the theorem which states that $\mathcal{E} + \mathcal{H}$ has its minimum for the Maxwellian distribution with modulus Θ, to arrive at this result. We minimize this expression with respect to f^* now with the constraints that the $\mathcal{H}$ function for each tube of force is constant to first order (see Eq. (134)) and the number of particles in each tube is constant. That is, using Eqs. (134) and (101) we may minimize subject to the constraints

$$0 = \int\!\!\int\!\!\int_T d^3x\,d\nu\,\epsilon\,(\beta/q)\epsilon\left[q^2\vec{\nabla}\cdot\vec{\xi}\,g_\epsilon' + (-\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n}:\vec{\nabla}\vec{\xi})(q^2 - \nu\beta)g_\epsilon - f^*\right] \tag{136}$$

and

$$0 = \int\!\!\int\!\!\int_T d^3x\,d\nu\,d\epsilon\,(\beta/q)\left[q^2\vec{\nabla}\cdot\vec{\xi}\,g_\epsilon + (-\vec{\nabla}\cdot\vec{\xi} + \vec{n}\vec{n}:\vec{\nabla}\vec{\xi})(q^2 - \nu\beta)g_\epsilon - f^*\right]. \tag{137}$$

This leads to

$$\frac{5}{6}\int d^3x\,p\,\overline{\vec{\nabla}\cdot\vec{\xi}}^{\,2} \tag{138}$$

for the value of the integral involving f^{*2}. It follows at once that

$$\delta W_{\mathrm{Coll}} \geq \delta W_H \tag{139}$$

i.e., stability is not destroyed by the occurence of collisions. (This model of collisions is a "weaker" model than that giving rise to Ideal Hydromagnetics. We could make this statement more precise in terms of inequalities involving characteristic frequencies, but we shall not do so here.)

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3.8. *Charge neutrality*

We conclude our presentation with a few remarks on the charge neutrality condition which is also a consequence of the small m/e limit. This condition is

$$0 = \sum_i e \int f^i(\beta/q)\, d\nu\, d\epsilon . \tag{140}$$

We must now minimize Eq. (106) with respect to f^* subject to the present constraint as well as to Eq. (101). This leads to a coupled set of linear integral equations for the multipliers with which the constraints are introduced. We have not solved these equations and defer further discussion to further work. It is clear, however, that since we are now minimizing under an additional constraint, the formula for δW in terms of ξ alone can only be increased from Eq. (117). Elementary physical arguments can be adduced to show that δW still remains bounded above by δW_D.

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LINEAR OSCILLATIONS OF A COLLISIONLESS PLASMA

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1. APPROXIMATE DERIVATIONS OF THE VLASOV EQUATION

(a) *Introduction*

In ordinary neutral gas theory, one uses the Boltzmann equation to determine the oscillations occurring in a gas. This equation assumes that particles suffer binary collisions only, and that in between these collisions they move freely under the influence of external forces only. An extremely useful additional approximation is to assume that the collision terms are very large in magnitude compared to the "streaming" terms. In this case the distribution function is very close to a Maxwell-Boltzmann and one can take moments of the Boltzmann equation and obtain the considerably more convenient hydrodynamic equations.

A hot, fully ionized plasma does not allow us these luxuries. Because of the long range nature of the coulomb force, the assumption of binary collisions is clearly inadequate. Even more important, the collision terms are now considerably smaller than the "streaming" terms.

To compare these quantities, we now calculate the approximate oscillation frequency of a plasma (since this is a measure of the "streaming" terms) and the corresponding collision frequency.

(b) *Oscillation frequency versus collision frequency*

We estimate the oscillation frequency by considering the case of a cold but collisionless plasma. Now the velocity $\vec{v}$ of the particles is a unique function of position $\vec{x}$ and we can write

$$\frac{d\vec{v}}{dt} = \frac{e\vec{E}}{m}, \tag{1.1}$$

where e is the particle's charge, m its mass and $\vec{E}$ the electric field. The conservation equation for each species is

$$\frac{dn}{dt} + n\,(\vec{\nabla}\cdot\vec{v}) = 0, \tag{1.2}$$

where n is the number density.

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If we now linearize about an initial uniform state in which there are equal numbers of electrons and ions at a density n_0 and with no mass velocity ($\vec{v}_0 = 0$) and no electric field ($\vec{E}_0 = 0$) we have:

$$\frac{dn_1}{dt} + n_0 \vec{\nabla} \cdot \vec{v}_1 = 0. \tag{1.3}$$

Now by Poisson's equation, we have

$$\vec{\nabla} \cdot \vec{E}_1 = 4\pi |e| (n_{1i} - n_{1e}). \tag{1.4}$$

Taking the second time derivative and using Eqs. (1.1) and (1.3) we have

$$\frac{d^2(\vec{\nabla} \cdot \vec{E}_1)}{dt^2} = -4\pi n_0 e^2 \left(\frac{1}{m_i} + \frac{1}{m_e}\right)(\vec{\nabla} \cdot \vec{E}_1).$$

Hence we see that the charge fluctuates with a natural frequency given by

$$\omega^2 = \omega_{pe}^2 + \omega_{pi}^2, \tag{1.5}$$

where the usual plasma frequencies have been introduced.

We must compare this with the coulomb collision rate. The cross-section for a coulomb collision is of the order of

$$\sigma \cong (e^2/mv^2)^2,$$

and the corresponding collision frequency is of the order of

$$\omega_c \sim n\sigma_c v .$$

Assume typical plasma parameters, such as

$$n \cong 10^{12} \text{cm}^{-3}$$

$$T = 100 \text{ eV}$$

then the electrons, which collide most frequently, yield such values as

$$\omega_c \cong 6 \times 10^3 \text{ s}^{-1},$$

while the electron plasma frequency is

$$\omega_p \cong 5 \times 10^{10} \text{ s}^{-1}.$$

We see that the collision terms can be ignored safely.

(c) What is a collision?

The estimate just given assumed that a "collision" was a coulomb scattering of one charged particle on another. It is important to distinguish between this "collision" and the motion of the particle under the influence of the averaged electric field of all the other particles. In fact, our estimate of the oscillation frequency tacitly assumed the existence of a smoothly varying electric field depending only on the co-ordinates of our particle and not on the precise location of the other particles in the plasma.

Thus, we define "collisions" to be due to large fluctuations of the electric field about a smoothly varying average electric field. This is the essential feature of the Vlasov equation in that we neglect the "collisions" and allow the particle's behaviour to be governed only by electric fields produced by the averages over the positions of the others. This is quite akin to the self-consistant field approximations used in atomic theory. It should also be noted that in a strict sense it is incorrect to call the Vlasov equation the "Collisionless Boltzmann Equation", as is often done, since this would imply neglect of the averaged field effects as well.

(d) Criterion for validity of Vlasov equation

It is convenient to have a numerical parameter available which tells us to what extent the use of the Vlasov equation for a plasma is justified. This is obtained by comparing the number of particles which make up the averaged field with those contributing to the field fluctuations.

At first glance we might think that the Vlasov equation is actually an exact description of a plasma. This is because there are an infinite number of particles contributing to the average field while only a finite number produce large fluctuations. However, the existence of a shielding limit on distant interactions makes this statement untrue and enables us to obtain our estimate. Let us estimate the shielding distance.

Assume that we have a potential field produced by ions and electrons. Assume the ions uniformly distributed and infinitely massive and that the electrons respond to the potential fluctuations as if they instantly relaxed to a Maxwell-Boltzmann distribution (we will make this implausible assumption somewhat more palatable in a few moments). Then Poisson's equation is (in one dimension)

$$-\frac{d^2\phi}{dx^2} = 4\pi\,|e|\left(N_0 - N_0 \exp|e|\frac{\phi}{kT}\right)$$

and for small fluctuations, we have

$$\frac{d^2\phi}{dx^2} = \frac{4\pi N_0 e^2}{kT}\,\phi.$$

Hence the fluctuation dies out exponentially with an e-folding length λ_D

$$\lambda_D = \left(\frac{kT}{4\pi N_0 e^2}\right)^{\frac{1}{2}} \tag{1.6}$$

which is the familiar Debye length.

Even when we do not make the assumption that the electrons are in equilibrium we still get a similar result. Suppose the electrons in a volume of radius r happen to be moving in such a way that a charge excess would occur. The time for buildup of this charge excess (and of the resultant electric field) is

$$\tau \cong r/\bar{v},$$

where $\bar{v}$ is the average electron velocity. On the other hand, the plasma's response to this field occurs in a time of the order of

$$\tau_{res} \cong 1/\omega_p.$$

Hence if the buildup time is short compared to the response we can create the charge excess. This condition is

$$\frac{r}{\bar{v}} < \frac{1}{\omega_p}$$

or

$$r < \left(\frac{mv^2}{4\pi ne^2}\right)^{\frac{1}{2}} = \lambda_D.$$

Now we can obtain a numerical estimate for the rate of the fluctuating field to the average field. We expect all particles which can cause appreciable coulomb deflection to contribute to large field fluctuations and that implies all particles within a distance d

$$d \cong e^2/kT.$$

Comparing this to the Debye shielding distance λ_D we have

$$\frac{R_{\text{fluctuation E}}}{R_{\text{average E}}} \cong \frac{d}{\lambda_D} = \frac{1}{4\pi n\lambda_D^3}$$

which is equal to the reciprocal of the number of particles in a Debye sphere. In usual practice this ratio is a very small number and hence we can use the Vlasov equation with good assurance.

(e) Derivation from the Liouville equation

Let us consider the statistics of a gas of N charged particles. We can describe the probability of finding our N particles with certain positions and velocities by the distribution function $f^N(\vec{x}_1, \vec{v}_1, \ldots\ldots \vec{x}_N\vec{v}_N, t)$. As we have just noted, when the number of particles in a Debye sphere is large we expect the particles to respond only to the averaged electric (and magnetic) fields of the other particles which, in turn, depend only on certain averages over the probability distribution of the other particles. In this first approximation we expect no dependence on the details of where the other particles actually are. Hence in first approximation we expect the individual particle probabilities to be independent or

$$f^N(\vec{x}_1\vec{v}_1,\ \vec{x}_2\vec{v}_2,\ \ldots\ \vec{x}_N\vec{v}_N t) = \prod_{i=1}^{N} f_i^1(\vec{x}_i,\ \vec{v}_i,t). \tag{1.7}$$

To next order in $(4\pi n\lambda_n^3)^{-1}$ we would expect the occurrence of terms involving correlations between a particle and the location of another, such as $g(\vec{x}_1\vec{x}_2,\ \vec{v}_1\vec{v}_2,t)$ but we shall not consider these here.

If we now make the assumption implied by Eq. (1.7) we can immediately derive the Vlasov equation [1]. Start with the Liouville equation

$$\left[\frac{\partial}{\partial t} + \sum_{i=1}^{N} (\vec{v}_i \cdot \vec{\nabla}_i + \vec{a}_i \cdot \vec{\nabla}_{vi})\right] f^N = 0$$

and substitute Eq. (1.7). Now integrate over the co-ordinates of all but the j-th particle. We find at once

$$\left(\frac{\partial}{\partial t} + \vec{v}_j \cdot \vec{\nabla}_j\right) f^1(\vec{x}_j\vec{v}_j t) + \sum_k \int \vec{a}_{jk} f^1(\vec{x}_k\vec{v}_k,t)$$
$$d\vec{x}_k d\vec{v}_k \cdot \vec{\nabla} v_j f^1(\vec{x}_j\vec{v}_j t) = 0,$$

where we have denoted $\vec{a}_{jk}$ as the acceleration of particle j due to its interaction with particle k. The quantity $\vec{a}_j$

$$\vec{a}_j = \sum_k \int \vec{a}_{jk} f^1(\vec{x}_k\vec{v}_k t) d\vec{x}_k d\vec{v}_k$$

is recognizable as simply the acceleration of the j-th particle due to the averaged fields produced by all the others. To within a term of higher order this is precisely the averaged electric and magnetic fields due to all particles.

The entire set of Vlasov equations can be written as:

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} + \vec{a}\cdot\vec{\nabla}_v\right) f = 0, \tag{1.8}$$

where

$$\vec{a} = \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) \tag{1.9}$$

and

$$\vec{\nabla}\cdot\vec{E} = 4\pi \sum_i e_i \int f_i d^3\vec{v}_i\ , \tag{1.10}$$

$$\vec{\nabla}\times\vec{B} = 4\pi\sum_i e_i\int f_i\vec{v}_i d^3\vec{v}_i + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}, \tag{1.11}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \tag{1.12}$$

$$\vec{\nabla}\cdot\vec{B} = 0, \tag{1.13}$$

and f has been normalized so that its integral over all positions and velocities is unity. Further details on this type of derivation can be found in Ref. [1].

(f) Entropy conservation

Before turning to consideration of linearized oscillations, we should point out an essential feature of the Vlasov equation. If we define a Boltzmann H-function in the usual fashion

$$H = \sum_i\int f_i \ln f_i \, d^3\vec{v}\, d^3\vec{x} \tag{1.14}$$

and take its time derivative, we have

$$\frac{dH}{dt} = \sum_i\int\left(\frac{df_i}{dt} + \ln f_i \frac{df_i}{dt}\right) d^3\vec{x}\, d^3\vec{v}.$$

By conservation of probability, the first term vanishes. Substituting from Eq. (1.8) for the time derivative in the second term and integrating by parts, we obtain

$$\frac{dH}{dt} = -\sum_i\int \ln f_i\,(\vec{v}_i\cdot\vec{\nabla} + \vec{a}_i\,\vec{\nabla}_{vi})\, f_i\, d^3\vec{x}\, d^3\vec{v}$$

$$= \sum_i\int (\vec{v}_i\cdot\vec{\nabla} + \vec{a}_i\nabla_{vi})\, f_i\, d^3\vec{x}\, d^3\vec{v} = 0.$$

Thus the entropy of our system is conserved.

2. LINEAR OSCILLATIONS

(a) Longitudinal waves

Since the mass of the ions is so much larger than the electrons, their intrinsic plasma frequency is much lower and in a mixture this usually represents a small correction to the dispersion relation. Furthermore, almost all of the interesting features in the dispersion relation are already present in the behaviour of the electrons themselves. Hence, in the following we shall assume that the ions are infinitely massive, at rest, and uniformly distributed. We also no longer distinguish between individual particles of the same species (electrons, ions, etc.) and now write the Vlasov equations in the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{e}{m}\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right) \cdot \vec{\nabla}_v f = 0 , \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{E} = 4\pi e\left(\int f d^3\vec{v} - n_i\right) , \tag{2.2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t} , \tag{2.3}$$

$$\vec{\nabla} \cdot \vec{H} = 0 , \tag{2.4}$$

$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c} e \int \vec{v} f d^3\vec{v}, \tag{2.5}$$

where f is normalized to the number of particles in a given species and now refers to the electron distribution only; n_j is the fixed ion number density and e is the electron charge.

Let us assume that the actual electron distribution function f departs only slightly from a zero-order distribution f_0 which is uniform in space and time and which corresponds to an electron density equal in number to that of the fixed ion background. Furthermore, it is also assumed that there is no net electric or magnetic field in the plasma in zero-order. This implies no externally applied electric or magnetic fields as well as the requirement that

$$\int \vec{v} f_0(\vec{v}) d^3\vec{v} = 0. \tag{2.6}$$

With these assumptions it is clear that Eqs. (2.1) through (2.5) are identically zero in zero'th order. If we write

$$f = f_0(\vec{v}) + f_1(\vec{x}, \vec{v}, \vec{t}) \tag{2.7}$$

the first-order equations become:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla} f_1 + \frac{e}{m}\left(\vec{E}_1 + \frac{\vec{v}}{c} \times \vec{H}_1\right) \cdot \vec{\nabla}_v f_0 = 0 , \tag{2.8}$$

$$\vec{\nabla} \cdot \vec{E}_1 = 4\pi e \int f_1 d^3\vec{v} , \tag{2.9}$$

$$\vec{\nabla} \times \vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{H}_1}{\partial t} , \tag{2.10}$$

$$\vec{\nabla} \cdot \vec{H}_1 = 0 , \tag{2.11}$$

$$\vec{\nabla}\times\vec{H}_1 = \frac{1}{c}\frac{\partial\vec{E}_1}{\partial t} + \frac{4\pi e}{c}\int\vec{v}\, f_1\, d^3\vec{v}. \tag{2.12}$$

We now specialize to the case of purely longitudinal oscillations by assuming that $\vec{\nabla}\times\vec{E}_1 = 0$, $\vec{H}_1 = 0$ and hence that

$$\vec{E}_1 = -\vec{\nabla}\phi_1 . \tag{2.13}$$

(This decomposition of oscillations into purely longitudinal and transverse modes is exact only if the distribution function $f_0(\vec{v})$ is isotropic.) Our equations now become:

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\vec{\nabla}f_1 - \frac{e}{m}\vec{\nabla}\phi_1\cdot\vec{\nabla}_v f_0 = 0, \tag{2.14}$$

$$\vec{\nabla}^2\phi_1 = -4\pi e\int f_1\, d^3\vec{v}. \tag{2.15}$$

We now solve by a Fourier expansion in both space and time. Let

$$f_1(\vec{x},\vec{v},t) = \int f_1(\vec{k},\vec{v},\omega)\, e^{i\vec{k}\cdot\vec{x}-i\omega t}\, d^3\vec{k}\, d\omega, \tag{2.16}$$

$$\phi_1(\vec{x},t) = \int \phi_1(k,\omega)\, e^{i\vec{k}\cdot\vec{x}-i\omega t}\, d^3\vec{k}\, d\omega, \tag{2.17}$$

then by Eq. (2.14) we have

$$f_1(\vec{k},\vec{v},\omega) = \frac{(e/m)\vec{k}\cdot\vec{\nabla}_v f_0(\vec{v})}{\vec{k}\cdot\vec{v}-\omega}\,\phi_1(k,\omega) \tag{2.18}$$

and substituting into Eq. (2.15) we find the relation:

$$1 = \frac{4\pi e^2}{mk^2}\int\frac{\vec{k}\cdot\vec{\nabla}_v f_0(\vec{v})}{\vec{k}\cdot\vec{v}-\omega}\, d^3\vec{v}. \tag{2.19}$$

This can always be reduced to a one-dimensional integral by choosing the z-axis in the $\vec{k}$ direction. Let us denote the z-component of $\vec{v}$ by u and the partial integral of $f_0(\vec{v})$ over the two directions of velocity perpendicular to $\vec{k}$ by $F_0(u)$. Thus

$$\vec{k}\cdot\vec{v} = ku,$$

$$\int f_0(\vec{v})\, dv_x dv_y = F_0(u)$$

and Eq. (2.19) becomes:

$$1 = \frac{4\pi e^2}{mk}\int_{-\infty}^{+\infty}\frac{\partial F_0(u)/\partial u}{ku-\omega}\, du. \tag{2.20}$$

Equation (2.20) is the dispersion relation first given by Vlasov. Note however that the integral is not well defined since the integrand is singular for velocities such that $u = \omega/k$. The resolution of this uncertainty will be discussed immediately below. For a zero-temperature plasma, however, there is no singularity for any finite frequency ω and the integral can be evaluated. Let us assume then that

$$F_0(u) = n\,\delta(u).$$

We then integrate by parts and find that

$$1 = 4\pi ne^2/m\omega^2$$

or

$$\omega^2 = 4\pi ne^2/m \equiv \omega_0^2.$$

Thus a perturbation in a zero-temperature electron gas oscillates with a frequency which is independent of wavelength. There is no propagation of the perturbation since the group velocity $d\omega/dk = 0$.

Returning to the case of a finite temperature plasma we are faced with the question of what to do about the singular integrand in Eq. (2.20). Vlasov arbitrarily decreed that the principle value of the integral was to be used. LANDAU [2] then showed that this result was incorrect; although in the case of long wavelengths, Vlasov's prescription does yield the principle frequency with which the plasma oscillates. Landau's method is to use complex values of ω, and hence Laplace transforms, to solve Eqs. (2.14) and (2.15).

Let us follow Landau's treatment. We define the spatial Fourier transforms

$$f_1(\vec{k},\vec{v},t) = \int f_1(\vec{x},\vec{v},t)\,e^{i\vec{k}\cdot\vec{x}}\,d^3\vec{x}\,, \tag{2.21}$$

$$\phi_1(\vec{k},t) = \int \phi_1(\vec{x},t)\,e^{i\vec{k}\cdot\vec{x}}\,d^3\vec{x}\,, \tag{2.22}$$

and then their Laplace transforms as

$$f_1(\vec{k},\vec{v},s) = \int_0^\infty e^{-st}\,f_1(\vec{k},\vec{v},t)\,dt \quad (\mathrm{Re}\ s > x_0), \tag{2.23}$$

$$\phi_1(\vec{k},s) = \int_0^\infty e^{-st}\,\phi_1(\vec{k},t)\,dt \quad (\mathrm{Re}\ s > x_0), \tag{2.24}$$

where ϕ_1 and f_1 are assumed to be of order $e^{x_0 t}$ for $t \geq 0$. A Laplace-Fourier transform of Eq. (2.14) then yields:

$$(s + i\vec{k}\cdot\vec{v})\,f_1(\vec{k},\vec{v},\vec{s}) - g(\vec{k},\vec{v}) = \frac{ie}{m}\,\phi_1(\vec{k},\vec{s})k\cdot\vec{\nabla}_v f_0(\vec{v}). \tag{2.25}$$

Here

$$g(\vec{k},\vec{v}) = f_1(\vec{k},\vec{v},t=0) \tag{2.26}$$

is the Fourier transform of the initial perturbation. Upon substitution of Eq. (2.25) in the transform of Eq. (2.15) we obtain:

$$\phi_1(\vec{k}, s) = \frac{\frac{4\pi e}{k^2}\int \frac{g(\vec{k}, \vec{v})}{s + i\vec{k}\cdot\vec{v}}\, d^3\vec{v}}{1 - \frac{4\pi e^2 i}{mk^2}\int \frac{\vec{k}\cdot\vec{\nabla}_v f_0}{s + i\vec{k}\cdot\vec{v}}\, d^3\vec{v}} .$$

Once again, this can be simplified by choosing the z-axis in the direction of $\vec{k}$ and defining $\vec{k}\cdot\vec{v} \equiv ku$, $g(k,u) = \int g(\vec{k},\vec{v})dv_x dv_y$ and $F_0(u) = \int f_0(\vec{v})\, dv_x dv_y$.

Hence we obtain:

$$\phi_1(k, s) = \frac{-\frac{4\pi e i}{k^2}\int_{-\infty}^{+\infty} \frac{g(k,u)}{ku - is}\, du}{1 - \frac{4\pi e^2}{mk}\int_{-\infty}^{+\infty} \frac{\partial F_0(u)/\partial u}{ku - is}\, du} . \tag{2.27}$$

This is the Laplace-Fourier transform of the perturbed scalar potential. The corresponding transform of the perturbed distribution function is found by substituting Eq. (2.27) in Eq. (2.25) which yields:

$$f_1 = \frac{g(\vec{k},\vec{v})}{s + i\vec{k}\cdot\vec{v}} + \frac{ie}{m}\,\frac{\vec{k}\cdot\vec{\nabla}_v f_0}{s + i\vec{k}\cdot\vec{v}}\, \phi_1(k, s). \tag{2.28}$$

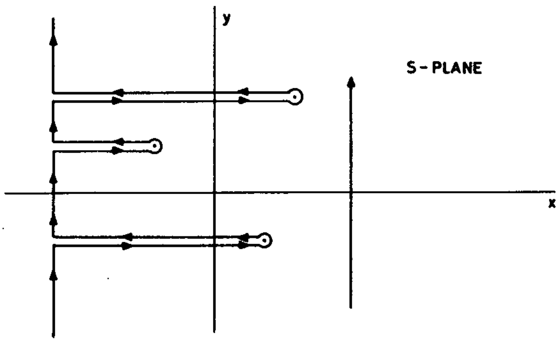
Note that ϕ_1 in Eq. (2.27) is composed of two parts. The integral in the numerator is dependent only on details of the initial perturbation $g(k, u)$ while the denominator depends only on details of the zero'th-order velocity distribution.

The actual variation of the perturbation in time is obtained from the inverse Laplace transform.

$$\phi_1(k, t) = \frac{1}{2\pi i}\int \phi_1(k, s)\, e^{st}\, ds, \tag{2.29}$$

where the integral is taken along the Bromwich path in the complex s-plane, which is a contour parallel to the imaginary axis and to the right of all singularities of ϕ_1, including the imaginary axis itself. Eq. (2.29), after Fourier inversion, represents a complete solution to the initial value problem. There are now no singularities associated with the ku-is term in the integrals of Eq. (2.27) since the inverse transform uses values of s which are to the right of the imaginary axis in the complex s-plane.

Although Eq. (2.29) represents a complete solution to the problem, it is not in a convenient form. We would prefer to deform the integral along the Bromwich path into the equivalent contour shown in Fig. 1(a) and then pass to the limit where the vertical part of the contour moves toward $-\infty$.



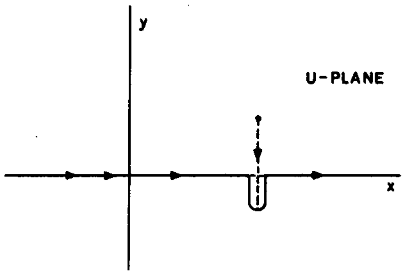


Fig. 1

Landau contour

In this case the contributions from all parts of the contour vanish, for any finite time, except for the integrals around the poles of ϕ_1 in the s-plane. These last contributions are just the residues at these poles (denoted by α) and the result is

$$\phi_1(k,t) = \sum_{\alpha} e^{\alpha t} \operatorname{Res}_{\alpha}[\phi_1(\vec{k},s)] . \tag{2.30}$$

In order that the integral of $\phi_1(\vec{k},s)$ be the same along either of these contours, it is necessary that $\phi_1(\vec{k},s)$ be analytic everywhere in the region between the two contours. This requires that we define $\phi_1(\vec{k},s)$ for $\operatorname{Re} s \leq x_0$ such that it is the proper analytic continuation of the function $\phi_1(\vec{k},s)$ defined so far only for $\operatorname{Re} s > x_0$. This can be accomplished by deforming the contour in the u-plane as shown in Fig. 1(b). When Re $s > 0$, the u-integral is taken along the real axis, as before. Assuming $k > 0$, there is a pole in the upper u-plane at $u = is/k$. Now, as s moves to the left of the imaginary axis in the s-plane, the corresponding pole in u-space moves across the real u-axis. If we deform the contour as indicated, the new function $\phi(\vec{k},s)$ is automatically the proper analytic continuation of the function in the right-hand plane. It is assumed in all this that $g(k,u)$ and $\partial F_0(u)/\partial u$ are themselves analytic functions in the complex u-plane (or more generally, are such that ϕ_1 itself is analytic except at isolated poles).

Let us denote the contour indicated in Fig. 1(b) by the subscript c. We can now write

$$\phi_1(k,s) = \frac{-\dfrac{4\pi ei}{k^2}\displaystyle\int_c \frac{g(k,u)}{ku-is}\,du}{1-\dfrac{4\pi e^2}{mk}\displaystyle\int_c \frac{\partial F_0(u)/\partial u}{ku-is}\,du} \tag{2.31}$$

and the time variation of ϕ_1 is entirely determined by its poles in the entire complex s-plane.

(b) Some properties of the dispersion equation

Consider first the numerator in Eq. (2.31). If we assume that the initial perturbation is a non-singular function on the real axis then it is easy to show that this numerator can only contribute poles which are decaying in time. For , if Re s > 0, we can properly take the integral along the real axis only. The numerator becomes (for s = a + ib)

$$\sim \int_{-\infty}^{+\infty} \frac{g(k,u)}{ku+b-ia}\,du$$

and thus obviously cannot be singular. Even if Re s = 0, we can show the non-existence of a solution. In this case let s = ib. Now we can write the numerator as

$$\sim P\int_{-\infty}^{+\infty} \frac{g(k,u)}{ku+b}\,du + \pi i g\left(k, -\frac{b}{k}\right), \tag{2.32}$$

where P denotes the principal part of the integral. Since g(k, u) is non-singular on the real axis the principal part integral can be no worse than logarithmically infinite. Hence the numerator can only contribute poles corresponding to time-decaying solutions.

We now turn to poles contributed by the denominator. These roots are in a sense the "natural frequencies" of the system since they do not depend on the details of the initial perturbations (except in so far as their amplitude is concerned). These poles of Eq. (2.31) occur when the denominator vanishes, or when

$$1 = \frac{4\pi e^2}{mk^2}\int_c \frac{dF_0/du}{u-is/k}\,du. \tag{2.33}$$

We shall call this our dispersion equation.

One important property of this dispersion equation can be readily derived. This is the demonstration that the Maxwell-Boltzmann distribution and more generally "single-humped" distributions have only time-decaying roots.

Let us assume that a pole does exist on the right-hand side of the complex s-plane. This implies then that for some $s = x + iy$, with $x > 0$,

$$1 = \frac{4\pi e^2}{mk} \int_{-\infty}^{+\infty} \frac{\partial F_0/\partial u}{ku + y - ix} du, \tag{2.34}$$

where it is now legitimate to take the integral along the real axis. Separating this into real and imaginary parts, we have:

$$1 = \frac{4\pi e^2}{mk} \int_{-\infty}^{+\infty} \frac{\partial F_0/\partial u\,(ku+y)}{(ku+y)^2 + x^2} du, \tag{2.35}$$

$$0 = \frac{4\pi e^2 x}{mk} \int_{-\infty}^{+\infty} \frac{\partial F_0/\partial u}{(ku+y)^2 + x^2} du. \tag{2.36}$$

Suppose now that $F_0(u)$ is a "single-humped" distribution, peaked at $u = U$, as shown in Fig. 2. By a "single-humped" distribution we mean that $\partial F_0/\partial u > 0$ for $u < U$ and $\partial F_0/\partial u < 0$ for $u > U$. Now, since Eq. (2.36) is zero, we can multiply it by $(kU + y)/x$ and subtract it from Eq. (2.35). The result is:

$$1 = -\frac{4\pi e^2}{mk} \int_{-\infty}^{+\infty} \frac{(U-u)\partial F_0/\partial u\, du}{(ku+y)^2 + x^2}. \tag{2.37}$$

Note, however, that the numerator of the integrand is positive definite by virtue of being "single-humped". Hence the right-hand side is negative definite and the equality cannot occur. This proof is not valid when s lies to the left of the imaginary axis in s-space, since then the integral in Eq. (2.34) cannot be confined entirely to the real axis.

(c) Landau damping

We now specialize to the case where f_0 is a Maxwell–Boltzmann distribution,

$$F_0 = \frac{n_0}{\sqrt{\pi}\,\alpha} \exp - (u/\alpha)^2$$

Eq. (2.33) can be written as

$$1 = \frac{\omega_p^2}{\sqrt{\pi}\,k^2\alpha^2} \int_C \frac{-2t \exp -t^2 dt}{t - is/k\alpha}$$

which can easily be put in the form

$$k^2\lambda_D^2 + 1 + i\sqrt{\pi}\, z\, W(z) = 0, \tag{2.38}$$

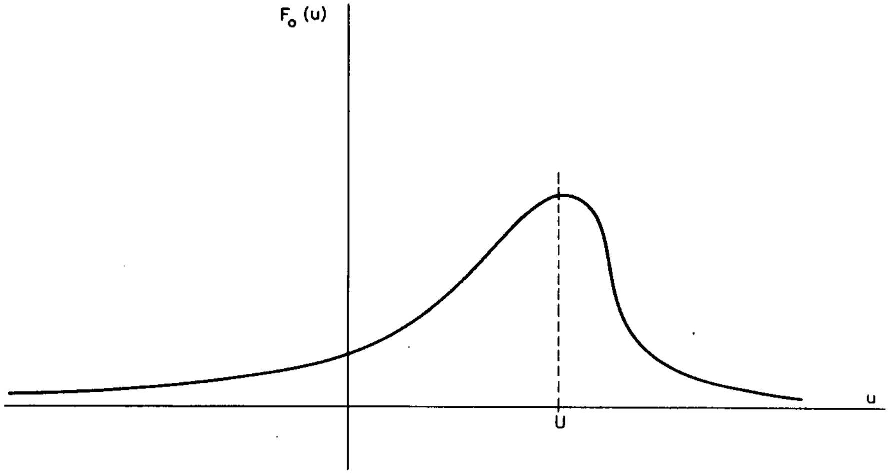


Fig. 2

"Single-humped" distribution function

where z = is/kα and

$$W(z) = \frac{i}{\pi}\int_C \frac{\exp -t^2}{z-t}\,dt. \tag{2.39}$$

The W-function defined above has been extensively tabulated [3].

Let us solve Eq. (2.38) above in the limit of very long wavelengths. If we assume that s does not vanish as rapidly as k, when $k \to 0$, we then must assume that as

$$k \to 0, \ z \to \infty.$$

This leads us to consider asymptotic expansions of the W-function. First suppose that z, although large, is real. In this case we have

$$W(z) = \frac{i}{\pi}\, P\int_{-\infty}^{+\infty} \frac{\exp -t^2}{z-t}\,dt + \exp -z^2.$$

Performing a Taylor series expansion of the denominator of the integral, we obtain:

$$W(z) \to \frac{i}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \frac{3}{4z^5} + \dots\right) + \exp -z^2.$$

The series just given is an asymptotic one, which means that it is only unique to within exponentially small quantities. Since the residue term itself is exponentially small, it is often nonsense to include this contribution. However, there are some special regions of the z-plane in which keeping the added term makes sense. Right on the real axis, of course, we note that the residue term is purely real while the series is purely imaginary. Hence, any exponentially small terms in the series do not contribute to the real part of W and it makes sense to include this residue as long as we stay in the immediate vicinity of the real axis and are interested in the exponentially small real part of the function.

The only other regions of the z-plane where inclusion of such a term makes sense are when $|\mathrm{Im}\, z| > |\mathrm{Re}\, z|$. In this case, the exponential term is no longer exponentially small, but exponentially large. For $\mathrm{Im}\, z > 0$, however, there is no such term since the contour goes along the real axis only. For $\mathrm{Im}\, z < 0$, we pick up a $2\pi i$ from the residue. Summarizing we have the following expansions:

$$
\begin{aligned}
W(z) &\to \frac{i}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \dots\right) && \mathrm{Im}\, z > |\mathrm{Re}\, z| \\
W(z) &\to 2 \exp -z^2 + \frac{i}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} \dots\right) && \mathrm{Im}\, z < -|\mathrm{Re}\, z| \qquad (2.40) \\
W(z) &\to \exp -z^2 + \frac{i}{\sqrt{\pi}}\left(\frac{1}{z} + \frac{1}{2z^3} + \dots\right) && \mathrm{Im}\, z \cong 0 .
\end{aligned}
$$

Now let us return to our long wavelength limit. Eq. (2.38) goes into

$$
k^2\lambda_D^2 + 1 - \left(1 + \frac{1}{2z^2} + \dots\right) + i\delta\sqrt{\pi}\, z \exp -z^2 = 0,
$$

where the value of δ depends on $\mathrm{Im}\, z$. We note that if we ignore the exponential term, then the series itself can only yield real values of z^2. This would not be consistent with assuming $\mathrm{Im}\, z > |\mathrm{Re}\, z|$. If instead we assume $\mathrm{Im}\, z < -|\mathrm{Re}\, z|$, we have a term $2\sqrt{\pi}\, z \exp -z^2$ which becomes infinite with z while all other terms remain finite. Hence our only possible solution is for $|\mathrm{Im}\, z| < |\mathrm{Re}\, z|$ and in this case, by the nature of our expansion we see that the imaginary part of z will be exponentially small. Thus a proper expansion is

$$
k^2\lambda_D^2 - \frac{1}{2z^2} + i\sqrt{\pi}\, z \exp -z^2 = 0.
$$

To lowest order $z_0^2 = 1/2k^2\lambda_D^2 = \omega_p^2/k^2\alpha^2$ the usual result. The small imaginary contribution z_i is found by iteration

$$
k^2\lambda_D^2 - \frac{1}{2z_0^2}\left(1 - \frac{2iz_i}{z_0}\right) + i\sqrt{\pi}\, z_0 \exp -z_0^2 = 0 \qquad (2.41)
$$

or

$$
z_i = -\sqrt{\pi}\, z_0^4 \exp -z_0^2 .
$$

We note at once that this result corresponds to damping, since the value of s is

$$\frac{is}{k\alpha} \equiv z_0 + iz_i\,,$$

consequently

$$s = -\sqrt{\pi}\, k\alpha\, z_0^4 \exp -z_0^2 - i k\alpha z_0.$$

In more familiar units we have

$$s_{real} = -\sqrt{\pi}\, k\alpha \left(\frac{\omega_p}{k\alpha}\right)^4 \exp -\left(\frac{\omega_p}{k\alpha}\right)^2 \tag{2.42}$$

or, in terms of the Debye length

$$s_{real} = -\frac{\sqrt{\pi}}{8}\frac{\omega_p}{(k\lambda_D)^3} \exp -\left(\frac{1}{2k\lambda_D}\right)^2, \tag{2.43}$$

while further iteration provides the imaginary part as:

$$s_{imag} = \pm\left(\omega_p^2 + k^2 \frac{3kT}{M} + \ldots\right)^{\frac{1}{2}}. \tag{2.44}$$

The result we have just obtained can also be found without using the asymptotic representations of the W-function [4]. An alternative procedure as well as further details on Landau damping can be found in Ref. [4].

Next we turn to the short wavelength limit. Again we assume that s stays finite or does not blow up as rapidly as k. Then as $k \to \infty$, $z \to 0$. Using a power series expansion for W(z)

$$W(z) = 1 + \frac{2i}{\sqrt{\pi}} z - z^2 - \frac{4i}{3\sqrt{\pi}} z^3 + \ldots \tag{2.45}$$

we see at once that Eq. (2.38) leads to a contradiction. Hence we must assume that $z \to \infty$ as $k \to \infty$. Using the asymptotic expansion we have:

$$k^2\lambda_D^2 + 1 - \left(1 + \frac{1}{2z^2} + \ldots\right) + i\delta\sqrt{\pi} z \exp -z^2 = 0.$$

In this case, since the first term becomes infinite, we can only have a solution if $|\mathrm{Im}\, z| > |\mathrm{Re}\, z|$ and indeed we must have $\mathrm{Im}\, z < -|\mathrm{Re}\, z|$ since otherwise $\delta = 0$. Thus our expansion is

$$k^2\lambda_D^2 + i2\sqrt{\pi}\, z \exp -z^2 = 0$$

or

$$k^2\lambda_D^2 = 2\sqrt{\pi}\,(z_i - i z_r) \exp (z_i^2 - 2i z_r z_i - z_r^2).$$

If we now assume that $|z_i| \gg |z_r|$ this expression reduces to:

$$k^2\lambda_D^2 = 2\pi z_i \exp(-z_i^2 - 2i z_r z_i).$$

However, $z_i < 0$. Hence we must also require that

$$2 z_r z_i = \pi. \tag{2.46}$$

There are also solutions for higher odd multiples of π but these decay even more rapidly. Now we find

$$k^2\lambda_D^2 = -2\pi z_i \exp -z_i^2. \tag{2.47}$$

Hence we see that roughly

$$z_i \to (\ln k)^{\frac{1}{2}} \to \infty$$

$$z_r \to \frac{1}{z_i} \to 0$$

verifying our assumptions. The result is a heavily damped mode. Thus we see that collective oscillations with wavelengths appreciably shorter than a Debye length cannot exist in a plasma.

All the results we have obtained so far are for the root with the smallest damping. Of course, other roots exist, but these are more heavily damped.

(d) The origins of Landau damping

We have seen that small disturbances of the electrostatic field in a plasma having a Maxwell-Boltzmann distribution of velocities oscillate with a frequency close to the plasma frequency and also decay exponentially in time (for long wavelengths; short wavelengths decay very rapidly). What is the origin of this damping and how do we reconcile it with entropy conservation which must be present?

Turning to the first question, we consider the form of the damping as shown in Eq. (2.42). The exponential arises from the distribution function $F_0(u)$. The exponential's coefficient is $\omega_p/k\alpha$. Since ω_p/k is the phase velocity of the oscillating part of the plasma disturbance, we see that the damping is proportional to the number of particles in the distribution function moving with the phase velocity of the wave. This can be made more precise by the following simplified derivation of the Landau damping.

Consider the linearized Vlasov equation with our usual assumptions.

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla} f_1 + \frac{e}{m} \vec{E}_1 \cdot \vec{\nabla}_v f_0 = 0.$$

We regain the "natural oscillations" result if we take Fourier transforms in time (remembering however to go around the pole in the proper Landau fashion). Thus taking transforms, we find

$$f_1 = \frac{-(e/m)E_1\, df_0/du}{i\,(-\omega + ku)},$$

where our transform is $\exp[-i(\omega t - \vec{k}\cdot\vec{x})]$.
The perturbed current j_1 is euf_1. Hence

$$j_1 = i\frac{e^2}{m}E_1 \frac{u\,df_0/du}{-\omega + ku}$$

and the work done on the particles by the electrostatic field is

$$W = j_1E_1 = \frac{ie^2}{m}E_1^2 \frac{udf_0/du}{-\omega + ku}.$$

Integrating over all velocities, we have

$$W = i\frac{e^2}{m}E_1^2 \int_c \frac{udF_0/du}{-\omega + ku}\,du. \qquad (2.48)$$

Assuming that our pole lies close to the real axis we see that the integral can be written as the sum of a real principal part and an imaginary contribution from the residue. Comparing with Eq. (2.48) we see that the only resulting real contribution to the work is provided by the residue term and is

$$W_r = -\pi \frac{e^2}{m|k|} E_1^2 \left(u\frac{dF_0}{du}\right)_{u=\omega/k}, \qquad (2.49)$$

where the subscript denotes that the brackets are evaluated at $u = \omega/k$. Note that the work done is proportional to E_1^2, the energy in the electrostatic field. The value of ω comes out of the real part of the dispersion equation and has been shown to be ω_p^2 in first approximation. In any event ω/k is the phase velocity of the oscillating part of the wave. We see that the damping term is proportional to the slope of the distribution function at the phase velocity of the wave.

For a Maxwell-Boltzmann distribution, we have

$$\frac{dF_0}{du} = -\frac{2}{\alpha}\left(\frac{u}{\alpha}\right)F_0$$

and thus

$$W_r = 2\sqrt{\pi}\frac{n_0e^2}{m|k|\alpha} E_1^2 \left(\frac{\omega}{k\alpha}\right)^2 \exp -\left(\frac{\omega}{k\alpha}\right)^2.$$

Hence the work done on the particles is positive (damping of the electrostatic field) and exactly of the form given in Eq. (2.42).

The result in Eq. (2.49) also allows us a simple physical interpretation of the damping. Particles moving a little more slowly than the wave are pushed by the wave and gain energy. Those moving just a little faster give up energy to the wave. The net energy transfer is proportional to the difference, which is the slope of the distribution function at the phase velocity of the wave.

Historically there have been a number of physical explanations of Landau damping starting from the observations above. One theory invoked electron trapping, since it is just those electrons moving nearly at the velocity of the wave which have very little energy in the wave frame and which can be trapped in the potential well formed by E_1. This mechanism is pretty well rejected by most people, today, since it requires non-linear effects for its operation and since we derive the Landau damping in a purely linear theory.

Other theories have invoked "phase mixing" and "resonant transfer". There are a number of special examples which are relevant to these theories and which may be found in Ref. [4]. We will not repeat them here except to note their results. One can show that a spread of velocities alone (phase mixing) will not give damping if there are no particles at the phase velocity of the wave. Similarly, if we have no spread of velocities there will be no damping (all particles move together and wave form persists). Hence one might say that some mixture of the "phase mixing" and "resonant transfer" produces the Landau damping since we do require some spread of velocities about the phase velocity of the wave.

(e) Electron distribution function and entropy conservation

Having found that the electrostatic potential ϕ_1 damps away in time, we now turn to the long time behaviour of the perturbed distribution function f_1 itself. By Eq. (2.38), we see that after a long time

$$f_1 \to \frac{g\,(\vec{k},\vec{v})}{s + i\,\vec{k}\cdot\vec{v}} \tag{2.50}$$

and that a purely oscillating pole remains. Hence f_1 does not damp away.

In fact, we can easily evaluate it.

$$f_1(\vec{k},\vec{v},t) \to e^{-i\vec{k}\cdot\vec{v}t}\;g(\vec{k},\vec{v}).$$

Inverting the Fourier transform, we have

$$f_1(\vec{x},\vec{v},t) \to \int e^{\,i\vec{k}\cdot(\vec{x}-\vec{v}t)}\;g(\vec{k},\vec{v})d^3\vec{k}.$$

But

$$g(\vec{k},\vec{v}) = \int e^{-i\vec{k}\cdot\vec{x}'}\;f_1(\vec{x}',\vec{v},t=0)d^3\vec{x}'.$$

Hence

$$f_1(\vec{x},\vec{v},t) = \iint e^{i\vec{k}\cdot(\vec{x}-\vec{x}'-\vec{v}t)} f_1(\vec{x}',\vec{v},t=0)\, d^3\vec{x}'\, d^3\vec{k}$$

$$= \int \delta(\vec{x}-\vec{x}'-\vec{v}t) f_1(\vec{x}',\vec{v},t=0)\, d^3\vec{x}'. \qquad (2.51)$$

Consequently

$$f_1(\vec{x},\vec{v},t) = f_1(\vec{x}-\vec{v}t,\vec{v},t=0).$$

Hence we see that, after a long time, the distribution function settles down to the value it would have if only free streaming of particles had occurred. Since the initial distribution function contained all the information on the problem (the electrostatic field is entirely determined by f_1) we see that information (i.e. entropy) has been conserved.

(f) Van Kampen modes and Fourier analysis

Let us return again to the numerator in Eq. (2.31). In section 2 (b) we noted that the numerator could contribute only decaying roots if g(k, u) was a non-singular function on the real axis. Suppose now that we allow g(k, u) to be singular on the real axis but, of course, insist that it be integrable. That is

$$\int_{-\infty}^{+\infty} g(k,u)\,du = \text{finite quantity.}$$

Again, if Re $s > 0$, we can integrate along the real axis and our numerator becomes

$$\int_{-\infty}^{+\infty} \frac{g(k,u)}{ku+b-ia}\,du,$$

where $s = a + bi$. If g (ku) is singular at $u = u_0$, we can approximate the integral near that point by

$$\frac{1}{ku_0+b-ia}\int g(k,u)\,du$$

which remains finite. Hence we still cannot obtain a growing root from the numerator.

However, an oscillating root is now possible. If $a = 0$, our numerator becomes

$$P\int \frac{g(k,u)}{ku+b}\,du + \frac{i\pi}{|k|}\, g\left(k, -\frac{b}{k}\right).$$

Obviously if we choose our root at the singular point of g we can have a pole. Van KAMPEN [5] has gone further than this. He showed that it is in fact possible to choose the form of the initial perturbation such that the only solution is a purely oscillatory mode. In fact, one can obtain a solution which oscillates with any ω for any given k. The perturbation which will do this is the following:

$$g(k,u) = \frac{k^3\phi_0(k)}{4\pi e}\left\{\frac{4\pi e^2}{mk^2}\,\frac{\partial F_0/\partial u}{(ku-\omega)} + [1-\psi(\omega)]\,\delta(ku-\omega)\right\}, \tag{2.52}$$

where

$$\psi(\omega) = \frac{4\pi e^2}{mk}\int_c \frac{\partial F_0/\partial u}{ku-\omega}\,du \tag{2.53}$$

and $\phi_0(k)$ is the Fourier transform of the initial perturbation of the potential. Here c denotes the Landau contour which avoids the poles on the real axis. To verify this, let us substitute Eq. (2.52) into Eq. (2.31). We have

$$\int_c \frac{(\partial F/\partial u)\,du}{(ku-\omega)\,(ku-is)} = \frac{1}{\omega - is}\int_c \frac{\partial F}{\partial u}\left(\frac{1}{ku-\omega} - \frac{1}{ku-is}\right)du\,.$$

Consequently

$$\frac{4\pi e^2}{mk^2}\int_c \frac{(\partial F/\partial u)\,du}{(ku-\omega)\,(ku-is)} = \frac{1}{k(\omega-is)}\,[\psi(\omega)-\psi(is)].$$

The second part of the right-hand side of Eq. (2.52) yields

$$\int_c \frac{[1-\psi(\omega)]\,\delta(ku-\omega)}{ku-is}\,du = \frac{1-\psi(\omega)}{k(\omega-is)}\,.$$

Hence the numerator of Eq. (2.31) becomes:

$$\sim \frac{1-\psi(is)}{k(\omega-is)}.$$

Now the denominator of Eq. (2.31) is simply $1-\psi(is)$, hence we have:

$$\phi_1(k,x) \sim \frac{1}{\omega - is}$$

which has only a single pole corresponding to a solution which oscillates with frequency ω.

Of course, the perturbation used by Van Kampen must be carefully tailored. It must have a singularity (the δ-function) and must depend upon the details of the zero-order velocity distribution in such a manner that it cancels out the poles of the denominator and contributes in addition a pure oscillating solution. Nevertheless, the existence of these purely oscillating solutions enabled Van Kampen to define normal modes and to treat the plasma by Fourier analysis rather than as an initial value problem with Laplace transforms. Either the Landau or the Van Kampen treatment may be used, as taste dictates. The final result, of course, is always the same.

(g) Transverse oscillations

Let us return to Eqs. (2.8) through (2.12) and obtain pure transverse oscillations by setting the scalar potential equal to zero. Now

$$\vec{E}_1 = -\frac{1}{c}\frac{\partial \vec{A}_1}{\partial t}, \tag{2.54}$$

$$\vec{H}_1 = \vec{\nabla}\times\vec{A}_1, \tag{2.55}$$

where $\vec{A}_1$ is the perturbed vector potential and our basic equations become:

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\vec{\nabla} f_1 - \frac{e}{m}\left[\frac{1}{c}\frac{\partial \vec{A}_1}{\partial t} - \frac{\vec{v}}{c}\times(\vec{\nabla}\times\vec{A}_1)\right]\cdot\vec{\nabla}_v f_0 = 0, \tag{2.56}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A}_1 = \frac{4\pi e}{c}\int \vec{v} f_1 d^3\vec{v}. \tag{2.57}$$

It is also necessary that

$$\int f_1 d^3\vec{v} = 0. \tag{2.58}$$

Once again, we can solve Eq. (2.56) by Laplace-Fourier transforms and substitute in Eq. (2.57). To simplify matters, let us assume that $f_0(v)$ is an isotropic function. In this case $\vec{\nabla}_v f_0 \sim \vec{v}$ and the last term in Eq. (2.56) vanishes. If we again let $g(k, v)$ denote the Fourier transform of the initial perturbation, $\vec{h}(k)$ the Fourier transform of the initial vector potential and let f_1 and A_1 now represent the Fourier-Laplace transforms, we have

$$f_1 = \frac{(e/mc)\, s\, \vec{A}_1\cdot\vec{\nabla}_v f_0}{s + i\vec{k}\cdot\vec{v}} + \frac{g(k,\vec{v}) - (e/mc)\vec{h}\cdot\vec{\nabla}_v f_0}{s + i\vec{k}\cdot\vec{v}}. \tag{2.59}$$

Now let $\vec{\epsilon}$ denote a unit vector perpendicular to $\vec{k}$ and parallel to $\vec{A}_1$. We have

$$\int (\vec{\epsilon}\cdot\vec{v}) f\, d^3\vec{v} = -\frac{e}{mc} s A_1 \int \frac{f_0}{s + i\vec{k}\cdot\vec{v}} d^3\vec{v}$$

$$+ \int \frac{[(\vec{\epsilon}\cdot\vec{v})\, g(k,\vec{v}) + (e/mc)\, h(k) f_0(v)]}{s + i\vec{k}\cdot\vec{v}} d^3\vec{v}$$

and since the left-hand side of Eq. (2.57) becomes

$$\left(k^2 + \frac{s^2}{c^2}\right) A_1 - \frac{s}{c^2} h - \frac{\dot{h}}{c^2},$$

where $\dot{h}$ is the initial time derivative of A, we find that

$$A_1 = \frac{\frac{s}{c^2} h(k) + \frac{1}{c^2}\dot{h}(k) + \frac{4\pi e}{c}\int \frac{[(\vec{\epsilon}\cdot\vec{v})g + (e/mc)h f_0]}{s + i\vec{k}\cdot\vec{v}} d^3\vec{v}}{k^2 + \frac{s^2}{c^2} + \frac{4\pi e^2 s}{mc^2}\int \frac{f_0}{s + i\vec{k}\cdot\vec{v}} d^3\vec{v}}. \tag{2.60}$$

Once again, we continue this into the left-hand side of the s-plane by using the Landau contour. Note also that as before the numerator in Eq. (2.60) depends on the initial perturbation while the denominator contains the "natural" frequencies. Let us concentrate our attention to these "natural" frequencies. The "dispersion relation" becomes:

$$1 = -\frac{4\pi e^2 s}{m(s^2 + k^2c^2)} \int_c \frac{F_0(u)\, du}{s + iku}. \tag{2.61}$$

If we now assume that F_0 is a Maxwell-Boltzmann distribution, we can readily write the above relation as

$$s^2 + k^2c^2 - i\sqrt{\pi}\omega_p^2 z W(z) = 0, \tag{2.62}$$

where W is the same function defined earlier and, as before, $z = is/k\alpha$. Now consider the long wavelength limit. Assume s remains finite as $k \to 0$. Then $z \to \infty$. Neglecting the residue term for the moment we have

$$s^2 + k^2c^2 - i\sqrt{\pi}\omega_p^2 \frac{i}{\sqrt{\pi}} = 0$$

or

$$s^2 = -(\omega_p^2 + k^2c^2); \tag{2.63}$$

consequently

$$s = \pm i(\omega_p^2 + k^2c^2)^{\frac{1}{2}}.$$

But this first approximation to s says that

$$|s| > kc$$

which means

$$|z| > \frac{c}{\alpha}.$$

Hence, if we include the residue term, we are including a portion of the particle distribution function for which

$$\frac{u}{\alpha} = z > \frac{c}{\alpha}$$

or

$$u > c.$$

If we had used a properly relativistic particle distribution function, there could be no contribution from such values of u. Hence it is entirely self-consistent to drop the residue term and we have Eq. (2.63) as the proper result. There is no Landau damping.

Turning next to the short wavelength limit assume first that s remains finite as $k \to \infty$. Then $z \to 0$. This obviously leads to a contradiction in Eq. (2.62). Hence we must assume that $s \to \infty$. If we assume that $s/k \to 0$ we still get a contradiction with Eq. (2.62). Hence we must assume that $s/k \to$ finite value or infinity. The second choice also leads to a contradiction. Hence we assume s/k remains finite and hence that z remains finite. Now the first approximation to s is:

$$s_0^2 = -k^2c^2.$$

The next approximation gives

$$(s_0 + s_1)^2 = -k^2c^2 + i\sqrt{\pi}\,\omega_p^2\left(\pm\frac{c}{\alpha}\right) W\left(\pm\frac{c}{\alpha}\right),$$

where we have replaced z in the last term by

$$z = \frac{is_0}{k\alpha} = \pm\frac{c}{\alpha}.$$

But $c/\alpha \gg 1$, hence

$$(s_0 + s_1)^2 \cong -k^2c^2 - \omega_p^2 + \dots$$

or

$$s^2 \cong -(\omega_p^2 + k^2c^2),$$

which is the same result as for long wavelengths. Again there is no damping of a purely transverse mode.

3. SOME EXAMPLES OF "VLASOV INSTABILITIES"

We have noted that the Landau damping is proportional to the slope of the distribution function at the phase velocity of the wave. Obviously then if we can create distribution functions which are reversed in slope at the phase velocity of a possible wave we will have an unstable situation present. In this section we will consider some simple examples of this type of behaviour.

(a) Penrose criterion

We shall consider purely electrostatic oscillations with our previous conditions ($E_0 = 0$, $B_0 = 0$, infinite uniform plasma) and ask for necessary and sufficient conditions for the existence of an unstable root [6]. We write the dispersion relation of Eq. (2.33) in the form

$$\frac{k^2}{\omega_p^2} = \int_c \frac{F_0'(u)}{u - t}\, du, \tag{3.1}$$

where $t = is/k$ and define the function

$$Z(t) = \int_c \frac{F_0'(u)}{u - t}\, du. \tag{3.2}$$

Suppose that an unstable root exists for some value of k. This means that there exists a t_0 lying in the positive imaginary half-plane for which Eq. (3.1) is satisfied, or that is, the function $Z(t_0)$ has a particular positive real value. Conversely, if the function $Z(t)$ has a positive real value for some value of t in the positive imaginary t half-plane, then our system is unstable since we can always choose a k such that the left side of Eq. (3.1) equals that particular value. Hence a necessary and sufficient condition for instability is that the function $Z(t)$ have a positive real value for some values of t lying in the positive imaginary t half-plane.

One can readily test for the existence of such roots by the Nyquist method. We allow t to move around a contour in the t-plane which lies just above the real axis and then closes by a very large arc, as shown in Fig. 3. We then plot the corresponding contour of $Z(t)$ in the Z-plane and observe whether or not any portion of the real z-axis is included in that contour.

The general slope of the Z-contour is easily determined. The outer contour in Fig. 3, by Eq. (3.2) is seen to map into the origin in the Z-plane. At the point marked 1 in Fig. 3 the function $Z(t)$ takes the value

$$\begin{aligned} Z(t_1) &= \int_{-\infty}^{+\infty} \frac{F_0'(u)}{u + a - i\epsilon}\, du \\ &= \int_{-\infty}^{+\infty} \frac{F_0'(u)\,(u + a + i\epsilon)}{(u + a)^2 + \epsilon^2}\, du. \end{aligned}$$

Since a is large and positive, ϵ positive and since $F_0'(-a)$ is surely positive for large a, we can see that the region denoted by 1 maps into a portion of the Z-contour which begins at the origin and moves up into the first quadrant. Similarly the region denoted by 2 maps into a portion which begins at the origin and moves into the fourth quadrant. Hence the general form of the contour shown in Fig. 4. By the argument principle the phase of $Z(t)$ must increase in the same sense as t. Hence the direction of the contour shown in Fig. 4.

The example shown in Fig. 4 illustrates the case of a stable distribution, since no part of the positive real axis is included inside the contour. Two examples of unstable distributions are shown in Fig. 5.

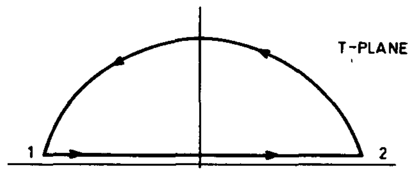


Fig. 3

Contour in the t-plane

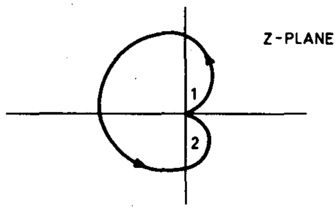


Fig. 4

Example of a contour in the Z-plane

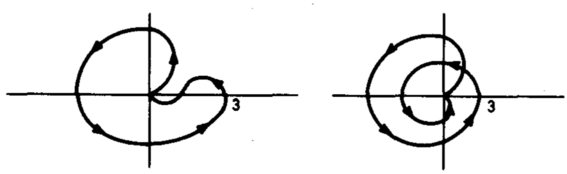


Fig. 5

Two examples of unstable solutions

We now note an essential feature of these results. If there exists an unstable root t_0 in the interior of the t-contour we will enclose a portion of the real axis in the Z-plane and some point on that axis corresponds to $Z(t_0)$. Figure 5 shows us that corresponding to the existence of any such root there must also exist a root corresponding to the point marked 3 in both diagrams in Fig. 5. This point has the property that the corresponding value of t lies on the contour between points 1 and 2 in the t-plane.

In other words, if there exists an unstable root with a finite imaginary value of t, there must also exist a root which is just marginally stable, and what is more, this root must be such that Im Z(t) changes from negative to positive as we move along the t-contour through that point.

This result can be made reasonably plausible. We can rewrite Eq.(3.1) in the form

$$\frac{k^2}{\omega_p^2} = \int_c \frac{F_0(u)}{(u - is/k)^2}\, du. \tag{3.3}$$

Now let $k \to 0$. The singularity in the denominator is moved out so far along the real axis that $F_0(u)$ is negligible there and we can ignore the residue.

The remaining term can be written as

$$\frac{k^2}{\omega_p^2} \cong \left(\frac{k}{is}\right)^2 \int F_0(u)\, du$$

or

$$s^2 = -\omega_p^2$$

which corresponds to a stable root. Hence our system is always stable for sufficiently long wavelengths and it seems plausible that one can find intermediate wavelengths for which an unstable system becomes marginally unstable.

Now suppose that t has the value corresponding to point 3 in Fig. 5. Since this is a marginally unstable root we can evaluate Eq. (3.1) by the usual residue method and obtain

$$\frac{k^2}{\omega_p^2} = P \int_{-\infty}^{+\infty} \frac{F_0'(u)}{u - t_R}\, du + i\pi F_0'(t_R). \tag{3.4}$$

In order that this equation be satisfied it is necessary that the imaginary part vanish or

$$F_0'(t_R) = 0 \tag{3.5}$$

and furthermore since the imaginary part of Z goes from negative to positive around this point, we also require

$$F_0''(t_R) > 0. \tag{3.6}$$

Hence a necessary condition for the existence of an unstable root is the presence of a minimum point in the distribution function.

The complete condition also requires that the real part of Eq. (3.4) be satisfied. Since $F_0'(u)$ vanishes at t_R, we can drop the principal part requirement, and integrating by parts, we can write

$$\int_{-\infty}^{+\infty} \frac{F_0(u) - F_0(t_R)}{(u - t_R)^2}\, du > 0. \tag{3.7}$$

Eqs. (3.5), (3.6) and (3.7) constitute a necessary and sufficient criterion for the existence of an unstable root and are known as the Penrose criterion.

(b) Two-stream instability

We will illustrate a Vlasov instability by the two-stream instability. However, we shall only deal with one special case and this only approximately. Consider two equal streams of electrons having the same density and temperature but displaced in average velocity by V. The distribution function is

$$F_0(u) \sim \left[\exp -\left(\frac{u}{\alpha}\right)^2 + \exp -\left(\frac{u - V}{\alpha}\right)^2\right].$$

By symmetry we see that this function is flat at $u = V/2$. Under what conditions is this a minimum in the distribution function?

The second derivative is

$$F_0''(u) \sim \left[\frac{4u^2}{\alpha^2} \exp - \frac{u^2}{\alpha^2} + \frac{4\,(u - V)^2}{\alpha^2} \exp - \frac{(u - V)^2}{\alpha^2}\right.$$

$$\left. - 2 \exp - \frac{\omega^2}{\alpha^2} - 2 \exp - \frac{(u - V)^2}{\alpha^2}\right]$$

and at $u = V/2$ this is

$$F_0''(u) \sim \left(\frac{V^2}{\alpha^2} - 2\right),$$

having omitted all positive coefficients. Hence we obtain a minimum in the distribution function only if

$$V > \sqrt{2}\,\alpha$$

or, in terms of the thermal velocity in this direction

$$V > \frac{2}{\sqrt{3}}\,\overline{v}_{TH}.$$

This is, of course, only a necessary condition. We must also satisfy Eq. (3.7) and more careful work shows that the two-stream instability actually sets in at

$$V > 2\,\overline{v}_{TH}. \tag{3.8}$$

This is a rather rigid requirement, but under certain circumstances the condition for instability is considerably relaxed. Consider the extreme case of zero temperature ions and electrons moving with an average velocity V. The ion distribution function is

$$F_{0i} = \delta(u) \tag{3.9}$$

and that of the electrons

$$F_{0e} = \frac{1}{\sqrt{\pi}\,\alpha_e} \exp - \left(\frac{u - V}{\alpha_e}\right)^2 \tag{3.10}$$

The dispersion equation becomes:

$$1 = \frac{\omega_{pi}^2}{k^2}\frac{(ik)^2}{s^2} + \frac{\omega_{pe}^2}{k^2}\int_c \frac{F_{0e}'(u)\,du}{u + s/ik}\,.$$

Assuming a marginally unstable root at $s = \epsilon - ibk$ we have

$$\frac{k^2}{\omega_{pe}^2} = \frac{m}{M}\frac{1}{b^2} + P\int_{-\infty}^{+\infty}\frac{F_{0e}'(u)}{u-b}\,du + i\pi F_{0e}'(b), \tag{3.11}$$

Comparing with Eq. (3.10), we see that we must choose $b = V$ in order that the imaginary part vanish. The real part becomes:

$$\frac{k^2}{\omega_{pe}^2} = \frac{m}{M}\frac{1}{V^2} + \int_{-\infty}^{+\infty}\frac{F_{0e}'(u)}{u-V}\,du \tag{3.12}$$

$$= \frac{m}{MV^2} - \frac{2}{\alpha_e^2}$$

or, in terms of the average velocity in the k-direction

$$V^2 = \frac{m/M}{k^2/\omega_{pe}^2 + 1/\overline{V}_e^2}\,. \tag{3.13}$$

Equation (3.13) gives the boundary between stable and unstable solutions for a given k. Obviously since there will be only a single humped distribution if $V = 0$, the instability region lies at the upper values of V.

For $k = 0$, the instability velocity is

$$V > \sqrt{\frac{m}{M}}\,\overline{v}_e = \sqrt{\frac{T_e}{M}} \tag{3.14}$$

while for larger values of k it goes to $V^2 = \omega_{pi}^2/k^2$ which goes to zero with k. Note that the electron mass drops out in this limit (hence the name ion wave instability). Thus a system with zero-temperature ions is actually unstable for any electron streaming velocity. For finite ion temperatures it goes over smoothly to the result in Eq. (3.8) when the temperatures of ions and electrons are comparable. When $T_e/T_i > 10$, we have instability conditions closer to that of Eq. (3.14).

(c) Inclusion of a magnetic field

In this section we shall make an extension of the previous results to the case of a plasma immersed in a uniform magnetic field. This requires a formal proof which we give in a very brief manner. Details can be found in THOMPSON's book [7] and an extension to the finite Larmor radius case in a paper by ROSENBLUTH and SIMON [8]. We shall assume that the uniform field is in the z-direction and that our plasma is low β which implies $B_1 \equiv 0$. Also we restrict ourselves to purely electrostatic waves, $\vec{\nabla} \times \vec{E} = 0$. Consider the Vlasov equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \left(\frac{e\vec{E}}{m} + \frac{e}{mc} \vec{v} \times \vec{B} \right) \cdot \vec{\nabla}_v f = 0. \tag{3.15}$$

The first term is of order ω, which in the hydrodynamic limit is usually

$$\omega \cong 0 \left(\frac{a}{L} \Omega_i \right) \equiv 0 \ (\epsilon \Omega_i),$$

where a is the ion Larmor radius, L a characteristic length in the gas (density variation distance, k^{-1} etc.) and Ω_i is the ion cyclotron frequency. ϵ is generally a small quantity. The second term is of order

$$\frac{v_{TH}}{L} = \frac{a}{L} \Omega_i = \epsilon \, \Omega_i .$$

The third term is of order

$$\frac{eE}{mv_{TH}} = \frac{1}{v_{TH}} \frac{cE}{B} \Omega_i$$

and since the drift velocity is of order v_{TH} this term is of order Ω_i. The last term is obviously of order Ω_i.

If we now transform to the drift frame, let

$$\vec{V} = c \frac{\vec{E} \times \vec{B}}{B^2} \tag{3.16}$$

and $\vec{v}' = \vec{v} - \vec{V}$. In this frame our equation becomes

$$\frac{\partial f}{\partial t} + (\vec{v}' + \vec{V}) . \vec{\nabla} f + \frac{eE_z}{m} . \frac{\partial f}{\partial v_z'} + \frac{e}{mc} \vec{v}' \times \vec{B} . \vec{\nabla}_{v'} f$$

$$- \left[\frac{\partial \vec{V}}{\partial t} + (\vec{v}' + \vec{V}) \cdot \vec{\nabla}(\vec{V}) \right] \cdot \vec{\nabla}_{v'} f = 0. \tag{3.17}$$

If we now restrict $c\, E_z / B$ to be of order ϵv_{TH}, then it is easy to show that all terms are of order $\epsilon \Omega_i$ or smaller except for the term

$$\frac{e}{mc} \vec{v}' \times \vec{B} \cdot \vec{\nabla}_{v'} f$$

which is still of order Ω_i.

This now permits us to iterate our solution in powers of ϵ. To lowest order we have

$$\vec{v}' \times \vec{\Omega} \cdot \vec{\nabla}_{v'} f_0 = 0 .$$

Changing to cylindrical co-ordinates in velocity space this is

$$\frac{\partial f_0}{\partial \phi} = 0, \tag{3.18}$$

where ϕ is the azimuthal angle. The general solution of Eq. (3.18) is

$$f_0 = f_0(v_\perp^2, v_z, \vec{x}, t). \tag{3.19}$$

The next order equation is obtained by replacing f by f_0 in all terms in Eq. (3.17) except for the large term which takes the form $\partial f_1/\partial\phi$. Now however, we average this equation over the phase angle ϕ. This last term vanishes. Remembering that $v_z \equiv 0$ and that $\vec{\nabla} \cdot \vec{V} = 0$ for an electrostatic wave we find that our equation reduces to:

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \vec{V} \cdot \vec{\nabla} f + \frac{eE_z}{m} \frac{\partial f}{\partial v_z} = 0 . \tag{3.20}$$

Now we linearize about an unperturbed state which is uniform in space and with $\vec{E}_0 = 0$. Taking transforms of Eq. (3.20) we find at once that

$$f_1 = \frac{(ie/m)\, E_z\, \partial f/\partial v_z}{-\omega + k_z v_z},$$

where the transform was of the form $\exp(-i\omega t + i\vec{k} \cdot \vec{x})$. Substituting this in the Poisson equation, we at once obtain

$$k^2 = \omega_p^2 \int_c \frac{dF_0/dv_z}{v_z - \omega/k_z}\, dv_z \tag{3.21}$$

which is seen to have exactly the same form as the dispersion relation [Eq. (2.33)] for the zero magnetic field case except that k_z appears instead of k on the right-hand side.

All of our previous considerations in this chapter (Penrose criterion, Two-stream instability) go through as before. The important point is that we can have an instability in which $k_\perp \neq 0$. Hence there can exist an unstable wave with E perpendicular to B which may cause large particle drifts and anomalous diffusion. There has been some attempt to use the two-stream instability for $T_i \ll T_e$ (Ion wave instability) to explain pumpout in the stellarator.

(d) Drift-wave (universal) instability

Again we just briefly touch on a topic which will be covered in detail by other speakers. Return to Eq. (3.20) and linearize about an unperturbed state which has $\vec{E}_0 = 0$ but now has a density varying in the x-direction. Our equation becomes

$$\frac{\partial f_1}{\partial t} + v_z \frac{\partial f_1}{\partial z} + V_{1x} \frac{\partial f_0}{\partial x} + e \frac{E_{z1}}{m} \frac{\partial f_0}{\partial v_z} = 0 .$$

Now

$$V_{1x} = c \frac{E_{1y}}{B} = -\frac{ic}{B} k_y \phi_1 = \frac{c k_y}{B k_z} E_{1z} .$$

Consequently

$$f_1 = i \frac{\left(\frac{c}{B} \frac{k_y}{k_z} \frac{\partial f_0}{\partial x} + \frac{e}{m} \frac{\partial f_0}{\partial v_z}\right) E_{1z}}{-\omega + k_z v_z} .$$

Hence the perturbed current in the z-direction is

$$j_z = \frac{\frac{ie^2}{m} v_z \left(\frac{k_y}{k_z \Omega} \frac{\partial f_0}{\partial x} + \frac{\partial f_0}{\partial v_z}\right) E_{1z}}{-\omega + k_z v_z}$$

and the work done on the electrons by the electrostatic field is

$$W \sim j_z E_z = \frac{ie^2}{m} E_{1z}^2 \int_c \frac{v_z \left(\frac{\partial F_0}{\partial v_z} + \frac{k_y}{k_z \Omega} \frac{\partial F_0}{\partial x}\right) dv_z}{-\omega + k_z v_z} .$$

The only real part of this comes from the residue, hence

$$W_R = -\frac{\pi e^2}{m|k_z|} E_{1z}^2 \left[v_z \left(\frac{\partial F_0}{\partial v_z} + \frac{k_y}{k_z \Omega} \frac{\partial F_0}{\partial x}\right)\right]_{v_z = \omega/k_z} .$$

If we now approximate F_0 as

$$F_0 = n_0(x) \exp -\left(\frac{v}{\alpha}\right)^2$$

this can be written as

$$W_R \sim \left[2\left(\frac{v_z}{\alpha}\right)^2 \left(F_0 - \frac{n'_0}{n_0} F_0 \frac{k_y \alpha^2}{k_z \Omega 2 v_z}\right)\right]_{v_z = \omega/k_z} ,$$

where n'_0 is the x-derivative of n_0. Hence

$$W_R \sim \left(\frac{\omega}{k_z}\right)^2 \left[\exp - \left(\frac{\omega}{k_z \alpha}\right)^2\right] \left(1 - \frac{KTk_y}{M\Omega\omega} \frac{n_0'}{n_0}\right). \tag{3.22}$$

Actually there are two such expressions as Eq. (3.22), one for each of the species (ions and electrons) and the total work done is the sum of the two. Assuming equal densities in the unperturbed state we have

$$W_R \sim \left(\frac{\omega}{k_z}\right)^2 \left\{ \left[\exp - \left(\frac{\omega}{k_z \alpha_e}\right)^2\right] \left(1 + \frac{cKTk_y}{|e|B\omega} \frac{n_0'}{n_0}\right) \right.$$

$$\left. + \left[\exp - \left(\frac{\omega}{k_z \alpha_i}\right)^2\right] \left(1 - \frac{cKTk_y}{|e|B\omega} \frac{n_0'}{n_0}\right)\right\}. \tag{3.23}$$

The first terms in each of the brackets represents the usual Landau damping. The second term can sometimes dominate, leading to an instability which is caused by a density gradient.

Before ending we note some qualitative features of this instability. If

$$\frac{\omega}{k_z} \gg \alpha_e,$$

i.e.: if the phase velocity is large compared to the electron thermal velocity, then instability, even if it occurs, is very small due to the exponential factor. On the other hand, if

$$\frac{\omega}{k_z} \ll \alpha_i,$$

the exponential factors vanish and the destabilizing terms cancel. Hence this instability is most dangerous for

$$\alpha_e > \omega / k_z > \alpha_i .$$

We obviously would also like a large value of k_y/k_z, but not ∞, since $k_z = 0$ would be damped by the exponential factor.

Further details of this instability requires evaluation of the real part of the dispersion equation and will be treated in other papers.

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BINARY PROCESSES IN PLASMA

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For many aspects of plasma physics it is sufficient to consider only the interaction between particles and electromagnetic fields; but certain fundamental processes do require a knowledge of the effects of particle collisions. In particular, to establish a plasma it is usually necessary to ionize a neutral gas, which is most easily done by bombarding it with electrons. Once a plasma has been formed collisions between electrons and ions lead to an equipartition of energy, while collisions with ions, especially with partially-stripped ions, lead to energy radiation. Finally, when a plasma decays, it usually does so by the recombination of ions and electrons.

I. INELASTIC COLLISIONS BETWEEN ELECTRONS AND ATOMS

(a) Ionization

The cross-section for ionization is zero until the electron energy reaches the ionization energy $\mathscr{E}_i$. It then increases rapidly reaching a value of the order of 10^{-16} cm^2 at 4-5$\times\mathscr{E}_i$, then falls slowly with increasing energy. Very roughly

$$\sigma_i(\mathscr{E}) \simeq 4\pi a_0^2 \frac{(\mathscr{E}/\mathscr{E}_i - 1)}{(\mathscr{E}/\mathscr{E}_i)^2}, \tag{1}$$

where $\pi a_0^2 \simeq 10^{-16}$ cm^2, $\mathscr{E}$ is the energy of the incident electron. This form of the cross-section has two consequences. If an electron field is used to accelerate electrons to the ionization energy, then the field must be strong enough to supply this energy in an electron mean free path, i. e. $eE\lambda > \mathscr{E}_i = eV_i$, where V_i is the ionization potential. Since $\lambda = 1/(n\sigma_{el})$ is proportional to inverse of gas pressure, E/p must exceed some critical value depending on the ionization potential, before the gas starts to ionize. If the electric field is increased above this level, the ionization rate at first increases, but because of the decrease in cross-section at high energies (> 80 eV) begins to drop again, and the maximum possible ionization rate depends only on the number density of the gas. For hydrogen

$$\left.\frac{dn}{dt}\right|_{max.} \simeq 2\times10^{-8}\, n^2 s^{-1}\, cm^{-3}. \tag{2}$$

and

$$\frac{1}{n}\frac{dn}{dt} \sim n_g \langle \sigma V_{max} \rangle \sim 10^{-8}\, n_g, \tag{3}$$

where n_g is the number density of gas atoms.

(b) Excitation

If an electron does not strip an electron from an atom, it may knock it into a higher energy state, thus exciting the atom. The cross-section for the process is often larger than the maximum ionization cross-section, and depends on the energy of the state excited, (measured by the wave-length λ in microns of the emitted radiation for the inverse process), and on the oscillator strength f for the transition. Roughly

$$\sigma = 1740\, \pi a_0^2 \lambda^2 \left(\frac{\mathscr{E}^*}{\mathscr{E}}\right) f b \tag{4}$$

b is a numerical factor increasing from 0 at $\mathscr{E} = \mathscr{E}^*$ to a maximum of ~ 1.3 at $\mathscr{E} \simeq 5\,\mathscr{E}^*$, and $\mathscr{E}^*$ is the excitation energy. This cross-section in plasmas containing incompletely stripped atomic systems, can be responsible for most of the energy radiated.

It is sometimes important to observe that an excited atom, instead of emitting its energy as a photon, can give it up to an electron on collision, thus increasing the electron kinetic energy. In cool plasmas such collisions of the second kind may be important in raising the electric temperature.

(c) Recombination

The simplest recombination process, radiative recombination in which an electron is captured by an ion, emitting its energy as a photon, has a very small cross-section. Indeed for recombination into any particular state of the atom $\sigma \sim 10^{-20}$ cm^2 at low energy ($\ll 1$ eV) and decreases rapidly as the electron energy is increased. The sum over all possible states still has a low value of the order of 10^{-19} cm^2 and only in pure systems at low pressures is radiative recombination significant.

If molecular ions, or partially stripped atoms, are present in the system, an electron can recombine and release its energy not by producing a photon but by exciting an internal electron, or by dissociating the molecule. Dissociative recombination, in particular, has a high cross-section $\sim 10^{-16}$ cm^2 and is important in cool plasmas.

At high density the most probable type of recombination involves an interaction between three particles and is inverse to the ionization by electrons. It proceeds, therefore, at a rate proportional to the cube of the electron density and, if characterized by a cross-section, must have a cross-section proportional to the density. For low-energy electrons

$$\sigma \simeq (10^{-33}\, n)\mathrm{cm}^2, \tag{5}$$

and for densities greater than 10^{14} cm^{-3}, three-body recombination is more important than radiative recombination.

(d) Bremsstrahlung

When an electron collides with an atom it is accelerated and emits radiation, bremsstrahlung, which is an important way in which a plasma can lose energy. The rate of energy loss can be roughly estimated by the following argument. The dipole radiation field produced by a particle is proportional to its acceleration; hence, that produced by an electron at point $\vec{x}_i$

$$\sim \frac{e_i}{m_i}\vec{E}(\vec{x}_i) = \frac{e_i}{m_i}\sum_j \frac{e_j(\vec{x}_i - \vec{x}_j)}{|\vec{x}_i - \vec{x}_j|^3} \tag{6}$$

and the total field

$$\sim \sum_{i,j} \frac{e_i e_j}{m_i} \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|^3}. \tag{7}$$

From this sum, contributions from like particles disappear for, if $\vec{x}_2 - \vec{x}_1$ appears in the sum, so does $\vec{x}_1 - \vec{x}_2$; the sum need be taken only over ions. The radiated power is proportional to the square of the field strength, hence to the square of the acceleration, i. e.

$$\sum_{\bar{i}\bar{k}}\sum_{\overset{+}{j}l}\left(\frac{e_-}{m_-}\right)^2 e_+^2 \frac{(\vec{x}_i - \vec{x}_k)(\vec{x}_j - \vec{x}_l)}{|\vec{x}_i - \vec{x}_k|^3\,|\vec{x}_j - \vec{x}_l|^3}, \tag{8}$$

and if particle positions are uncorrelated, and the only terms surviving in the sum are those for which $i = k$, $j = l$ and if $r_{ij} = |\vec{x}_i - \vec{x}_j|$, the sum becomes

$$a^2 = \left(\frac{e_-}{m_-}\right)^2 e_+^2 \sum_{i,j} \frac{1}{r_{ij}^4}. \tag{9}$$

This may be written in terms of the electron number N and the ion number density n as

$$a^2 = N\left(\frac{e_-}{m_-}\right)^2 e_+^2 \int \frac{n(r)}{r^4}\,d\tau = 4\pi N \frac{e_-^2}{m_-^2} e_+^2 n \frac{1}{r_{min}}, \tag{10}$$

$1/r_{min}$ here may in turn be identified with the de Broglie wave number $k = mv_e/\hbar$ of the electron, and the power loss per unit volume of plasma written

$$P = \sum \frac{2}{3}\frac{e^2}{c^3} a^2 = \frac{8\pi}{3} Nn \frac{e_-^4}{m_-^2 c^3} e_+^2 \frac{mv_e}{\hbar} \tag{11}$$

$$= \frac{8\pi}{3}\left(\frac{e^2}{m_- c^2}\right)^2 \left(\frac{e^2}{\hbar c}\right) m_- c^2 n^2 c \frac{v_e}{c}, \tag{12}$$

where we have used the fact that $e_+ = e_-$ and $n_+ = N$.

An integration of the low-energy cross-section given by Heitler over a Maxwellian distribution leads to a slightly different numerical factor. The power radiated, with T measured in keV is

$$P \simeq 4\times10^{-31}\, n^2 T^{\frac{1}{2}}\ \mathrm{W/cm^3\,s}. \tag{13}$$

II. INELASTIC COLLISIONS BETWEEN IONS AND ATOMS

(a) Ionization

To a large extent the stripping of an electron from an atom is produced by the electric field at the atom, and since an ion travelling with the same velocity will produce roughly the same field at an atom as would an electron, it is not surprising that the cross-section for ionization by a fast ion is, when large, close to that for ionization by an electron of the same speed, and indeed the ionization cross-section for atoms by ions reaches a value of the order of 10^{-16} cm^2 for ions of energy of the order of 20 keV.

(b) Charge exchange

A second phenomenon when an ion strikes an atom is the transfer of a bound electron from the atom to the ion. This too has a huge cross-section, $\sim 10^{-16}$ cm^2, when the ion energy is ~ 100 keV. The effect of charge exchange when a fast ion strikes a slow atom is that the fast ion becomes neutralized, and the slow atom ionized, and in magnetically-trapped hot ion plasmas the presence of a small amount of neutral gas can lead to catastrophic energy losses, the energy being carried away by fast neutrals produced by charge exchange.

The charge exchange process can also be used for filling magnetic traps with energetic ions, for a beam of fast ions can be passed through a region in which a high pressure of gas is maintained, whereupon many ions will charge exchange with gas atoms, and a beam of fast neutrals will be produced. The neutrals can then pass into a region of high magnetic field where they may be ionized.

A useful method of ionizing such a beam employs the Lorentz force $e\vec{v}\times\vec{B}/c$. For energetic particles moving across a strong magnetic field this can be equivalent to a very strong electric field $\sim 0.3\sqrt{\mathscr{E}}\,B$ volts/cm, where $\mathscr{E}$ is the atom's energy in keV, and B the field in Gauss; $\simeq 0.16\times10^6$ volts per centimetre for 40 keV ions in an 80-kG field; this can give rise to a significant potential drop across an atom. If the atom is in a highly excited state with quantum number $n \gtrsim 7$, the disturbing field is strong enough to remove the electron, and ionize the atom. Since the charge exchange process has a reasonably high chance of producing a highly excited atom, the Lorentz trapping process is quite efficient.

III. ELASTIC COLLISION BETWEEN CHARGED PARTICLES

(a) *Scattering cross-section*

If two charged particles of masses m_1, m_2, and charges e_1, e_2, approach one another along a pair of straight lines with minimum separation b, then as they approach, the Coulomb interaction between them will cause their orbits to be deflected. In the centre of mass system, the angle of deflection θ satisfies

$$\sin\frac{\theta}{2} = \left[1 + m^2 g^4 b^2/(e_1 e_2)^2\right]^{-1/2}, \tag{14}$$

where m is the reduced mass $m_1 m_2/(m_1 + m_2)$, and $\vec{g}$ the relative velocity of approach, $g = |\vec{v}_1 - \vec{v}_2|$.

The rate at which particles pass one another with an impact parameter between b and b+db is $2\pi g b db$ and in any azimuthal range $d\varphi$ is $g b db d\varphi$, hence the differential scattering cross-section σ_c in the centre of mass system defined in terms of the frequency $\upsilon(\theta)$ of collision resulting in a deflection θ is

$$\upsilon(\theta) d\Omega = n^2 g \sigma_c(\theta) d\Omega \tag{15}$$

becomes

$$\sigma_c(\theta) d\Omega = \sigma_c(\theta) \sin\theta d\theta d\varphi = b(\theta) db d\varphi; \tag{16}$$

hence

$$\sigma_c(\theta) = \frac{b(\theta)}{\sin\theta} \frac{db}{d\theta}, \tag{17}$$

but

$$b^2 = \frac{e_1^2 e_2^2}{m^2 g^4}\left(1/\sin^2\frac{\theta}{2} - 1\right) \tag{18}$$

and

$$b = \frac{|e_1 e_2|}{mg^2} \cot\frac{\theta}{2}. \tag{19}$$

$$\sigma_c = \left(\frac{e_1 e_2}{mg^2}\right)^2 \frac{1}{4} \frac{\cot\frac{\theta}{2}}{\sin^3\frac{\theta}{2}\cos\frac{\theta}{2}} = \left(\frac{e_1 e_2}{2mg^2}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}, \tag{20}$$

which is Rutherford's cross-section.

(b) Kinematics of a collision

On a collision the incident and final energies and momenta must be unchanged, hence the centre of mass velocity

$$\vec{V} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \tag{21}$$

must be constant, and the entire change occurs in the relative velocity, so that on collision

$$\vec{v}_1 - \vec{v}_2 = \vec{g} \rightarrow \vec{g}' = \vec{v}_1' - \vec{v}_2', \tag{22}$$

$\vec{v}_1'$ and $\vec{v}_2'$ being the final velocities of particles 1 and 2. $\vec{v}_1$ and $\vec{v}_2$ can be expressed in terms of $\vec{V}$ and $\vec{g}$ as

$$\vec{v}_1 = \vec{V} + \frac{m_2}{m_1 + m_2}\vec{g}, \quad \vec{v}_2 = \vec{V} - \frac{m_1}{m_1 + m_2}\vec{g}, \tag{23}$$

while $\vec{v}_1'$ and $\vec{v}_2'$ are the same functions of $\vec{V}$ and $\vec{g}'$. The change in momentum and energy of each particle are thus

$$\Delta\vec{p}_1 = m_1\vec{v}_1' - m_1\vec{v}_1 = \frac{m_1 m_2}{m_1 + m_2}(\vec{g}' - \vec{g}) = -\overrightarrow{\Delta p_2} = m_r\overrightarrow{\Delta g}, \tag{24}$$

$$\Delta\mathscr{E}_1 = \frac{m_1 m_2}{(m_1 + m_2)}\vec{V}\cdot(\vec{g}' - \vec{g}) + \frac{m_1 m_2^2}{2(m_1 + m_2)^2}(g'^2 - g^2), \tag{25}$$

and

$$\Delta\mathscr{E}_2 = -\frac{m_1 m_2}{(m_1 + m_2)}\vec{V}\cdot(\vec{g}' - \vec{g}) + \frac{m_1^2 m_2}{2(m_1 + m_2)^2}(g'^2 - g^2). \tag{26}$$

Since $\Delta\mathscr{E}_1 + \Delta\mathscr{E}_2 = 0$, $g'^2 = g^2$ and the length of $\vec{g}$ does not change, hence

$$\Delta\mathscr{E}_1 = -\Delta\mathscr{E}_2 = m_r\vec{V}\cdot\overrightarrow{\Delta g}, \tag{27}$$

$m_r = m_1 m_2/(m_1 + m_2)$, the reduced mass.

Since $\vec{g}$ changes only in direction (Fig. 1), not in magnitude, $\vec{g}'$ expressed as its projection on an orthogonal triplet of unit vectors, one along the original direction $\hat{g}$, $\hat{g}$, $\hat{m}, \hat{n}$ is

$$\vec{g}' = g(\hat{g}\cos\theta + \hat{m}\sin\theta\cos\varphi + \hat{n}\sin\theta\sin\varphi) \tag{28}$$

$$\begin{aligned}\overrightarrow{\Delta g} = \vec{g}' - \vec{g} &= g\Big(\hat{g}(\cos\theta - 1) + \hat{m}\sin\theta\cos\varphi + \hat{n}\sin\theta\sin\varphi\Big)\\ &= 2g\sin\frac{\theta}{2}\Big(-\hat{g}\sin\frac{\theta}{2} + \hat{m}\sin\frac{\theta}{2}\cos\varphi + \hat{n}\sin\frac{\theta}{2}\sin\varphi\Big)\end{aligned} \tag{29}$$

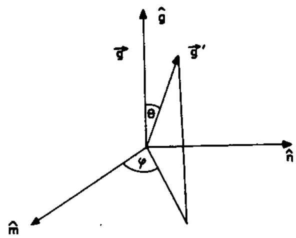

Fig. 1
Direction of $\vec{g}'$ versus $\vec{g}$

(c) Energy and momentum transfer on collision

If we ask for the transfer of energy and momentum on a collision between two particles at fixed relative velocity g, scattering angle θ, we must average over φ

$$\langle\Delta\vec{g}\rangle = -2g\sin^2\frac{\theta}{2}\hat{g} = -2\sin^2\frac{\theta}{2}\vec{g} \tag{30}$$

hence

$$\langle\Delta\vec{p}_1\rangle = -2m_r\sin^2\frac{\theta}{2}\vec{g} = -2m_r\sin^2\frac{\theta}{2}(\vec{v}_1-\vec{v}_2) \tag{31}$$

and

$$\langle\Delta\mathscr{E}_1\rangle = m_r\vec{V}\cdot\overrightarrow{\Delta g} = -2m_r\sin^2\frac{\theta}{2}\vec{V}\cdot\vec{g} \tag{32}$$

$$\begin{aligned} &= -4\sin^2\frac{\theta}{2}\frac{m_1m_2}{(m_1+m_2)^2}\left[\frac{1}{2}m_1v_1^2+(m_2-m_1)\vec{v}_1\cdot\vec{v}_2-\frac{1}{2}m_2v_2^2\right] \\ &= -4\sin^2\frac{\theta}{2}\frac{m_1m_2}{(m_1+m_2)^2}\left[\mathscr{E}_1-\mathscr{E}_2+(m_2-m_1)\vec{v}_1\cdot\vec{v}_2\right]. \end{aligned} \tag{33}$$

Now, we may average these quantities over all scattering angles, holding g fixed. Then

$$\begin{aligned} \frac{d\vec{p}}{dt} &= n\int\sigma_c(g)\,g\,\overrightarrow{\Delta p}(g,\theta)\sin\theta\,d\theta\,d\varphi \\ &= -4\pi n\left(\frac{e_1e_2}{2m_r g^2}\right)^2 m_r g\vec{g}\int_{\theta_0}^{\pi}\frac{\sin^2\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}\sin\theta\,d\theta \\ &= -4\pi n\left(\frac{e_1e_2}{2m_r g^2}\right)^2 m_r g\vec{g}\,4\log\sin\frac{\theta}{2}\Big|_{\theta_0}^{\pi/2} \\ &= -4\pi n\frac{(e_1e_2)^2}{m_r}\frac{\vec{g}}{g^3}\left(-\log\sin\frac{\theta_0}{2}\right). \end{aligned} \tag{34}$$

If θ_0 is allowed to vanish here, the momentum loss rate will appear infinite. Note, however, that

$$-\log\sin\frac{\theta_0}{2} = \frac{1}{2}\log\left[1 + m^2g^4b^2/(e_1e_2)^2\right]; \tag{35}$$

hence, the infinity arises from infinite values of the impact parameter b. In any physical system of interacting particles the maximum possible b will be limited in some way. In most plasmas this occurs at $b = \lambda\!\!\!^{-}_0 = v/\omega_p$ and approximating $m_r\,g^2 \simeq$ by kT yields

$$\frac{m^2g^4b^2}{e_1^2e_2^2} \simeq (kT)^2\frac{mv_e^2}{mne^2/m}\frac{1}{e^4} \simeq \frac{(kT)^3}{ne^6}. \tag{36}$$

$$-\log\sin\theta_0 \simeq \log\left[(kT)^{3/2}\,n^{-1/2}\;e^{-3}\right] = \log\Lambda \tag{37}$$

$$\log\Lambda \simeq 22.3 - 1.15\log_{10} n + 3.5\log_{10} W, \tag{38}$$

n being in cm^{-3} and W in eV. $\Lambda^{2/3} = kT/n^{1/3}e^2 \simeq$ ratio of mean kinetic energy/ mean potential between nearest neighbours.

For some purposes it is necessary to observe the relative importance of contributions from small and large b to the scattering process. The integral over θ which reduces to

$$\int_0^{\pi/2}\frac{\cos\frac{\theta}{2}\,d\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \int_0^{\pi/2}\frac{d(\sin\frac{\theta}{2})}{\sin\frac{\theta}{2}} = \frac{m^2g^4}{e^4}\int_0^{b_{max}}\frac{b\,db}{(1 + m^2g^4b^2/e^4)}, \tag{39}$$

and if one is interested in the contributions from various ranges of b, $\int_0^{b_0}$, $\int_{b_0}^{b_{max}}$, these become

$$\sim \log(1 + m^2g^4b_0^2/e^4),\ \log\left(\frac{1 + m^2g^4b_{max}^2/e^4}{1 + m^2g^4b_0^2/e^4}\right) \tag{40}$$

and the ratio

$$[\log(1 + m^2g^4b_{max}^2/e^4)/\log(1 + m^2g^4b_0^2/e^4) - 1] > 1, \tag{41}$$

hence, contributions to scattering for large b always exceed those due to small b, and for a Coulomb interaction, a particle changes its direction principally because of the cumulative effect of many small angle scatterings. To a good approximation, for large b_{max},

$$\log(1 + m^2 g^4 b_{max}^2 / e^4) \simeq 2 \log(m g^2 b_{max} / e^2). \tag{42}$$

Now the scattering angle θ, may be approximated, for small values, by using the impulse approximation, in which a linear approximation to the effect of the fields is used. The scattering angle then becomes

$$\sin\theta = \frac{\Delta g_\perp}{g} = \frac{1}{g}\int F_\perp \, dt = \frac{1}{g^2}\frac{e}{m_r}\int \vec{E}_\perp \, \overrightarrow{dx}, \tag{43}$$

the integral being carried along the unperturbed trajectory, a straight line passing the target at a distance b. Since

$$E_\perp = eb/\gamma^3 = eb/(b^2 + x^2)^{3/2},$$

$$\sin\theta = \frac{1}{g^2}\frac{e^2}{m_r}\int_{-\infty}^{\infty}\frac{b\,dx}{(b^2+x^2)^{3/2}} = \frac{1}{b}\frac{e^2}{m_r g^2}\int_{-\infty}^{\infty}\frac{dt}{(1+b^2)^{3/2}} = \frac{2}{b}\frac{e^2}{m_r g^2} \tag{44}$$

and $\sin\theta/2 \simeq \theta/2 \simeq e^2/(b m_\gamma g^2)$. If we were to use this value in the integral we would need the difference

$$\log(b_{max} m_r g^2/e^2) - \log(b_{min} m_r g^2/e^2) \tag{45}$$

which would agree with the exact result (for $b_{max} \gg m_r g^2/e^2$) if b_{min} were selected as $e^2/m_r g^2$.

Hence we have shown that if the maximum impact parameter $b_{max} \gg m_r g^2/e^2$, then the actual scattering is well approximated by a calculation linearized in the interaction, provided that the latter is cut off at an impact parameter $b_{min} = e^2/m_r g^2$.

(d) The Boltzman collision integral

If the particle distribution function $f(\vec{v})$ is given, it is sometimes important to ask how rapidly this is changed by collisions. It is clear that the rate at which particles are knocked out of a range $d^3\vec{v}$ about $\vec{v}$ is equal to the rate at which particles in that range make collisions, i. e. the sum of all collisions, at any scattering angle with particles having any velocity, i. e.

$$\int d\Omega \int d^3\vec{g}\, f(\vec{v}+\vec{g})\, g\,\sigma(g,\theta)\, f(\vec{v}). \tag{46}$$

After such a collision, the particles will have velocities

$$\vec{v} \to \vec{v} - \overrightarrow{\Delta g}; \quad \vec{v} + \vec{g} \to \vec{v} + \vec{g} + \overrightarrow{\Delta g}, \tag{47}$$

hence, for a given value of scattering angle, and relative velocity, there exists a pair of velocities $\vec{v}, \vec{v}'$ such that a collision between these will leave one of the particles in the velocity range $d^3\vec{v}$ about $\vec{v}$. These are just the final velocities, with the relative velocity changed in sign, hence, the net rate of increase of particles is the velocity range $d^3\vec{v}$ about $\vec{v}$ is

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \int d\Omega \int d^3\vec{v}\, g\, \sigma(g, \theta)[\, f(\vec{\underline{v}})\, f(\vec{\underline{v}}') - f(\vec{v})\, f(\vec{v}')]. \tag{48}$$

This is Boltzmann's collision integral.

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THE TRANSPORT EQUATION FOR A PLASMA

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I. TRANSPORT EQUATIONS, TEST PARTICLES AND PARTICLE CORRELATION IN PLASMA

If the distribution function $f(\vec{x}, \vec{v}, t)$, for each species in a plasma is known, the macroscopic properties of the plasma may be easily calculated. Since f is so defined that $f(\vec{x}, \vec{v}, t)d^3\vec{v}\, d^3\vec{x}$ represents the number of particles to be found at time t in the volume element $d^3\vec{x}$ about $\vec{x}$ having velocities in the range $d^3\vec{v}$ about $\vec{v}$, the macroscopic quantities, which are just mean values, are represented by moments over f, e.g. the mass density

$$\rho = \sum_{\text{species}} \int m_i f_i \, d^3\vec{v}, \tag{1}$$

and the charge density

$$q = \sum \int e_i f_i \, d^3\vec{v}. \tag{2}$$

The mass velocity $\vec{V}$ is given by

$$\rho\vec{V} = \sum \int m_i \vec{v} f_i \, d^3\vec{v}, \tag{3}$$

and the current [1] by

$$\vec{j} = \sum \int e_i \vec{v} f_i \, d^3\vec{v}. \tag{4}$$

A question of fundamental importance then is that of how the distribution function is determined. It may be deduced from an exact description of the plasma, given by an N body distribution function, the Liouville function $F(\vec{x}_1, \vec{x}_2, \ldots \vec{x}_N, \vec{v}_1, \vec{v}_2, \ldots \vec{v}_N, t)$ which gives the probability of finding the whole plasma in a given configuration, i.e. with the first particle at $\vec{x}_1$ with velocity $\vec{v}_1$, the second at $\vec{x}_2$ with velocity $\vec{v}_2$ etc. The function F develops because of the motion of the individual particles and if the plasma is at some time in a definite configuration $\vec{x}_1(0) \ldots \vec{x}_N(0)$, $\vec{v}_1(0) \ldots \vec{v}_N(0)$ so that

$$F(\vec{x}_1 \ldots \vec{x}_N, \vec{v}_1 \ldots \vec{v}_{N,0}) = \prod_i \delta[\vec{x}_i - \vec{x}_i(0)]\, \delta[\vec{v}_i - \vec{v}_i(0)] \tag{5}$$

then at a later time t the i-th particle will have moved to $\vec{X}_i[t, \vec{x}_i(0), \vec{v}_i(0)]$ and have a velocity $\vec{V}_i[t, \vec{x}_i(0), \vec{v}_i(0)]$ and $F = \prod_i \delta[\vec{x}_i - \vec{X}_i(t)]\, \delta[\vec{v}_i - \vec{V}_i(t)]$. If the configuration is unknown, but the probability of each configuration is given at some time, i.e., $F(\vec{x}_0 \ldots, \vec{v}_0 \ldots)$ then F propagates just as before and

$$F(\vec{x}_1 \ldots \vec{x}_N, \vec{v}_0 \ldots \vec{v}_N) = F(\vec{x}_i \ldots, \vec{v}_i \ldots) \prod_i \delta[\vec{x}_i - \vec{X}_i(t)]\, \delta[\vec{v}_i - \vec{V}_i(t)] \ . \qquad (6)$$

Clearly

$$\begin{aligned} \frac{\partial F}{\partial t} &= \sum_i \left[\frac{\partial F}{\partial \vec{X}_i} \cdot \dot{\vec{X}}_i \right|_{x_i v_i} + \left. \frac{\partial F}{\partial V_i} \cdot \dot{\vec{V}}_i \right|_{x_i v_i} \\ &= \sum_i \vec{v}_i \cdot \frac{\partial F}{\partial \vec{X}_i} + \sum_i \vec{a}_i \cdot \frac{\partial F}{\partial V_i} \qquad (7) \\ &= -\sum_i \vec{V}_i \cdot \frac{\partial F}{\partial \vec{x}_i} - \sum_i \vec{a}_i \cdot \frac{\partial F}{\partial \vec{v}_i} \end{aligned}$$

since F is a function of $\vec{x}_i - \vec{X}_i$, $\vec{v}_i - \vec{V}_i$. Hence F satisfies Liouville's equation

$$\frac{\partial F}{\partial t} + \sum_i \vec{v}_i \cdot \frac{\partial F}{\partial \vec{x}_i} + \sum_i \vec{a}_i \cdot \frac{\partial F}{\partial \vec{v}_i} = 0. \qquad (8)$$

To get an expression for f, which is approximately the probability of finding some particle at a point $\vec{x}$, $\vec{v}$ we can form the sum over all the particles of the probability of finding any one at $\vec{x}$, $\vec{v}$, that is

$$\begin{aligned} &\sum_i F\, d^3\vec{x}_1 d^3\vec{v}_1 \ldots d^3\vec{x}_{i-1} d^3\vec{v}_{i-1},\ d^3\vec{x}_{i+1} d^3\vec{v}_{i+1} \ldots d^3\vec{x}_N d^3\vec{v}_N \\ &= \sum_i \int F \,\Pi\, (d^3\vec{x}\, d^3\vec{v} \ldots)_{\text{not } i} \ . \qquad (9) \end{aligned}$$

If all particles are equivalent this is $N \int F\, d^3\vec{x}\, d^3\vec{v}$. To normalize f to the density $\int f\, d^3\vec{v} = n$

$$f = \frac{N}{V} \int F \prod_i (d^3\vec{x}\, d^3\vec{v})_{\text{not } 1} \ . \qquad (10)$$

To get an expression for the rate of change of f we can perform a similar operation on the Liouville equation and write

$$\begin{aligned} &\sum_i \frac{\partial}{\partial t} \int F \,\Pi\, (d^3\vec{x}\, d^3\vec{v})_{\text{not } i} + \sum_i \vec{v}_i \frac{\partial}{\partial x_i} \int F \,\Pi\, (d^3\vec{x}\, d^3\vec{v})_{\text{not } i} \\ &\qquad + \int \vec{a}_i \cdot \frac{\partial F}{\partial \vec{v}_i} \prod_{\text{not } i} d^3\vec{x}\, d^3\vec{v} = 0 \ . \qquad (11) \end{aligned}$$

The first terms here become just $V(\partial/\partial t + v\partial/\partial x)f$ but the last presents a problem, since in general $\vec{a}_i$ will depend on the position of all the particles, because of particle interaction. The macroscopic part of $\vec{a}$ i.e. that depending only on external forces, and on gross interactions, may indeed be written like the first terms as $V\vec{a}_i\cdot\partial f/\partial\vec{v}$ but the particle interaction requires knowledge of the position of the remaining particles. For two body forces, the only knowledge required is that of the correlation between pairs of particles, i.e. if $\vec{a}_i = \sum_i \vec{a}_i(x_i - x_j)$, the last integral becomes $\sum_{ij}\int \vec{a}_i(x_i - x_j)(\partial/\partial v_i)$ $F\,\Pi\,(d^3\vec{x}\,d^3\vec{v})$ and if all particles are equivalent:

$$N(N-1)\int \vec{a}(\vec{x}_1 - \vec{x}_2)\cdot\frac{\partial}{\partial\vec{V}_1} F(\vec{x}_1, \vec{x}_2, \vec{V}_1, \vec{V}_2) \prod_{\text{not } 1} (d^3\vec{x})(d^3\vec{v}). \qquad (12)$$

Introducing the pair correlation function, $f_2 = [(N-1)/V]\int F(\vec{x}_1, \vec{x}_2, \vec{V}_1, \vec{V}_2) \prod_{\text{not } 1} d^3x d^3V$ enables the equation for f_2 to be written:

$$\frac{\partial f}{\partial t} + \vec{V}\cdot\frac{\partial}{\partial\vec{x}} f + \vec{a}_M\cdot\frac{\partial f}{\partial\vec{V}} + \int \frac{\partial}{\partial\vec{V}}\cdot\int \vec{a}(1,2) f_2(1,2)\, d^3\vec{x}_2 d^3\vec{v}_2 = 0. \qquad (13)$$

To determine f then, calls for a knowledge of particle correlation f_2. A similar equation may be written for f_2, but it requires a knowledge of f_3, and a hierarchy of exact equations is obtained closing only at the original Liouville equation for F_N.

It is not to be expected that any simple exact equation for f should be obtained in this way, since, throughout, an exact mechanical description has been used. Instead, an approximate description of f is required. This can be obtained by coarse graining, i.e. by forcing f to vary smoothly on some time scale τ, an operation which can be performed by replacing f by its mean value over some small volume of phase space and some short time τ. For some systems, this time scale may be so selected that during τ, f_2 changes rapidly, but reaches some mean value which depends only on f. We shall present a physical argument, which shows how this might happen and how a closed, though approximate equation for the coarse-grained f can be obtained. An alternative method of approaching the problem is to assume that the f_2, f_3... depend only on f, then to terminate the hierarchy in some way, often by finding an expansion parameter which orders the terms so that a closed approximation can be found to some order. This leads to an equation for $f_2(f)$, hence to one for f.

Under some circumstances the obtaining of a transport equation can be much simplified. For example, if the plasma is so diffuse that the particle interactions may be neglected, and only $\vec{a}_M$ retained in the transport equation, it becomes

$$\frac{\partial f}{\partial t} + \vec{V}\cdot\frac{\partial f}{\partial\vec{x}} + \frac{e}{m}\vec{E}\cdot\frac{\partial f}{\partial\vec{v}} = 0 \qquad (14)$$

and $\vec{E}$ satisfies Poisson's equation

$$\vec{\nabla}\cdot\vec{E} = 4\pi q = 4\pi \sum \int e_i f \, d^3\vec{v}. \tag{15}$$

This is the collisionless Boltzmann or Vlasov equation [2].

On the other hand, it is possible to use the Boltzman collision integral to describe particle interactions, but its use runs into trouble because of the long range of the Coulomb force. One consequence of this is that particles do not interact just a pair at a time, but that many interactions occur simultaneously. However, as we have shown, the effect of the interaction can be well represented by a suitably interpreted form of the impact approximation. In this, the interaction occurs linearly, and if particle correlation is neglected, except insofar as it determines the effective minimum impact parameter, the effects may be summed, as though they occurred separately.

A second consequence is more serious, for the long range of the interaction leads to a singularity in the effective cross-section; a consequence of the neglect of a long range correlation between particles. This difficulty is often glossed over by the introduction of a more or less arbitrary cut-off in the range. Our main object will be to discuss these long range correlations, and show how they affect the transport equation.

Before proceeding to do that, we will make use of the dominance in the momentum change of multiple small angle scattering to reduce the Boltzmann equation to a simpler form, at least for continuous distribution functions.

II. LANDAU'S FORM FOR THE COLLISION INTEGRAL

Consider the Boltzmann collision integral for an ionized gas of like particles (for the moment). Using

$$\sigma = \left(\frac{e^2}{2m_r}\right)^2 \left(g \sin\frac{\theta}{2}\right)^{-4} = \left(\frac{e^2}{m}\right)^2 \left(g \sin\frac{\theta}{2}\right)^{-4} \tag{16}$$

since, for like particles $m_r = m/2$, $\vec{g} = \vec{v}' - \vec{v}$ and using the result

$$\underline{\vec{v}} = \vec{v} - \frac{1}{2}\Delta\vec{g}, \qquad \underline{\vec{v}}' = \vec{v}' + \frac{1}{2}\Delta\vec{g}, \tag{17}$$

where $\Delta\vec{g}$ is the change in $\vec{g}$ on collision, enables this to be written

$$I = \left(\frac{e^2}{m}\right)^2 \int d^3\vec{v}' \int d\varphi \, d\theta \, \sin\theta \, \frac{1}{g^3 \sin^4(\theta/2)} \cdot \left[f\left(\vec{v} + \frac{1}{2}\Delta\vec{g}\right) f\left(\vec{v} - \frac{1}{2}\Delta\vec{g}\right) - f(\vec{v}')f(\vec{v}) \right]. \tag{18}$$

We may write for $\Delta\vec{g}$ (See Eq. (29) of [3]).

$$\Delta\vec{g} = 2g \sin\frac{\theta}{2}\left[-\hat{g}\sin\frac{\theta}{2} + \hat{m}\cos\frac{\theta}{2}\cos\varphi + \hat{n}\cos\frac{\theta}{2}\sin\varphi\right]. \tag{19}$$

Now we may expand the quantity within the square brackets, thus:

$$f\left(\vec{v}' + \frac{1}{2}\Delta\vec{g}\right) f\left(\vec{v} - \frac{1}{2}\Delta\vec{g}\right) - f(\vec{v}')f(\vec{v}) = \left[f(\vec{v})\frac{\partial f(\vec{v}')}{\partial\vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial\vec{v}}\right]\cdot\frac{1}{2}\Delta\vec{g}$$

$$+\frac{1}{2}\left[f(\vec{v})\frac{\partial^2 f(\vec{v}')}{\partial\vec{v}'\,\partial\vec{v}'} + f(\vec{v}')\frac{\partial^2 f(\vec{v})}{\partial\vec{v}'\,\partial\vec{v}} - 2\frac{\partial f(\vec{v}')}{\partial\vec{v}'}\,\frac{\partial f(\vec{v})}{\partial\vec{v}}\right]\cdot\frac{1}{4}\Delta\vec{g}\Delta\vec{g} + 0(\Delta\vec{g})^3. \tag{20}$$

If we are interested only in scattering at small $\Delta\vec{g}$, i. e. small angle scattering, this Taylor expansion may be substituted in the collision integral (18) and if the series is cut off at the second term, the expansion (19) used for $\Delta\vec{g}$ and the integral over φ performed, there results:

$$I = 8\pi\left(\frac{e^2}{m}\right)^2\int d^3\vec{v}'\int_0^{\pi/2} d\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left\{-\frac{\vec{g}}{g^3\sin^4(\theta/2)}\right.$$

$$\cdot\left[f(\vec{v})\frac{\partial f(\vec{v}')}{\partial\vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial\vec{v}}\right]$$

$$+\frac{1}{2}\left[f(\vec{v})\frac{\partial^2 f(\vec{v}')}{\partial\vec{v}'\,\partial\vec{v}'} + f(\vec{v}')\frac{\partial^2 f(\vec{v})}{\partial\vec{v}\partial\vec{v}} - 2\frac{\partial f(\vec{v}')}{\partial\vec{v}'}\,\frac{\partial f(\vec{v})}{\partial\vec{v}}\right]$$

$$\left.:\left[\frac{\vec{g}\vec{g}}{g^2}\sin\frac{\theta}{2} + \frac{1}{2}(\hat{m}\hat{m} + \hat{n}\hat{n})\left(\frac{1}{\sin(\theta/2)} - \sin\frac{\theta}{2}\right)\right]\right\}. \tag{21}$$

In this integral

$$\int_0^{\pi/2} d\left(\frac{\theta}{2}\right)\cos\frac{\theta}{2}\sin\frac{\theta}{2} = 1 \tag{22}$$

while

$$\int_{\theta_0}^{\pi/2} d\left(\frac{\theta}{2}\right)\cos\frac{\theta}{2}\,\frac{1}{\sin(\theta/2)} = -\log\left(\sin\frac{\theta_0}{2}\right) \tag{23}$$

diverges; however we may, as before, cut this off at a maximum impact parameter $b_{max} \simeq v/\omega_0 \simeq \lambda_D$ and write

$$\log\,(n^{1/3}e^2/kT)^{3/2} = \log\Lambda. \tag{24}$$

Eq.(21) may now be reduced to

$$I = 4\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda \int d^3\vec{v}'\left\{-\frac{\vec{g}}{g^3}\cdot\left[f(\vec{v})\frac{\partial f(\vec{v}')}{\partial \vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial \vec{v}}\right]\right.$$

$$\left. + \frac{1}{2}\left[f(\vec{v})\frac{\partial^2 f(\vec{v}')}{\partial\vec{v}'\partial\vec{v}'} + f(\vec{v}')\frac{\partial^2 f(\vec{v})}{\partial\vec{v}\partial\vec{v}} - 2\frac{\partial f(\vec{v}')}{\partial\vec{v}'}\frac{\partial f(\vec{v})}{\partial\vec{v}}\right] : \frac{1}{2g}(\hat{m}\hat{m} + \hat{n}\hat{n})\right\}, \quad (25)$$

Note that $\hat{m}\hat{m}+\hat{n}\hat{n}$ is the unit tensor normal to $\hat{g}\hat{g}$, i.e. $\hat{m}\hat{m}+\hat{n}\hat{n} = \mathrm{I\!I} - gg$. If

$$\overleftrightarrow{W} = \frac{1}{g}(\mathrm{I\!I} - \hat{g}\hat{g}) = \frac{\hat{m}\hat{m}+\hat{n}\hat{n}}{g}; \quad (26)$$

$$\overleftrightarrow{W} = \frac{\partial}{\partial\vec{g}}\frac{\partial}{\partial\vec{g}}\left|\vec{g}\right|, \quad (27)$$

and

$$\frac{\partial}{\partial\vec{g}}\cdot\overleftrightarrow{W} = -2\frac{\vec{g}}{g^3}. \quad (28)$$

In Eq.(26) the second quantity in square brackets may be split up as follows:

$$\frac{1}{2}\left[f(\vec{v})\frac{\partial^2 f(\vec{v}')}{\partial\vec{v}'\partial\vec{v}'} + f(\vec{v}')\frac{\partial^2 f(\vec{v})}{\partial\vec{v}\partial\vec{v}} - 2\frac{\partial f(\vec{v}')}{\partial\vec{v}'}\frac{\partial f(\vec{v})}{\partial\vec{v}}\right]$$

$$= \frac{1}{2}\frac{\partial}{\partial\vec{v}'}\left[f(\vec{v})\frac{\partial f(\vec{v}')}{\partial\vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial\vec{v}}\right] - \frac{1}{2}\frac{\partial}{\partial\vec{v}}\left[f(\vec{v})\frac{\partial f(\vec{v}')}{\partial\vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial\vec{v}}\right] \quad (29)$$

$$= \frac{1}{2}\frac{\partial}{\partial\vec{v}'}\vec{F}(\vec{v},\vec{v}') - \frac{1}{2}\frac{\partial}{\partial\vec{v}}\vec{F}(\vec{v},\vec{v}'),$$

where

$$\vec{F}(\vec{v},\vec{v}') = f(\vec{v})\frac{\partial f(\vec{v}')}{\partial\vec{v}'} - f(\vec{v}')\frac{\partial f(\vec{v})}{\partial\vec{v}}. \quad (30)$$

The integral now becomes

$$I = 8\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda \int d^3\vec{v}'\left\{\frac{1}{2}\left(\frac{\partial}{\partial\vec{g}}\cdot\overleftrightarrow{W}\right)\cdot\vec{F} + \frac{1}{4}\overleftrightarrow{W} : \left[\frac{\partial}{\partial\vec{v}'}\vec{F} - \frac{\partial}{\partial\vec{v}}\vec{F}\right]\right\}, \quad (31)$$

and on performing a partial integration the third term becomes

$$-\frac{1}{4}\left(\frac{\partial}{\partial\vec{v}'}\cdot\overleftrightarrow{W}\right)\cdot\vec{F} = +\frac{1}{4}\left(\frac{\partial}{\partial\vec{v}}\overleftrightarrow{W}\right)\cdot\vec{F}. \quad (32)$$

Hence

$$I = -2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda \int d^3\vec{v}'\left[\frac{\partial}{\partial\vec{v}}\cdot(\overleftrightarrow{W})\cdot\vec{F} + \overleftrightarrow{W}:\frac{\partial}{\partial\vec{v}}\vec{F}\right]$$

$$= 2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda\,\frac{\partial}{\partial\vec{v}}\cdot\int d^3\vec{v}'\,\overleftrightarrow{W}\cdot\left[f(\vec{v}')\frac{\partial f}{\partial\vec{v}} - f(\vec{v})\frac{\partial f}{\partial\vec{v}'}\right]$$

$$= 2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda\,\frac{\partial}{\partial v_i}\int d^3\vec{v}'\,W_{ij}\left[f(\vec{v}')\frac{\partial f}{\partial v_j} - f(\vec{v})\frac{\partial f}{\partial v_j'}\right]. \tag{33}$$

This is the Landau's form for the collision integral. This may also be written as

$$I = -2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda\frac{\partial}{\partial v_i}\int d^3\vec{v}'\left[\frac{f(v)\partial f(\vec{v}')}{\partial v'_j}W_{ij} - f(\vec{v}')\frac{\partial f}{\partial v}W_{ij}\right].$$

$$= -2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda\frac{\partial}{\partial v_i}\int d^3\vec{v}'\left[f(\vec{v}')\frac{\partial W_{ij}}{\partial v_j}f(\vec{v}) - f(\vec{v}')\frac{\partial f}{\partial v_j}W_{ij}\right].$$

$$= \frac{\partial}{\partial v_i}\left[\left(\frac{\partial}{\partial v_j}D_{ij}\right)f(\vec{v}) - D_{ij}\frac{\partial f}{\partial v_j}\right], \tag{34}$$

where

$$D_{ij} = 2\pi\left(\frac{e^2}{m}\right)^2 \log\Lambda\int d^3\vec{v}'\,f(\vec{v}')W_{ij}\,. \tag{35}$$

III. THE FOKKER – PLANCK EQUATION

The form of the collision integral given by Eq. (33) may be approached by a somewhat different route and arises from a study of the rate of change of a Markovian probability distribution. Suppose that $P(x,t)$ represents the distribution function for a quantity x at time t: and suppose moreover, that during the interval $t \to t+\delta t$, x changes to x' with a probability $W(x, x'-x)$. Then, $P(x, t+\delta t)$ may be expressed in terms of $P(x', t)$ as

$$P(x, t+\delta t) = \int dx'\,W(x', x-x')P(x', t)$$

$$= \int d\xi\,W(x-\xi, \xi)P(x-\xi, t). \tag{36}$$

If W is a rapidly decreasing function of ξ, i.e. if P develops by small steps, then we may use a Taylor's expansion on the right hand side, so that

$$P(x, t+\delta t) = \int d\xi\, W(x, \xi)\, P(x, t)$$

$$-\frac{\partial}{\partial x}\int d\xi\, \xi\, W(x, \xi)\, P(x, t) + \frac{1}{2}\frac{\partial}{\partial x}\frac{\partial}{\partial x}\int d\xi\, W(x, \xi)\, \xi\xi P(x, t)\,. \qquad (37)$$

Now, the first integral here = 1, since

$$\int W(x, \xi)\, d\xi = 1, \qquad (38)$$

the second = $\langle \xi \rangle$, the third = $\langle \xi\xi \rangle$, thus

$$\frac{\partial P}{\partial t} = \frac{P(x, t+\delta t) - P(x, t)}{\delta t}$$

$$= -\frac{\partial}{\partial x}[D_1 P(x, t)] + \frac{1}{2}\frac{\partial^2}{\partial x\, \partial x}[D_2 P(x, t)] + \ldots, \qquad (39)$$

where

$$D_1 = \frac{1}{\tau}\langle \xi \rangle, \qquad D_2 = \frac{1}{\tau}\langle \xi\xi \rangle, \qquad \tau = \delta t\,. \qquad (40)$$

For I(f, f) we may now write

$$I(f, f) = -\frac{\partial}{\partial \vec{v}} \cdot \vec{D}_1 f(\vec{v}) + \frac{1}{2}\frac{\partial^2}{\partial \vec{v}\, \partial \vec{v}} : \overleftrightarrow{D}_2 f(\vec{v}), \qquad (41)$$

where

$$\vec{D}_1 = \frac{1}{\tau}\langle \Delta\vec{v} \rangle, \qquad \overleftrightarrow{D}_2 = \frac{1}{\tau}\langle \Delta\vec{v}\, \Delta\vec{v} \rangle\,. \qquad (42)$$

For an unmagnetized plasma the quantities $\vec{D}_1$ and $\overleftrightarrow{D}_2$ can be expressed in terms of the fluctuating microfield within the plasma, i.e. the change in velocity in a time τ of a particle initially at $\vec{x}_0$ having velocity $\vec{v}_0$ is

$$\Delta\vec{v} = \frac{e}{m}\int_{t-\tau}^{t} E(\vec{x}', t')\, dt', \qquad (43)$$

where

$$\vec{x}' = \vec{x}_0 + \int \vec{v}(\vec{x}', t')\, dt' \qquad (44)$$

and

$$\vec{v}(\vec{x},t) = \vec{v}_0 + \frac{e}{m}\int \vec{E}(\vec{x}', t')\,dt'. \tag{45}$$

If $(e/m)\vec{E}$ is small, we may expand $\Delta\vec{v}$ thus

$$\Delta v_i = \frac{e}{m}\int_{t-\tau}^{t} dt'\left[E_i(\vec{x}_0 + \vec{v}_0 t', t') + \frac{\partial E_i}{\partial x_j}\int_{t-\tau}^{t'} dt'' \int_{t-\tau}^{t''} \frac{e}{m} E_j(\vec{x}_0 + \vec{v}_0 t''', t''')dt'''\right] \tag{46}$$

while to the same order in (eE/m)

$$\Delta\vec{v}\Delta\vec{v} = \frac{e^2}{m_2}\int_{t-\tau}^{t} dt'\,\vec{E}(\vec{x}_0 + \vec{v}_0 t', t')\int_{t-\tau}^{t} dt''\,\vec{E}(\vec{x}_0 + \vec{v}_0 t'', t'')$$

$$= \frac{e^2}{m}\int_{t-\tau}^{t} dt' \int_{0}^{\tau} ds\,\vec{E}(\vec{x}_0 + \vec{v}_0 t', t')\,\vec{E}[\vec{x}_0 + \vec{v}_0(t' - s), t' - s]\,\cdot \tag{47}$$

This equation may be derived in an alternative way which requires, however, a return to the Liouville equation

$$\frac{\partial F}{\partial t} + \vec{v}_i \cdot \frac{\partial F}{\partial \vec{x}_i} + \frac{e}{m}\vec{E}_i \cdot \frac{\partial F}{\partial \vec{v}_i} = 0. \tag{48}$$

Now the Liouville function F can be written as

$$F = F_1(\vec{x}_1)\,\Psi(\vec{x}_2, \vec{x}_3 \ldots \vec{x}_N; \vec{x}_1) \tag{49}$$

and by integrating over $\vec{x}_2 \ldots \vec{x}_N$ an equation for $\vec{F}_1(\vec{x}_1)$ may be deduced.
Since

$$\vec{E}_i = -\frac{\partial}{\partial \vec{x}_i}\phi = -\frac{\partial}{\partial \vec{x}_i}\sum_j \frac{e}{|\vec{x}_i - \vec{x}_j|} \tag{50}$$

depends upon the $\vec{x}_j$, however, the integration cannot be carried out explicitly, but involves a term of the form

$$-\frac{\partial}{\partial \vec{v}}\cdot\sum_j \int \frac{e^2}{m}\frac{\partial}{\partial \vec{x}_1}\,\phi(\vec{x}_1, \vec{x}_j)\,\Psi(\vec{x}_2 \ldots \vec{x}_N; \vec{x}_1)F_1(\vec{x}_1)dx_2 \ldots dx_N. \tag{51}$$

For any given complexion

$$F = \prod_i \delta(\vec{x}_i - \vec{X}_i(t))\delta(\vec{v}_i - \vec{V}_i(t)) \tag{52}$$

is a rapidly varying function and we prefer to work with a smoothly varying quantity, f_1, which represents the probability of a particle being at $\vec{x}_1$, $\vec{v}_1$, given some initial probability distribution $p_0(\vec{x}_1, \vec{x}_2 \ldots \vec{x}_N, \vec{v}_1, \vec{v}_2 \ldots \vec{v}_N)$ for the entire distribution.

Then, F has the form

$$p_0(\vec{x}_1^{\circ} \ldots \vec{x}_N^{\circ}, \vec{v}_1^{\circ} \ldots \vec{v}_N^{\circ}) \prod_i \delta[\vec{x} - \vec{X}_i(t)]\, \delta[\vec{v} - \vec{V}_i(t)] \tag{53}$$

where

$$\vec{X}(t) = \vec{x}_1^{\circ} + \int \vec{V}_i(t')\, dt' \tag{54}$$

and

$$\vec{V}(t) = \vec{v}_1^{\circ} + \frac{e}{m} \int \vec{E}(\vec{X}(t'), t')\, dt'. \tag{55}$$

From this we may form F_1, again by integrating over $d\vec{x}_2 \ldots d\vec{x}_N$ and noting that if p_0 is smooth, we expect F_1 to be a slowly varying function. In fact we expect F_1 to satisfy an equation of the Boltzmann type, and to change significantly in times of order τ_i, the collision period. This means that instead of considering the equation of motion for F_1 we obtain an adequate description by considering the motion of the coarse-grained distribution

$$f = \frac{1}{\tau} \int_{t-\tau}^{t} F_1(t')\, dt', \tag{56}$$

where $\tau \ll \tau_i$. This satisfies

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e}{m} \frac{\partial}{\partial \vec{v}} \cdot \frac{1}{\tau} \int_{t-\tau}^{t} dt' \vec{E}(\vec{x}(t'), t')\, F_1(t')\, dt' = 0 \tag{57}$$

i.e.

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e}{m} \frac{\partial}{\partial \vec{v}} \cdot \frac{1}{\tau} \int_{t-\tau}^{t} dt' \vec{E}(\vec{x}(t'), t')\, \delta[\vec{x} - \vec{X}(t')]\, \delta[\vec{v} - \vec{V}(t')]\, F_1(t' - \tau) = 0, \tag{58}$$

where the last term is explicitly

$$-\frac{e^2}{m} \frac{\partial}{\partial \vec{v}} \cdot \frac{1}{\tau} \int_{t-\tau}^{t} dt'\, d^3\vec{x}_2 \ldots d^3\vec{x}_N d^3\vec{v}_2 \ldots d^3\vec{v}_N \sum_j \frac{\partial \phi}{\partial \vec{x}_i}(\vec{x}_i, \vec{x}_j)$$

$$\cdot \Psi(\vec{x}_2 \ldots \vec{x}_N, \vec{v}_2 \ldots \vec{v}_N; \vec{x}_1)\, F_1(\vec{x}_1, \vec{v}_1, t-\tau)\, \delta[\vec{x}_1 - \vec{X}_1(t')] \cdot \delta[\vec{v}_1 - \vec{V}_1(t')]. \tag{59}$$

Now the electric field may contain some mean part $\vec{E}_0$, independent of particle positions, a part $\vec{E}_1$, depending on particle position and velocity, and a rapidly fluctuating part $\vec{E}_f \cdot \vec{E}_0$ and $\vec{E}_1$ are easily handled, contributing terms

$$\frac{e}{m}\frac{\partial}{\partial \vec{v}} \cdot (\vec{E}_0 + \vec{E}_1) f . \tag{60}$$

To treat $\vec{E}_f$ we may observe that for most plasmas, where the ratio of kinetic to potential energy is small, the field dependent quantities $\vec{X}$ and $\vec{V}$ may be calculated in perturbation theory. Thus

$$\vec{V}(t) = \vec{V}_0 + \frac{e}{m}\int_0^t \vec{E}(\vec{x}_0 + \vec{v}_0 t', t')\, dt', \tag{61}$$

and

$$\vec{X}(t) = \vec{x}_0 + \vec{v}_0 t + \frac{e}{m}\int_0^t dt' \int_0^{t'} dt''\, \vec{E}(\vec{x}_0 + \vec{v}_0 t'', t''), \tag{62}$$

and the δ-functions appearing in Eq. (58) may be similarly expanded, thus

$$\begin{aligned}\delta[\vec{x} - \vec{X}(t')]\delta[\vec{v} - \vec{V}(t')] &= \delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t')\delta(\vec{v} - \vec{v}_0) \\ &- \delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t')\frac{\partial}{\partial \vec{v}}\delta(\vec{v} - \vec{v}_0 t', t')\frac{e}{m}\int^{t'} \vec{E}(\vec{x}_0 + \vec{v}_0 t'', t'')\, dt'' \\ &- \delta(\vec{v} - \vec{v}_0)\frac{\partial}{\partial \vec{x}}\delta(\vec{x} - \vec{x}_0 - \vec{v}_0 t')\frac{e}{m}\int^{t'} dt \int^{t''} \vec{E}(\vec{x}_0 + \vec{v}_0 t''', t''')\, dt''' .\end{aligned} \tag{63}$$

These may be handled, as usual, by a partial integration, and use made of the fact that F_1 is slowly varying so that $F(t-\tau, v) = f(t, v)$

whereupon Eq. (58) becomes

$$\begin{aligned}&\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{e}{m}\vec{E}_0 \cdot \frac{\partial f}{\partial \vec{v}} + \frac{e}{m}\frac{\partial}{\partial \vec{v}} \cdot (\vec{E}, f) + \frac{e^2}{m}\frac{\partial}{\partial v_i}\Big\langle \frac{1}{\tau}\int_{t-\tau}^{t} dt' \frac{\partial}{\partial x_j} E_i(\vec{x}_0 + \vec{v}_0 t', t') \\ &\qquad \times \int_{t-\tau}^{t'} dt'' \int_{t-\tau}^{t''} dt'''\, E_j(\vec{x}_0 + \vec{v}_0 t''', t''') \Big\rangle f(\vec{v}) \\ &- \frac{e^2}{m}\frac{\partial}{\partial v_i}\frac{\partial}{\partial v_j}\Big\langle \frac{1}{\tau}\int_{t-\tau}^{t} dt'\, E_i(\vec{x}_0 + \vec{v}_0 t', t') \int_{t-\tau}^{t} dt''\, E_j(\vec{x}_0 + \vec{v}_0 t'', t'') \Big\rangle f(\vec{v}) = 0\end{aligned} \tag{64}$$

and equation of the Fokker-Planck form with the coefficients given by Eqs. (46) and (47). These expanded forms may be written exactly as those in Eq. (59), i.e.

$$\langle \vec{E} \int_{t-\tau}^{t} \vec{E}\, dt' \rangle = \int d^3\vec{x}_2 \ldots d^3\vec{x}_N\, d^3\vec{v}_2 \ldots d^3\vec{v}_N \sum_{ij} \frac{\partial \phi}{\partial \vec{x}_1}(\vec{x}_1, \vec{x}_j)$$

$$\times \int_{t-\tau}^{t} dt' \frac{\partial \phi}{\partial x_1}(\vec{x}_1, \vec{x}_k)\Psi(\vec{X}_2, \ldots \vec{X}_N, \vec{V}_2 \ldots \vec{V}_N, \vec{x}_1), \tag{65}$$

i.e. as mean values $\langle E_i E_j \rangle$.

Now, the correlation functions

$$\langle \phi(t) \int_{t-\tau}^{t} \phi(t')dt' \rangle = \langle \int_{0}^{\tau} \phi(t)\, \phi\,(t-s)\, ds \rangle \tag{66}$$

may be simplified somewhat by assuming a property of $\langle \vec{E}(t)\vec{E}(t-s) \rangle$ which we will be able to demonstrate, namely, that the fields are strongly correlated for small times, but that the product $\vec{E}\vec{E}$ becomes small and fluctuates about zero for long times; indeed, in time of order $1/\omega_0$ the correlation is already small. This suggests that if the upper limit of integration is taken as $\tau \gg \omega_0^{-1}$, which implies $\tau_e \gg \omega_0^{-1}$, a condition which is usually satisfied, then the upper limit of integration in Eq. (66) may be extended to infinity and the required quantities become

$$\langle \int_0^\infty \phi(t)\, \phi(t-s)ds \rangle, \tag{67}$$

and if the field is derivable from a potential

$$\langle E_i E_j \rangle = \langle \frac{\partial \phi}{\partial x_i}\, \frac{\partial \phi}{\partial x_j} \rangle. \tag{68}$$

The correlation function may be expressed in terms of the spectrum of ϕ, for if

$$\phi(\vec{x}, t) = \frac{1}{(2\pi)^4} \int d^3\vec{k}\, d\omega\, \exp\,[i(\omega t + \vec{k}\cdot\vec{x})]\, \phi(\vec{k}, \omega), \tag{69}$$

then

$$\langle \int_0^\infty \phi(\vec{x}, t)\, \phi(\vec{x} - \vec{v}s,\, t-s)\, ds$$

$$= \mathrm{Re}\frac{1}{(2\pi)^8} \langle \int_0^\infty d^3\vec{k}\, d\omega \int_0^\infty d^3\vec{k}'\, d\omega' \int ds\, e^{i(\vec{k}\cdot\vec{x} + \omega t)}$$

$$\times\, e^{i[\vec{k}'\cdot(\vec{x}-\vec{v}s)+\omega'(t-s)]}\phi(\vec{k},\omega)\phi(\vec{k}',\omega')\rangle$$

$$= \mathrm{Re}\Big\langle\frac{1}{(2\pi)^4}\int ds\int d^3\vec{k}\,d\omega\; e^{-i(\vec{k}\cdot\vec{v}+\omega)s}$$

$$\times\frac{1}{(2\pi)^4}\int d^3\vec{k}'\,d\omega'\,e^{i(\vec{k}+\vec{k}')\cdot\vec{x}+i(\omega+\omega')t}\phi(k,\omega)\,\phi(k',\omega')\Big\rangle. \tag{70}$$

The inner integral however is the energy spectrum, $\langle\phi^*(\vec{k},\omega)\,\phi(\vec{k},\omega)\rangle$, hence, the correlation function is

$$\langle\phi(\vec{x},t)\phi(\vec{x}-\vec{v}s,\,t-s)\rangle$$

$$= \mathrm{Re}\,\frac{1}{(2\pi)^4}\int d^3\vec{k}\,d\omega\phi^*(\vec{k},\omega)\phi(\vec{k},\omega)e^{-i(\vec{k}\cdot\vec{v}+\omega)s}$$

$$= \frac{1}{2(2\pi)^4}\Big\langle\int d^3\vec{k}\;d\omega\,\phi^*(\vec{k},\omega)\,\phi(\vec{k},\omega)\cos(\omega+\vec{k}\cdot\vec{v})\Big\rangle \tag{71}$$

the cosine transform of the energy spectrum (the Wiener–Kinchin theorem). It follows that

$$\Big\langle\int_0^\infty ds\,\phi(\vec{x},t)\,\phi\,(\vec{x}-\vec{v}s,t-s)\Big\rangle$$

$$= \pi\frac{1}{(2\pi)^4}\int d^3\vec{k}\,d\omega\;\phi(\vec{k},\omega)\phi(\vec{k},\omega)\;\delta\,(\omega+\vec{k}\cdot\vec{v}). \tag{72}$$

IV. CALCULATION OF THE SPECTRUM

To calculate the spectrum we can again assume that the electric field within the plasma is weak, and the interactions are small. At the same time, the effect of the field on the distribution function must be retained in any calculation of the field, which otherwise diverges. Our first object then must be to calculate the response of the plasma to a field in that approximation in which the particle interaction is neglected. This however, requires a solution to the Vlasov equation,

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial\vec{x}}+\frac{e}{m}\vec{E}\cdot\frac{\partial f}{\partial\vec{v}}=0, \tag{73}$$

and since the fields are assumed small, we may use a perturbation solution of this about some distribution f_0, assumed known – whereupon, on Fourier transforming

$$f = f_0 + f', \qquad f'(\omega, \vec{k}) = -\frac{e}{m}\,\frac{\vec{E}(\omega,\vec{k})\cdot(\partial/\partial\vec{v})f_0}{i(\omega+\vec{k}\cdot\vec{v})} \tag{74}$$

or [2], if $\vec{E}$ is derivable from a potential $\vec{E} = -ik\phi$,

$$f' = \frac{e}{m}\,\frac{\vec{k}\cdot(\partial f_0/\partial\vec{v})}{(\omega+\vec{k}\cdot\vec{v})}\,\phi\,. \tag{75}$$

The charge induced by a potential ϕ, then becomes, if f_0^-, f_0^+ are the unperturbed distributions of electrons and ions,

$$q_{ind}(\omega,\vec{k}) = \int d^3v \frac{1}{(\omega+\vec{k}\cdot\vec{v})}\,\vec{k}\cdot\frac{\partial}{\partial\vec{v}}\left[\frac{e_-^2}{m_-}f_0^- + \frac{e_+^2}{m_+}f_0^+\right]\cdot\phi(\omega,\vec{k})$$

$$= -k^2 K(\omega,\vec{k})\,\phi(\omega,\vec{k})\,. \tag{76}$$

If now, a test charge e_1 is introduced into a plasma, the charge produced is

$$q^*(\omega,\vec{k}) = e_1\int \exp\,[-i(\omega t + \vec{k}\cdot\vec{x})]\,\delta(\vec{x}-\vec{v}t)\,d^3x\,dt = e_1\,\delta(\omega+\vec{k}\cdot\vec{v})\,, \tag{77}$$

and the induced potential may be written, from

$$k^2\phi = 4\pi q = 4\pi(q_{ind} + q^*) = -4\pi e_1\delta(\omega+\vec{k}\cdot\vec{v}) - 4\pi k^2 K\phi \tag{78}$$

whence

$$\phi = 4\pi\,\frac{e_1\delta(\omega+\vec{k}\cdot\vec{v})}{k^2(1+4\pi K)} = 4\pi\,\frac{e_1\delta(\omega+\vec{k}\cdot\vec{v})}{k^2\epsilon(\omega,\vec{k})}\,, \tag{79}$$

introducing the dielectric coefficient,

$$\epsilon(\vec{k},\omega) = 1 + 4\pi K(\vec{k},\omega) \tag{80}$$

$$= 1 + \frac{\omega_0^2}{k^2}\int \frac{dv_\parallel}{(v_\parallel - v_p)}\,\frac{\partial}{\partial v_\parallel}\left\{g_- + \frac{\omega_+^2}{\omega_0^2}\,g_+\right\} \tag{81}$$

where

$$-v_p = \omega/k, \qquad \omega_0^2 = 4\pi n_- e^2/m_-, \qquad \omega_+^2 = 4\pi n_+ e_+^2/m_+ \tag{82}$$

$$g_\pm = \frac{1}{n}\int d^2v_\perp f_0^\pm(\vec{v}), \qquad \vec{v}_\perp = \vec{v} - v_\parallel\hat{k}, \qquad v_\parallel = \hat{k}\cdot\vec{v}. \tag{83}$$

To handle the singular integrals in Eq. (81) requires some care; however several different arguments, e.g. solving an initial value problem, considering a perturbation which is adiabatically switched on, or considering a model of residual collision process, all lead to the conclusion that the Landau value sould be used, with P∫ meaning the Cauchy principle value [2],

$$\int dt \frac{g(t)}{t-x} = P\int dt \frac{g(t)}{t-x} + i\pi g(x) = \int_L dt \frac{g(t)}{t-x}, \tag{84}$$

hence ϵ is complex. In normal systems an imaginary part to ϵ represents the loss due to collisions; here the loss process is Landau damping.

The field at a charge e_i introduced by its own presence is

$$E(\vec{v}t, t) = \mathrm{Re}\int \frac{d^3k\, d\omega}{(2\pi)^4} 4\pi e_1 i\vec{k} \exp[i\vec{k}\cdot\vec{x}+\omega t] \frac{\delta(\omega+\vec{k}\cdot\vec{v})}{k^2\epsilon(\omega,\vec{k})}$$

$$= \frac{1}{4\pi^3} e_T \int \vec{k}\ \mathrm{Im}\ \frac{\{k^2\epsilon(\omega,k)\}}{|k^2\epsilon(\omega,\vec{k})|^2}\ \delta(\omega+\vec{k}\cdot\vec{v}) d^3k\, d\omega\,. \tag{85}$$

Using the definition of $\epsilon(\vec{k},\omega)$ and of the singular integrals given above, this may be written

$$\frac{e}{m}\vec{E}_{q_1} = -\frac{1}{4\pi^2}\frac{4\pi^2 e^4}{m^2}\int d^3\vec{k}\int d^3\vec{v}'\ \frac{\vec{k}}{|k^2\epsilon(\vec{k}\cdot\omega)|^2}$$

$$\times\ \vec{k}\cdot\frac{\partial}{\partial\vec{v}'} f(\vec{v}')\,\delta(\omega+\vec{k}\cdot\vec{v}')\,\delta(\omega+\vec{k}\cdot\vec{v}) \tag{86}$$

The fluctuating field in the plasma may be calculated by noting that each charge in the plasma itself produces a polarization, like that of a test charge; hence, if particles are distributed at points $\vec{X}_i(t)$:

$$q(\vec{x},t) = \sum e_i\,\delta(\vec{x}-\vec{X}_i(t)) \tag{87}$$

$$q(\omega,\vec{k}) = \sum_i e_i \int dt\ \exp[-i\vec{k}\cdot\vec{X}_i(t) - i\omega t], \tag{88}$$

$$\phi(\omega,\vec{k}) = 4\pi\sum_i e_i\int dt\,\frac{\exp[-i(\vec{k}\cdot\vec{X}_i(t)+\omega t]}{k^2\epsilon(\omega,\vec{k})}\,. \tag{89}$$

The motion of each particle is approximately constant, for times of order τ_c, provided the mean field is small, i.e. $n^{1/3}e^2 \ll kT$, and provided no close collision occurs; hence if

$$t' = t+s, \qquad \vec{x}(t') = \vec{x}(t)+\vec{v}s, \tag{90}$$

and

$$\phi(\omega, \vec{k})\,\phi(\omega', \vec{k}') = \frac{(4\pi)^2}{\vec{k}^2 \epsilon(\omega, \vec{k}) k'^2 \epsilon(\omega', \vec{k}')} \sum_{ij} \int dt \int ds \exp\,[i(\vec{k}\cdot\vec{x}_i(t) + \omega t)]$$

$$\cdot \exp\,[i\vec{k}'\cdot\vec{x}_j(t) + \vec{v}s]\,\exp\,[i\omega'(t+s)]\,. \tag{91}$$

Further, because the correlation between any pair of particles is small the random phase approximation may be used to reduce the double sum to a single one:

$$\left\langle \sum_i \exp\,[i(\vec{k}' + \vec{k})\cdot\vec{x}_i(t)] \right\rangle, \tag{92}$$

and the mean value of this (only quantity involving $\vec{x}_i$) is equal to

$$\frac{1}{v}\int d^3x_i \exp\,[i(\vec{k}' + \vec{k})\cdot\vec{x}_i] = (2\pi)^3 \delta(\vec{k} + \vec{k}'). \tag{93}$$

Further

$$\left\langle \int \exp\,[-i(\omega + \omega')t]\,dt \right\rangle = 2\pi\,\delta(\omega + \omega')\,, \tag{94}$$

$$\int ds\, \exp\,[i(\omega' + \vec{k}\cdot\vec{v})s] = 2\pi\,\delta(\omega + \vec{k}\cdot\vec{v}), \tag{95}$$

while

$$\sum_i \Psi(\vec{v}_i) = \int d^3v\, f(\vec{v})\,\Psi(\vec{v}). \tag{96}$$

Therefore

$$\langle \phi(\vec{k}, \omega)\,\phi(\vec{k}', \omega') \rangle = (2\pi)^5 (4\pi)^2 \frac{\delta(\vec{k} + \vec{k}')\,\delta(\omega + \omega')}{|k^2 \epsilon(\omega, \vec{k})|^2}\, e^2 \int d^3v\, \delta(\omega + \vec{k}\cdot\vec{v})\, f(v)\,\cdot \tag{97}$$

The power spectrum is then

$$\frac{1}{(2\pi)^4}\int d^3k'\, d\omega' \langle \phi(\vec{k}, \omega)\,\phi(\vec{k}', \omega') \rangle$$

$$= \frac{32\pi^3}{|k^2 \epsilon(\vec{k}, \omega)|^2} \int d^3v\, \delta(\omega + \vec{k}\cdot\vec{v}) \sum_{\pm} e_{\pm}^2 f_{\pm}(\vec{v}), \tag{98}$$

and the diffusion coefficient, using Eqs.(64) and (72) becomes

$$D_{ij} = \left(\frac{e}{m}\right)^2 \langle E_i E_j \rangle$$

$$= \frac{32\pi^3 e^2}{m^2} \int d^3v' \int \frac{d^3k\, d\omega\, k_i k_j}{(2\pi)^4} \frac{\delta(\omega + \vec{k}\cdot\vec{v})}{|k^2 \epsilon(\vec{k},\omega)|^2} \delta(\omega + \vec{k}\cdot\vec{v}') \sum_{\pm} e_{\pm}^2 f_{\pm}(\vec{v}'). \qquad (99)$$

The friction involves a term of the form

$$-\frac{e^2}{m^2} \langle \frac{\partial}{\partial x_j} E_i \int dt' \int^{t'} dt'' E_j(t'') \rangle, \qquad (100)$$

in using the property that the quantity under the first integral sign, ϵ = function of $t - s$, we may write this as

$$-\frac{e^2}{m} \langle \int ds\, E_j(x,t)\, s \frac{\partial}{\partial x_j} E_i(\vec{x} - \vec{v}s,\, t-s) \rangle$$

$$= \frac{e^2}{m^2} \frac{\partial}{\partial v_j} \langle \int ds\, E_j(\vec{x},t)\, E_i(\vec{x} - \vec{v}s,\, t-s) \rangle = \frac{\partial}{\partial v_j} D_{ij}. \qquad (101)$$

V. THE DOMINANT APPROXIMATION

The integrals required to evaluate the D_{ij} take the form

$$\int d^3g \int d^3k\, d\omega\, \vec{k} \frac{\delta(\omega + \vec{k}\cdot\vec{v})}{|k^2 \epsilon(\vec{k},\omega)|^2} \delta(\vec{k}\cdot\vec{g})\vec{k}\cdot \frac{\partial}{\partial \vec{v}} f(\vec{v} + \vec{g}), \qquad (102)$$

from Eq. (85) and

$$\int d^3g \int d^3k\, d\omega\, \vec{k}\vec{k} \frac{\delta(\omega + \vec{k}\cdot\vec{v})}{|k^2 \epsilon(\vec{k};\omega)|^2} \delta(\vec{k}\cdot\vec{g}) f(\vec{v} + \vec{g}), \qquad (103)$$

from Eq. (99); where $\vec{g} = \vec{v}' - \vec{v}$.
The integral over $d\omega$ is trivial, and on splitting $d^3k = k^2 dk\, d\Omega$ these become

$$\int d^3g \int d\Omega\, \delta(\hat{k}\cdot g)\hat{k}\hat{k} \int dk \frac{k^3}{|k^2 \epsilon(\vec{k},\, -\vec{k}\cdot\vec{v})|^2} \qquad (104)$$

Now

$$k^2 \epsilon(\vec{k},\omega) = k^2 + \omega_0^2 \int_L \frac{dv_\parallel}{(v_\parallel - v_p)} \cdot \frac{\partial}{\partial v_\parallel} \left(g_- + \frac{\omega_+^2}{\omega_0^2} g_+\right) \qquad (105)$$

$$= k^2 + k_D^2 \left[X\left(\frac{\omega}{kv_\theta}\right) + iY_\bullet\left(\frac{\omega}{kv_\theta}\right)\right]$$

$$\int \frac{dkk^3}{(k^2+k_0^2X)^2+k_0^4Y^2} = \log\left(\frac{k_{max}}{k_0}\right)+\frac{1}{4}\log\left\{\frac{[1+(k_0^2/k_{max}^2)X]^2+Y^2}{X^2+Y^2}\right\}$$

$$-\frac{x}{4}\left\{\tan^{-1}\left|\frac{X}{Y}\right| - \frac{\pi}{2}\right\} \tag{106}$$

In this integral k_{max} is an upper limit which is forced upon us by the existence of close encounters, for which the field strength becomes large and which may be identified with the inverse of the minimum impact parameter e^2/mv_θ^2, needed to make the linearized approximation agree with the exact treatment.
Since

$$\left(\frac{n^{1/3}}{k_D}\right)^2 = n^{2/3}\frac{mv_\theta^2}{4\pi ne^2} = \frac{1}{2\pi}\frac{\frac{1}{2}mv_\theta^2}{n^{1/3}e^2} = \frac{1}{2\pi}\frac{T}{V}, \tag{107}$$

the large value of the ratio T/V insures that (k_{max}/k) is large. In the dominant approximation only this term is retained; and the integrals become

$$\log\Lambda\int d^3g\int d\Omega\,\hat{k}\,\delta(\hat{k}\cdot\vec{g})\hat{k}\cdot\frac{\partial f}{\partial\vec{v}}(\vec{v}+\vec{g}), \tag{108}$$

$$\log\Lambda\int d^3g\int d\Omega\,\hat{k}\hat{k}\,\delta(\hat{k}\cdot\vec{g})\,f(\vec{v}+\vec{g})\ . \tag{109}$$

Note that $(108) = \partial/\partial\vec{v}\ (109)$, consequently $(e_1/m_1)\vec{E} = -\partial/\partial\vec{v}\cdot\vec{D}$, while in (108) the angular integration merely selects those parts of $\vec{kk}$ orthogonal to $\vec{g}$ i.e.

$$\int d\Omega\,\hat{k}\,\hat{k}\,\delta(\hat{k}\cdot\vec{g}) = \frac{\pi}{g}(1-\hat{g}\,\hat{g}) = \pi\overleftrightarrow{W}, \tag{110}$$

cf. Eq. (26), hence

$$D_{ij} = 2\pi\left(\frac{e^2}{m}\right)^2\log\Lambda\int d^3v' w_{ij} f(\vec{v}'). \tag{111}$$

Now collecting the terms in the fluctuating field, the Fokker-Planck equation becomes

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\vec{\nabla}f+\frac{e}{m}\vec{E}_0\cdot\frac{\partial f}{\partial v}-\frac{\partial}{\partial\vec{v}_i}\left[\left(\frac{\partial}{\partial v_j}D_{ij}\right)f\right]$$

$$-\frac{\partial}{\partial v_j}\left(\frac{\partial}{\partial v_j}D_{ij}\right)f+\frac{\partial}{\partial v_i\,\partial v_j}(D_{ij}f) = 0. \tag{112}$$

Using the symmetry of D_{ij}, the last terms here may be written as:

$$\frac{\partial}{\partial v_i}\left[-2\left(\frac{\partial f}{\partial v_j} D_{ij}\right) f + \left(\frac{\partial}{\partial v_j} D_{ij}\right) f + D_{ij}\frac{\partial f}{\partial v_j}\right] = \frac{\partial}{\partial v_i}\left[D_{ij}\frac{\partial f}{\partial v_j} - \left(\frac{\partial D_{ij}}{\partial v_j}\right) f\right]. \quad (113)$$

This however is Landau's form Eq. (33), which thus represents the dominant approximation to a kinetic equation which includes both the static correlation effects which produce screening with the dynamic effects that represent the production of plasma oscillations. It is of particular interest to observe that our approach to this has required that f_0 be substantially uniform over times $\gg \omega_0^{-1}$; thus, if we wish to discuss the attenuation of radio-frequency oscillations propagating through the plasma, the usual Boltzmann treatment is inadequate; instead correlation effects must be determined for the distribution function perturbed by the incident r-f field [1].

VI. CORRELATION FUNCTIONS AND SCATTERING OF RADIATION FROM A PLASMA

1. *The correlation functions in a plasma*

Since our procedure has given a value for the potential produced by a particle in a plasma, namely (in the absence of a magnetic field)

$$\phi(\vec{k}, \omega) = 8\pi^2 e_i \frac{\delta(\omega + \vec{k}\cdot v)}{k^2 \epsilon(\vec{k}, \omega)} \exp - i\vec{k}\cdot\vec{x}_i\,, \quad (114)$$

we may now use the Vlasov equation to calculate the disturbance that this produces in the distribution function, i.e.

$$f' = 8\pi^2 e_1 \frac{\delta(\omega + \vec{k}\cdot\vec{v})}{k^2\epsilon(\vec{k},\omega)} \exp\,[-i\vec{k}\cdot\vec{x}_i]\, \frac{e_2}{m_2}\frac{\vec{k}\cdot(\partial f_0/\partial\vec{v})}{\omega + \vec{k}\cdot\vec{v}}\,. \quad (115)$$

Now the probability of finding a particle at $(\vec{x}_1, \vec{v}_1, t_1)$ and a particle at $(\vec{x}_2, \vec{v}_2, t_2)$ is clearly

$$\phi(1,2) = f\cdot(v_1, x_1, t_1)\, f(v_2, x_2, 0) + f(x_2, v_2, 0)\, f(1, 2, x_1, v_1, t_1)\,. \quad (116)$$

However, the last term is just (115); i.e. f' the change in f(1) produced by a particle at 2, hence

$$f(\vec{x}_1, \vec{v}_1, t_1, 2) = \frac{e_1 e_2}{m_1}\frac{1}{2\pi^2}\int d^3k\, d\omega\, \frac{\delta(\omega + \vec{k}\cdot\vec{v}_2)\exp\,[i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)]}{k^2\epsilon(\vec{k},\omega)(\omega + \vec{k}\cdot\vec{v}_1)}$$

$$\exp\,[i\omega t]\,\vec{k}\cdot\frac{\partial f_0(1)}{\partial v_1}\,. \quad (117)$$

Having the (space-dependent) perturbation induced in the distribution function by the potential of a charged particle, it is possible to calculate the spatial distribution of electrons and ions in a plasma, in this approximation. If the zero-order positions of electrons and ions are x_i and X_i, then the plasma potential is

$$\phi = 8\pi^2 \sum_i e_- \frac{\delta(\omega+\vec{k}\cdot\vec{v}_i)\exp[-i\vec{k}\cdot\vec{x}_i] + e_+\delta(\omega+\vec{k}\cdot\vec{v}_i)\exp[-i\vec{k}\cdot\vec{X}_i]}{k^2\epsilon(\vec{k},\omega)} \tag{118}$$

and

$$n_-(\vec{k},\omega) = \sum_i \exp(-i\vec{k}\cdot\vec{x}_i)\cdot\delta(\omega+\vec{k}\cdot\vec{v}_i)$$

$$+\frac{e_-^2}{m_-}\int d^3v\,\frac{\delta(\omega+\vec{k}\cdot\vec{v}_i)\exp(-i\vec{k}\cdot\vec{x}_i)}{k^2\epsilon(\vec{k},\omega)(\omega+\vec{k}\cdot\vec{v})}\,\vec{k}\,\frac{\partial f_0}{\partial\vec{v}}\,. \tag{119}$$

From this, the charge density is

$$q_1 = e_-n_-(\vec{k},\omega) + e_+n_+(\vec{k},\omega)$$

$$= 2\pi\sum\left\{e_-\exp[-ik\cdot x_i]\,\delta(\omega+k\cdot v_i) + e_+\exp[-ik\cdot X_i]\,\delta(\omega+k\cdot V_i)\right\}$$

$$\cdot\left\{\frac{1+\left[\omega_0^2\int\frac{d^3v}{\omega+\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_-}{\partial\vec{v}} + \omega_+^2\int\frac{d^3v}{(\omega+\vec{k}\cdot\vec{v})}\,\vec{k}\cdot\frac{\partial f_+}{\partial\vec{v}}\right]}{k^2\epsilon(\vec{k},\omega)}\right\}. \tag{120}$$

But

$$k^2\epsilon(\vec{k},\omega) = k^2 - \omega_0^2\left[\int\frac{d^3v}{(\omega+\vec{k}\cdot\vec{v})}\,\vec{k}\cdot\frac{\partial f_-}{\partial\vec{v}} + \frac{\omega_+^2}{\Omega_0^2}\int\frac{d^3v}{(\omega+\vec{k}\cdot\vec{v})}\vec{k}\cdot\frac{\partial f}{\partial\vec{v}}\right] \tag{121}$$

hence

$$4\pi q_1 = 8\pi^2\overline{\sum}\left[\frac{e_-\,\delta(\omega+\vec{k}\cdot\vec{v}_i)\exp(-i\vec{k}\cdot\vec{x}_i) + e_+\delta(\omega+\vec{k}\cdot\vec{V}_i)\exp(-i\vec{k}\cdot\vec{X}_i)}{\epsilon(\vec{k},\omega)}\right] \tag{122}$$

and the calculation is self-consistent.

2. Scattering of radiation from a plasma

It is of interest to note an observable phenomenon which depends on the details of the electron correlation function; this is the scattering of radiation by a plasma. To treat this, we consider a plasma in which the distribution function may be written $f_0(v) + f_1(x, v, t)$ and consider the effect on this of an electric field $E(x, t)$. After Fourier-transforming we obtain the induced currents

$$j_{ind}(\omega, \vec{k}) = -\frac{e^2}{m}\int d^3v \left[\frac{\vec{E}(\omega, k)\cdot(\partial f_0/\partial\vec{v})\vec{v} + \Sigma\vec{E}(\Omega, K)\cdot(\partial/\partial\vec{v})f(\omega-\Omega, \vec{K}-\vec{k})\vec{v}}{i(\omega + \vec{k}\cdot v)}\right]. \tag{123}$$

If the phase velocity is high $\omega/k \gg v_\theta$; then

$$j_{ind}(\omega, k) = \frac{1}{4\pi}\frac{\omega_0^2}{i\omega}\left[E(\omega, k) + \frac{1}{\omega_0^2}\sum \Delta\omega_0^2(\omega-\Omega, k-K)E(\Omega, K)\right]. \tag{124}$$

Maxwell's equations for this field become

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial j}{\partial t} + \vec{\nabla}(\vec{\nabla}\cdot\vec{E}), \tag{125}$$

or, on Fourier-transforming, and using the equation of continuity

$$\left(\frac{\omega^2}{c^2} - k^2\right)\vec{E}(\omega, \vec{k}) = -\frac{4\pi i\omega}{c^2}\vec{j} - \vec{k}(\vec{k}\cdot\vec{E}), \tag{126}$$

or

$$\left(\frac{\omega^2 + \omega_0^2}{c^2} - k^2\right)\vec{E}(\omega, \vec{k}) = \left[\frac{c^2}{\omega^2} - \vec{k}\vec{k}\right]\sum \frac{\Delta\omega_0^2}{\omega^2}(\omega - \Omega, \vec{K}-\vec{k})\vec{E}(\Omega, \vec{K}). \tag{127}$$

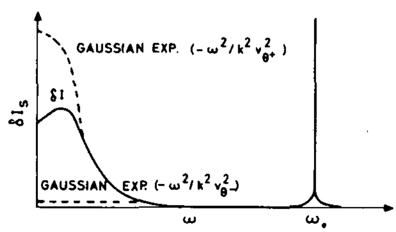

Fig. 1

Scattering of radiation from a plasma: form of δI versus ω.

Now, if there is incident on the plasma a field of frequency Ω and wave number $\vec{K}$ ($K^2 = \Omega^2/c^2$), a scattered wave will be produced given by

$$E(\omega, k) = \frac{(\omega^2 - c^2\vec{k}\vec{k})\,\Delta\omega_0^2(\omega - \Omega, k-K)\vec{E}(\Omega, K)}{(\omega^2 + \omega_0^2)/c^2 - k^2}. \tag{128}$$

For scattered and incident waves well above the plasma frequency, this yields an expression for the intensity at large distances in the form

$$\frac{d^2I_s(\omega', \vec{k}')}{d\omega d\Omega} = I_0(\Omega, \vec{K})\, 4\pi \left(\frac{e^2}{mc^2}\right)^2 |\Delta n(\omega, \vec{k})|^2 [1-(\sin\theta\cos\varphi)^2], \tag{129}$$

where θ, φ are the scattering angles, with respect to incident direction and polarization and the last factor $= \frac{1}{2}(1+\cos^2\theta)$ for an unpolarized incident beam. If the only polarization of the electron density is that produced by random fluctuations Δn is given by Eq. (119), i.e.

$$\Delta n(\omega, k) =$$

$$= \frac{\sum_i (1+G_-)\exp(-i\vec{k}\cdot\vec{x}_i)\,\delta(\omega+\vec{k}\cdot\vec{v}_i) + G_+\left(\frac{e_+}{e_-}\right)\delta(\omega+\vec{k}\cdot\vec{V}_i)\exp(-i\vec{k}\cdot\vec{X}_i)}{1-G_- - G_+} \tag{130}$$

where

$$G_\pm = 4\pi\left(\frac{e^2}{m}\right)_\pm \frac{1}{k^2}\int \frac{\vec{k}\cdot\partial f_0^\pm/\partial\vec{v}}{\omega+\vec{k}\cdot v}\, d^3v \sim \frac{k_D^2}{k^2}. \tag{131}$$

Now, to form $(\Delta n)^2$, the random phase approximation may be invoked whereupon

$$\left|\sum \exp(-i\vec{k}\cdot\vec{x}_i)\delta(\omega+\vec{k}\cdot\vec{v}_i)\right|^2 = \int d^3v\, f(v)\, \delta(\omega+\vec{k}\cdot\vec{v}) \tag{132}$$

and

$$|\Delta n_-(\omega, k)|^2 = N\left\{ \left|\frac{1-G_+}{1-G_- -G_+}\right|^2 \int d^3v\, f_-(\vec{v})\, \delta(\omega+\vec{k}\cdot\vec{v}) \right.$$

$$\left. + \left|\frac{G_-}{1-G_- - G_+}\right|^2 \int d^3v\, f_+(\vec{v})\, \delta(\omega+\vec{k}\cdot\vec{v}) \right\}. \tag{133}$$

The scattered radiation is then given by Eqs. (129) and (133). If $(k_D/k)^2 \gg 1$ several interesting features appear. Since f_+ is much narrower than f_-, the second term dominates near $\Delta\omega = 0$; but since G_- increases with ω, the scattered wave first increases, then decreases, with a half-width determined by $f_+(\omega/d)$; however, at large frequency shifts the first term dominates. When $\Delta\omega = \omega_0$ the dielectric coefficient $(1-G_- - G_+)$ becomes small over a narrow region and a sharp narrow peak corresponding to the emission of plasma oscillation appears.

If $T_- \gg T_+$, ion sound waves may appear, and a sharp peak appears at $(\Delta\omega/\Delta k) \sim (KT_-/m_+)$ from the centre. If $K_D \ll k$ the G's are small, and only the first terms persist.

The interest in this process lies in the possibility of exploring the correlation function directly; the difficulty lies in the small value of the

Thomson cross-section $8\pi(e^2/mc^2)^2 \approx 10^{-25}$ cm^2 which determines the scale of the phenomenon. If because of an instability $\Delta n(\omega, \vec{k})$ becomes very large for some narrow range of $\omega, \vec{k}$ a much more spectacular effect would be expected.

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HIGH-FREQUENCY CONDUCTIVITY AND THE EMISSION AND ABSORPTION COEFFICIENTS OF A FULLY IONIZED PLASMA

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1. INTRODUCTION

Most previous computations of the resistivity of a fully ionized plasma assume the duration of a two-particle encounter to be shorter than any other time in the problem, and, in particular, shorter than any macroscopic time variation. In a plasma this assumption is often violated because the dominant collisional process arises from distant collisions where the encounter duration time can be of the order of the reciprocal plasma frequency, whereas the plasma is capable of sustaining macroscopic oscillations at and above the plasma frequency.

It is the purpose of this paper to investigate by means of a simple model [1, 2] the frequency dependence of the resistivity, and the associated processes of absorption and emission of radiation, for frequencies greater than $\nu_c \equiv 2\pi/t_D$ (t_D is the cumulative 90° deflection time [3]).

The validity of this simple treatment in a large part rests upon the fact that the electron motion is inertia dominated, or in other words, to lowest approximation the current response to the driving $\vec{E}$ field is mainly reactive. Collective effects are properly included and so our results should be valid for all frequencies greater than $\sim \nu_c$ (including the vicinity of the plasma frequency) for which a classical treatment is valid. Most previous results have not treated the collective aspects consistently and so are incorrect particularly near the plasma frequency.

Indeed the validity of this model is borne out by an exact treatment (in the limit of infinite ion mass) stemming from the BBGKY (Bogoliubov, Born, Green, Kirkwood and Yvon) hierarchy [4]. Further, the model has been generalized to include a uniform magnetic field [5] and quantum effects [6], but these generalizations will not be discussed here.

2. PLASMA MODEL AND DERIVATION OF BASIC EQUATIONS

We begin by adopting the Vlasov set of equations to describe the electron dynamics, while the ions are regarded as fixed point scatterers (but not necessarily randomly distributed!). There is, in addition to the self-consistent field, a prevailing spatially uniform electric field oscillating in time at frequency ω. (We take these uniform-field results to apply for long

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wavelength, $k<\omega_p/u_0$, transverse or longitudinal waves in the dipole approximation [7]). In the rest frame of the ions the equations read

$$\frac{\partial F}{\partial t}(\vec{r},\vec{v},t)+\vec{v}\cdot\frac{\partial F}{\partial \vec{r}}-\frac{e}{m}(\vec{E}_0 e^{i\omega t}-\vec{\nabla}\Phi)\cdot\frac{\partial F}{\partial \vec{v}}=0, \tag{1}$$

$$\nabla^2\Phi=4\pi\,[e\int d^3vF-Ze\sum_j\delta(\vec{r}-\vec{r}_j)]\,. \tag{2}$$

Under the transformation

$$\vec{\rho}=\vec{r}+\vec{\epsilon}e^{i\omega t}\,,\ \vec{u}=\vec{v}+i\omega\vec{\epsilon}e^{i\omega t}\,,\ t=t,$$

Eqs. (1) and (2) become

$$\frac{\partial F}{\partial t}+\vec{u}\cdot\frac{\partial F}{\partial \vec{\rho}}+\frac{e}{m}\frac{\partial \Phi}{\partial \vec{\rho}}\cdot\frac{\partial F}{\partial \vec{u}}=0, \tag{3}$$

$$\frac{\partial}{\partial \vec{\rho}}\cdot\frac{\partial \Phi}{\partial \vec{\rho}}=4\pi\,[e\int d^3uF-Ze\sum_j\delta(\vec{\rho}-\vec{\epsilon}e^{i\omega t}-\vec{r}_j)], \tag{4}$$

where $\vec{\epsilon}=e\vec{E}_0/m\omega^2$.

We shall linearize these equations about a spatially uniform Maxwell distribution

$$f_0=(2\pi)^{-\frac{3}{2}}\,u_0^{-3}\,\exp\,(-u^2/2u_0^2), \tag{5}$$

under the assumption that the discrete nature of the ions causes a small perturbation in the electron distribution. The resultant equations for the first-order quantities f, ϕ are

$$\frac{\partial f}{\partial t}+\vec{u}\cdot\frac{\partial f}{\partial \vec{\rho}}+\frac{e}{m}\frac{\partial f}{\partial \vec{\rho}}\cdot\frac{\partial f_0}{\partial \vec{u}}=0 \tag{6}$$

and

$$\frac{\partial}{\partial \vec{\rho}}\cdot\frac{\partial \phi}{\partial \vec{\rho}}=4\pi e\,[n_0\int d^3uf+n_0-Z\sum_j\delta(\vec{\rho}-\vec{\epsilon}e^{i\omega t}-\vec{r}_j)]. \tag{7}$$

In reality, Eq. (7) contains only first-order quantities, since we take the average ion density to cancel the average electron density n_0.

Upon Fourier analysis of the spatial variable we have

$$\frac{\partial f_{\vec{k}}}{\partial t}+i\vec{k}\cdot\vec{u}f_{\vec{k}}+\frac{ie}{m}\phi_{\vec{k}}\,\vec{k}\cdot\frac{\partial f_0}{\partial \vec{u}}=0 \tag{8}$$

and

$$\phi_{\vec{k}}=-\frac{4\pi e}{k^2}\{n_0\int d^3uf_{\vec{k}}-(2\pi)^{-3}\,Z\sum_j\,\exp\,[-i\vec{k}\cdot(\vec{r}_j+\vec{\epsilon}e^{i\omega t})]\}, \tag{9}$$

where $k \equiv |\vec{k}|$. Since we are interested only in the long-time steady-state behaviour, we shall solve these equations under the assumption of vanishing perturbations in the remote past. These equations can be formally integrated to yield

$$f_{\vec{k}} = -\frac{ie}{m}\vec{k}\cdot\frac{\partial f_0}{\partial \vec{u}}\int_{-\infty}^{t} d\tau\,\phi_{\vec{k}}(\tau)\,\exp\left[-i\vec{k}\cdot\vec{u}(t-\tau)\right] \tag{10}$$

and

$$\phi_{\vec{k}} = \omega_p^2 \int_{-\infty}^{t} d\tau(\tau - t)\,\exp\left[-\tfrac{1}{2}k^2u_0^2(\tau-t)^2\right]\phi_{\vec{k}}(\tau) + S_{\vec{k}}(t), \tag{11}$$

where $\omega_p^2 = 4\pi ne^2/m$ and $S_{\vec{k}}(t)$ is the last term on the right-hand side of Eq. (9). In obtaining Eq. (11) we have substituted Eq. (10) into Eq. (9) and made use of Eq. (5) to explicitly carry out the velocity integration.

We are interested in the solution of these equations in the limit $\vec{\epsilon}\to 0$; indeed it is only in this limit that the usual concept of resistivity is defined. However, we must make sure, because of the singular nature of the ion potential, to have properly included the effects for large $\vec{k}$. This large $\vec{k}$ contribution has been investigated and it has been shown that Eq. (16) for the electron-ion interaction, which is derived from the assumption $\vec{k}\cdot\vec{\epsilon} < 1$, is actually valid to lowest order in $\vec{\epsilon}$ for all $\vec{k}$.

Correspondingly we expand the source term,

$$S_{\vec{k}}(t) \cong \frac{Ze}{2\pi^2k^2}\left(1 - i\vec{k}\cdot\vec{\epsilon}e^{i\omega t}\right)\sum_j e^{-i\vec{k}\cdot\vec{r}_j} \tag{12}$$

and then solve Eq. (11). This integral equation is readily solved to yield

$$\phi_{\vec{k}}(t) = \frac{4\pi Ze}{(2\pi)^3k^2}\left[\frac{1}{D(k,0)} - \frac{i\vec{k}\cdot\vec{\epsilon}e^{i\omega t}}{D(k,\omega)}\right]\sum_j e^{-i\vec{k}\cdot\vec{r}_j}, \tag{13}$$

where $D(k,\omega)$ is the dielectric function

$$D(k,\omega) = 1 + \omega_p^2\int_0^{\infty} d\theta\,\theta\,\exp\left(-i\omega\theta - \tfrac{1}{2}k^2u_0^2\theta^2\right). \tag{14}$$

The static term gives the usual Debye shielding while the oscillating term gives the dynamic collective response of electrons to the oscillations of the ions (since we are in the co-ordinate frame oscillating with the electrons).

We have now to compute the average electric field which an ion feels due to the electrons.

$$\vec{\bar{E}}_{ie} = \frac{1}{N_i}\sum_l \vec{E}(\vec{r}_l + \vec{\epsilon}e^{i\omega t}), \tag{15}$$

or

$$\vec{\bar{E}}_{ie} = -\frac{i}{N_i}\sum_l\int d^3k\,\vec{k}\,\phi_k\,\exp\left[i\vec{k}\cdot(\vec{r}_l + \vec{\epsilon}e^{i\omega t})\right]. \tag{16}$$

If now the expression (13) for ϕ is inserted in (16), and again the fact that $\vec{k}\cdot\vec{\epsilon}<1$ is utilized, there results (after dropping terms of quadratic degree or higher in $\vec{\epsilon}$)

$$\vec{E}_{ie} = -\frac{4\pi Ze^2}{(2\pi)^3 N_i m\omega^2}\left\{\int d^3k\,\frac{\vec{k}\vec{k}}{k^2}\left[\frac{1}{D(k,0)} - \frac{1}{D(k,\omega)}\right]\sum_{j,l}\exp\,[i\vec{k}\cdot(\vec{r}_l-\vec{r}_j)]\right\}\cdot\vec{E}_0 e^{i\omega t}\ . \tag{17}$$

The purely static term (independent of $\vec{\epsilon}$) gives no contribution, as it should not, since there can be no net force on the ions due to the electrons, in equilibrium.

Equation (17) applies to a definite ion configuration. We have yet to take the ensemble average of (18) (denoted by $\langle\,\rangle$) over a distribution of configurations. Let us assume for the moment that this has been done. Now, so far we have worked in the oscillating-ion frame, since the ion-electron interaction is most easily computed in this frame. However, the impedance (or conductivity) is most straightforwardly calculated in the ion rest frame. From the equation of motion for the electrons it follows at once that

$$\frac{\partial\langle\vec{J}_e\rangle}{\partial t} = -\frac{e}{m}\,\langle\vec{G}_e\rangle, \tag{18}$$

where $\langle\vec{J}_e\rangle$ is the average current density and $\langle\vec{G}_e\rangle$ is the average total force density on the electrons and is given by

$$\langle\vec{G}_e\rangle = -n_0 e(\vec{E}_0 e^{i\omega t} + \langle\vec{E}_{ie}\rangle), \tag{19}$$

where the term $\langle\vec{E}_{ie}\rangle$ arising from the ion-electron interaction follows from Newton's third law and the invariance of this quantity under the co-ordinate transformation $\vec{r}\rightarrow\vec{\rho}$. We may thus write for the average current density

$$\vec{J}(\omega) = (\omega_p^2/4\pi i\omega)(\vec{I}+\tilde{\vec{\sigma}}_1) \equiv (\tilde{\vec{\sigma}}_0+\vec{\sigma}_1)\cdot\vec{E}, \tag{20}$$

or, since $\tilde{\vec{\sigma}}_1$ is small, for the specific impedance

$$\vec{Z}(\omega) = (4\pi i\omega/\omega_p^2)(\vec{I}-\tilde{\vec{\sigma}}_1). \tag{21}$$

We shall treat now only those situations where the ion distribution is isotropic hence $\vec{\sigma}_1$ is a scalar times the unit dyadic $\vec{I}$. In particular we shall deal now with situations where the ions are in thermal equilibrium.

3. THERMAL EQUILIBRIUM ION CORRELATIONS

For thermal equilibrium we can evaluate the ion density spectrum. That is, we wish to find

$$(2\pi)^6 \langle |n(k)|^2 \rangle = \sum_{l,j} \langle \exp [i\vec{k} \cdot (\vec{r}_l - \vec{r}_j)] \rangle. \tag{22}$$

The terms for $l = j$ sum to N_i, the total number of ions. There are $N_i(N_i - 1)$ terms for $l \neq j$ and they all have the same ensemble average since the ions are identical. We can evaluate one of these terms by making use of the probability of finding ion j at a distance $\vec{r}$ from ion l.

$$\langle \exp [i\vec{k} \cdot (\vec{r}_l - \vec{r}_j)] \rangle = \int d^3r P(r) e^{-i\vec{k}\cdot\vec{r}}. \tag{23}$$

It is well known that in equilibrium the probability of finding a particular ion in a volume element d^3r at a distance r from a given ion is

$$P(r) d^3r = \frac{d^3r}{V} \exp(-Ze\phi/\Theta) \cong \frac{d^3r}{V}(1 - Ze\phi/\Theta), \tag{24}$$

where V is the plasma volume. Here ϕ is the shielded potential of an ion and is given by

$$\phi = (Ze/r) e^{-K_T r}, \tag{25}$$

where K_T is the Debye number for both ions and electrons, $K_T^2 = (Z+1)4\pi ne^2/\Theta \equiv (Z+1)K^2$, and Θ is the temperature in energy units. If we now substitute Eq. (25) in Eq. (24) and Eq. (24) in Eq. (23) we find

$$\begin{aligned} \langle \exp i\vec{k} \cdot \vec{r} \rangle &= \frac{-2\pi Z^2 e^2}{V\Theta} \int\int dr d\theta \sin\theta e^{-(ikr\cos\theta + K_T r)} \\ &= \frac{-4\pi Z^2 e^2}{V\Theta} \frac{1}{k^2 + K_T^2}. \end{aligned} \tag{26}$$

Hence

$$\vec{\sigma}_1 = \frac{4\pi Z e^2}{(2\pi)^3 m\omega^2} \int d^3k \frac{\vec{k}\vec{k}}{k^2} \left[\frac{1}{D(k,0)} - \frac{1}{D(k,\omega)}\right] \left[\frac{K^2 + k^2}{(1+Z)K^2 + k^2}\right]. \tag{27}$$

For this isotropic situation the angular integrations are readily performed to obtain

$$\vec{\sigma}_1 = \frac{2}{3\pi} \frac{Ze^2}{m\omega^2} \vec{I} \int dk k^4 \frac{k^2 + K^2}{k^2 + (1+Z)K^2} \left(\frac{1}{D(k,0)} - \frac{1}{D(k,\omega)}\right). \tag{28}$$

4. LIMITING FORMS

We shall now write down and discuss the limiting forms for $\omega^2 \ll \omega_p^2$ and $\omega^2 \gg \omega_p^2$. Since in this limit, $\omega \gg \nu_c$, the large reactive term dominates the small reactive collisional correction, it is the small ($\sim 1/n\lambda_0^3$) resistive piece we shall discuss.

For $\omega^2 \ll \omega_p^2$, and only to dominant order,

$$\mathrm{Re}\vec{Z}(\omega) = \frac{4\pi\omega}{\omega_p^2} \vec{I} \left\{ \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Ze^2\omega_p^2}{6mu_0^3\omega} \ln\left[\frac{k_{max}^2}{(1+Z)K^2}\right] \right\}. \tag{29}$$

This result is in agreement, as it should be, with the large ω limit of "conventional" kinetic theory (i.e. that derived from Guernsey-Balescu-Lenard kinetic equation [8] under the "Bogoliubov Ansatz" that the decorrelation time for the pair correlation function ($\sim 1/\omega_p$) is short compared to that time, $1/\omega$, in which f changes).

For $\omega^2 \gg \omega_p^2$

$$\mathrm{Re}\vec{Z}(\omega) = \frac{4\pi\omega}{\omega_p^2} \vec{I} \left[\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{Ze^2\omega_p^2}{6mu_0^3\omega} \ln(2k_{max}^2 u_0^2/\omega^2) \right]. \tag{30}$$

Notice that the upper effective impact parameter cut-off has changed from $\sim \lambda_D$ to $\sim u_0/\omega$, showing that collisions at large distances take too long a time compared to the time of oscillation of the field and are hence rendered ineffective. This result (30) leads to the spatial coefficient of energy absorption for transverse waves as given by SCHEUER [9].

For frequencies near the plasma frequency the ion graininess gives rise to the generation of long wavelength longitudinal oscillations. Part of the resistance is due to this effect. For frequencies slightly exceeding the plasma frequency the imaginary part of the integrand, as shown in Fig. 1, exhibits a spike at k_0 due to the excitation of these longitudinal excitations. The value k_0 at which the spike occurs is determined by the vanishing of $\mathrm{Re}D(k,\omega)$. For ω close to ω_p, $D(k,\omega)$ is given approximately by

$$D(k,\omega) \cong \left[1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\,\frac{k^2u_0^2}{\omega^2}\right)\right] - i\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\omega_p^2\omega}{k^3u_0^3} \exp\left(-\omega^2/2k^2u_0^2\right). \tag{31}$$

Hence k_0 is given by

$$k_0^2 \cong (\omega^2 - \omega_p^2)/3u_0^2. \tag{32}$$

The plot of the resistivity versus frequency in Fig. 2 reflects this contribution as a bump just above the plasma frequency. For this problem, where the ion density spectrum is thermal the contribution to the resistance is small. But, if for some reason non-thermal ion correlations exist the wave resistance may be increased many times. See Ref. [2] for a more detailed discussion of this point.

5. EMISSIVITY

The power radiated per unit volume can be computed from the time rate of absorption of transverse wave energy making use of Kirchhoff's law. To do this we find the energy density in transverse waves for thermal equilibrium. We then equate the time rate of absorption to the power emitted.

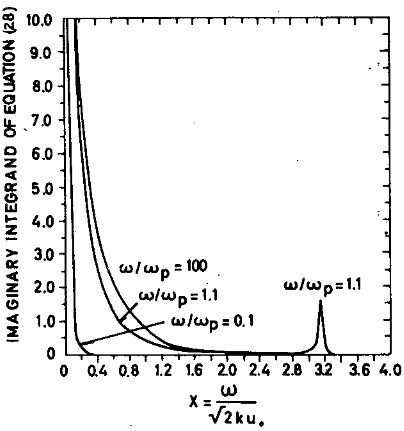

Fig. 1

Imaginary part of integrand Eq. (28)

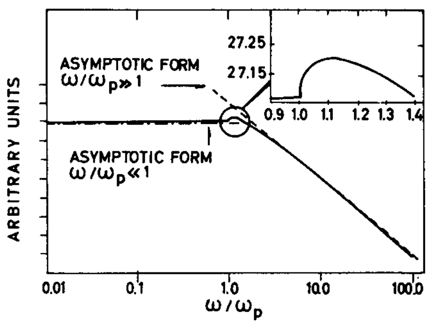

Fig. 2

Resistivity versus frequency

To find the energy density in transverse waves we imagine the plasma enclosed in a volume V. The number of modes in a shell between k and k + dk, when both polarizations are taken into account, is given by

$$dn = \frac{V}{\pi^2} k^2 d\, k. \tag{33}$$

In terms of ω we have

$$dn = \frac{V}{\pi^2} k^2(\omega)(dk/d\omega)d\omega \equiv \rho(\omega)d\omega. \tag{34}$$

Now the dispersion relation for transverse waves neglecting absorption is

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) + 0\left(\frac{u_0^2}{c^2}\right) \tag{35}$$

Neglecting the relativistic corrections we find

$$dn = \frac{V}{\pi^2}\left[(\omega^2 - \omega_p^2)^{\frac{1}{2}}/c^3\right]\omega d\omega. \tag{36}$$

To compute the rate of energy absorption we must now include the effect of the resistance. That is, from Maxwell's equations and the relation $\vec{J} = \vec{\sigma}\cdot\vec{E}$ it follows that

$$k^2 = \frac{\omega^2}{c^2}\left[1 - 4\pi i\sigma(\omega)/\omega\right]. \tag{37}$$

Using (20), we find

$$\omega = \pm(\omega_p^2 + c^2k^2)^{\frac{1}{2}}\left[1 + \frac{\omega_p^2\,\widetilde{\vec{\sigma}}_1}{2(c^2k^2 + \omega_p^2)}\right]. \tag{38}$$

Now twice the imaginary part of (38) gives the reciprocal of the decay time, $\tau(\omega)$, for the energy density in modes with frequency ω.

$$\frac{d\mathscr{E}(\omega)}{d\tau} = -\frac{\mathscr{E}(\omega)}{\tau(\omega)} = -\Theta\rho(\omega)/\tau(\omega). \tag{39}$$

This must just balance the power emitted. Thus

$$P(\omega)d\omega = \Theta\omega_p^2\,\mathrm{Im}\widetilde{\vec{\sigma}}_1(\omega)\ \left[(\omega^2 - \omega_p^2)^{\frac{1}{2}}/\pi^2c^3\right]d\omega. \tag{40}$$

This formula agrees with Scheuer's result [9] for $\omega \gg \omega_p$. Notice the cut-off at the plasma frequency.

In closing we must say that, although we have computed the power emitted, not all this energy may escape from the plasma. Aside from internal absorption much of the energy may be reflected from the surface. This is particularly true near the plasma frequency. However, if the plasma density goes to zero slowly as one approaches the boundary, one may expect that reflection can be neglected and the total radiation computed from (40) and the absorption coefficient per unit length.

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CYCLOTRON RADIATION

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I. INTRODUCTION

The problem of cyclotron radiation from a thermonuclear reactor can be divided into two parts. First, we leave the question of how much energy is radiated in cyclotron radiation from a reactor of a given geometry and electron temperature. This is basically a physics question and can be obtained directly. Second, we have the question of how large a reactor must be in order that the radiation loss, which is essentially a surface loss, be no bigger than the thermonuclear energy production which is a volume effect. This second question involves assumptions about how a thermonuclear reactor could be operated which are somewhat "economic" in character. The best we can do at this time is to make estimates based upon a set of "optimistic" assumptions and upon a set of "pessimistic" assumptions and thus try to bracket the actual results.

The radiation from a single electron moving in its cyclotron orbit around the magnetic field can be obtained by calculating the vector potential, $\vec{A}_\omega(r)$.

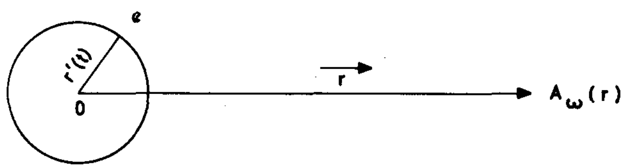

Fig. 1

Geometry for calculation of radiation of an electron moving in a Larmor orbit

Referring to Fig. 1 we have

$$\vec{A}_\omega(r) \propto \frac{e}{r} \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, e^{i\vec{k}\cdot\vec{r}'(t)}\, \vec{v}(t) \tag{1}$$

where $A_\omega(r)$ is the vector potential at r for radiation at frequency, ω, $\vec{k}$ is the propagation vector in the direction of $\vec{r}$, $\vec{r}'(t)$ is the position of the electron in its orbit at time, t, and $\vec{v}(t)$ is the velocity of the electron, e.g.,

$$\vec{r}'(t) = \frac{v}{\Omega}(\vec{1}_x \cos\Omega t + \vec{1}_y \sin\Omega t)$$

$$\vec{v}(t) = v(-\vec{1}_x \sin\Omega t + \vec{1}_y \cos\Omega t) \tag{2}$$

where Ω is the electron cyclotron frequency, and $\vec{v}$ is the electron velocity.

In the dipole approximation $\vec{k}\cdot\vec{r}' \ll 1$ is neglected and we see from Eq.(1) that we only have radiation at the cyclotron frequency, Ω. However, if we keep the term $\exp[i\vec{k}\cdot\vec{r}(t')]$ we can expand it in multiples of the cyclotron frequency, e.g.,

$$\exp\left(i\,\frac{kv_0}{\Omega}\right) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{kv_0}{\Omega}\right) e^{in\omega t},$$

and we have radiation at all harmonics of the cyclotron frequency. However, the amplitude of the n^{th} harmonic is of the order of $J_n\,(\omega v/\Omega c) \propto (v/c)^n \ll 1$ so that the energy drops off radically with increasing harmonic number.

If we now have a Maxwellian distribution of electrons the total energy radiated per unit volume, $S(\omega)$, can be found by integrating the energy radiated by a single electron of velocity $\vec{v}$ over the Maxwellian distribution.

TRUBNIKOV [1] was the first to point out that this total energy radiated per unit volume, $S(\omega)$, was greater than the expected energy production per unit volume by D-D (deuterium-deuterium) thermonuclear reactions. He further pointed out that since the expected electron temperatures were in the range of 50-100 keV the cyclotron frequency of the electrons would be shifted by the relativistic change in mass so that after averaging over a distribution of electron velocities the radiated energy would not be simply at multiples of the cyclotron frequency but, at the higher multiples, would be smeared out into a continuum.

The only way that a thermonuclear reactor can thus produce more energy than is lost by cyclotron radiation is for the reactor to be big enough so that most of the cyclotron radiation is absorbed within the plasma, i.e., the plasma must be at least as thick as the mean free path for the absorption of the cyclotron radiation. Thus the essential result that is to be obtained is the "critical size" of a reactor which is the size at which the cyclotron radiation loss is just balanced by the thermonuclear energy production.

II. RADIATION LOSS

In deriving the radiation loss, Trubnikov and Kudryavtsev assumed that the electrons in the plasma radiate as though they were in a vacuum; this assumption has been the subject of some controversy. In this paper we examine this question by solving the relativistic Boltzmann equation and conclude that for the frequency range of interest, i.e. for frequencies large compared to the plasma frequency, the electrons do indeed radiate as though they were in a vacuum.

In the plasmas of interest the quantity

$$\beta_H = \left[\frac{nK(T_e + T_i)}{B_0^2/8\pi}\right] \tag{3}$$

is a number which varies from 0.1 to 1. Here n is the electron (or ion) density, K is the Boltzmann constant, T_e and T_i are the electron and ion temperatures, respectively, and B_0^2 is the static magnetic field. This can be re-written as

$$\frac{\omega_p^2}{\omega_c^2} = \beta_H \frac{mc^2}{2K(T_e + T_i)}, \tag{4}$$

where $\omega_p^2 = 4\pi ne^2/m$ is the electron plasma frequency, e is the electronic charge, m is the electronic mass, c is the speed of light, and $\omega_c = eB_0/mc$ is the electron cyclotron frequency. We shall be interested in radiation at harmonics of the cyclotron frequency, i.e., $\omega = m\omega_c$; thus,

$$\beta_H \frac{mc^2}{2K(T_e + T_i)} = \frac{\omega_p^2}{\omega^2} m^2. \tag{5}$$

In situations of interest, the left-hand side of Eq. (5) is usually less than 1, and thus the inequality

$$\omega_p^2 \ll \omega^2 \tag{6}$$

is satisfied for all but the lowest harmonics. As will be seen below, this materially simplifies the calculation of the cyclotron radiation.

The collisionless Boltzmann equation is an exact result of the Liouville theorem in the limit that $e = m = 1/n = KT = 0$, with e/m, ne, and nKT remaining infinite. This is the fluid limit and there can be no individual-particle radiation in this limit. If one regards e, m, 1/n, and KT as small quantities of the same order (we denote the order of any of these by g), individual-particle effects such as cyclotron radiation enter when we consider quantities of first order in g.

The straightforward way to calculate the cyclotron radiation from an individual particle in a plasma is to introduce the concept of a test charge. The test charge is considered to move with a prescribed orbit in the plasma, and thus it provides a current source for radiation. However, there will be an additional current due to the reaction of the plasma to the test charge. This reaction current includes two-particle correlations as well as the fluid response of the plasma. The test charge will be surrounded by a co-moving current cloud of opposite sign, and the radiation may be considerably reduced. However, we shall show below that radiation at harmonics for which the inequality (6) is satisfied is essentially unaltered. The reason for this is that the radius of the transverse shielding current is of the order of c/ω_p and thus will not materially affect radiation at wave-lengths, λ, that are much less than c/ω_p.

The source function, $S(\omega,\Omega)$, is defined as the energy radiated per unit solid angle per unit frequency, per unit volume, per second. To obtain this, one simply calculates the energy radiated by a test charge and averages over a Maxwellian distribution of test charges. In terms of our small quantities, this is of order $ne^2 = g$. However, the same results can be accomplished in a simpler way by calculating the absorption length, $\alpha(\omega,\Omega)$, to zero order in g from the collisionless Boltzmann equation. Then $S(\omega,\Omega)$ is obtained from Kirchoff's law $S(\omega,\Omega) = I_{RJ}\ \alpha(\omega,\Omega)$, where $I_{RJ} = KT\omega^2/8\pi^3c^2$

(the Rayleigh-Jeans distribution) is of first order in g. This is the procedure we use to obtain $S(\omega, \Omega)$.

To obtain the radiation intensity in a given direction with a given polarity in an infinite slab of thickness L we need only solve

$$\frac{dI}{ds} = - \alpha I + S = - \alpha (I - I_{RJ}) \tag{7}$$

subject to the boundary conditions I(1) = 0, see Figs. 2a and 2b.

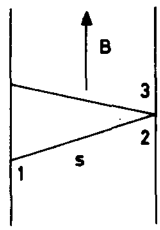

Fig. 2a

Geometry of slab

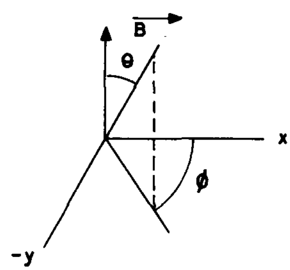

Fig. 2b

Co-ordinate system showing the propagation vector at angle θ relative to the magnetic field

Here s is a co-ordinate along the direction of propagation, $s(x) = x / \sin\theta\cos\phi$. $S(\theta, \omega)$ is the source function for the frequency ω and is related to the absorption coefficient $\alpha(\theta, \omega)$ by Kirchoff's law $S(\theta) = \alpha(\theta) I_{RJ}$, ($I_{RJ} = (KT)\omega^2 / 8\pi^3 c^2$), which is valid provided that the polarization we are considering is an eigenmode for propagation at this angle. Integrating Eq.(7), we obtain for x = L

$$I(\omega, \theta, \phi) = I_{RJ} \{1 - \exp[-\alpha s(L)]\}. \tag{8}$$

The total radiation/unit area leaving the slab in the polarization is thus given by

$$W = \int_0^\infty d\omega \int d\Omega \hat{k} \cdot \hat{n} I(\omega, \theta, \phi) \tag{9}$$

where $\hat{k}$ and $\hat{n}$ are unit vectors in the direction of propagation and normal to the surface respectively, and the integration is carried over all outwardly directed $\hat{k}$.

Now $\alpha(\omega)$ turns out to be a rapidly varying function of ω which is large for small ω and very small for large ω. Thus as a function of ω, $\exp[-\alpha(\omega)S] \cong 0$ for all ω up to some critical frequency, ω^*, and then rises immediately to 1 for $\omega > \omega^*$. Thus, as Trubnikov has pointed out

$$I(\omega) \cong I_{RJ} \text{ for } \omega < \omega^*$$

$$\cong 0 \text{ for } \omega > \omega^*$$

and the critical frequency ω^* is defined by

$$\alpha(\omega^*)L \cong 1.$$

The problem is thus to find $\alpha(\omega)$. This is done as follows. From Maxwell's equations we have, for the propagation of waves,

$$\left|(s^2 + c^2k^2)\overleftrightarrow{I} - c^2\vec{k}\vec{k} + 4\pi s\overleftrightarrow{\sigma}\right| = 0. \tag{10}$$

Here $s = -i\omega-\gamma, \omega$ is the frequency of the wave, γ is the damping constant (in time), $\overleftrightarrow{I}$ is the unit dyadic, $\vec{k}$ is the propagation vector of the wave, c is the velocity of light, and $\overleftrightarrow{\sigma}$ is the conductivity tensor of the plasma and is obtained from the relativistic Vlasov equation

$$\sigma_{ij} = -e^2 \int d^3p \frac{p_i}{\sqrt{1+p^2}} \int_{-\infty}^{0} dt' \exp[st' + i\vec{k}\cdot\vec{r}(t')] \frac{\partial}{\partial p_j(t')} f_0\,(p^2(t')) \tag{11}$$

where $\vec{p}$ is the electron momentum, $f_0(p^2)$ is the relativistic distribution function $f_0 = (\mu n/4\pi K_2(\mu)) \exp -\mu(1+p^2)^{\frac{1}{2}}$ with $\mu = mc^2/KT$, K_2 is the usual Bessel function and

$$\vec{p}(t') = \vec{p}\,(\phi - \omega_0 t')$$

$$\omega_0 = \frac{\Omega}{\sqrt{1+p^2}} \tag{12}$$

$$\Omega = \frac{eB}{mc}$$

$$\vec{r}(t') = \int_0^{t'} \frac{\vec{p}(t'')}{\sqrt{1+p^2}}\, dt''$$

and for convenience we have set $m = c = 1$ in $\overleftrightarrow{\sigma}$.

It is convenient to express the dispersion relation, Eq. (10), in a co-ordinate system the axes of which are along $\vec{k}$, $\vec{k}\times\vec{B}$, and $\vec{k}\times(\vec{k}\times\vec{B})$. $\vec{B}$ is assumed to lie along the z axis and without loss of generality we take k to lie in the x, z plane with $k_{\parallel} = k\cos\theta$, $k_{\perp} = k\sin\theta$ (see Fig. 3).

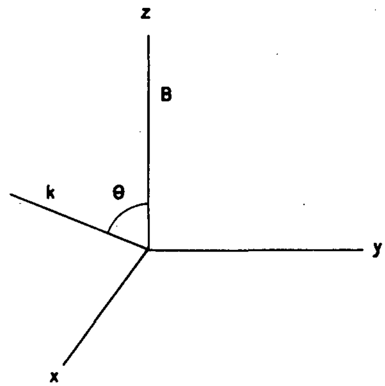

Fig. 3

Co-ordinate system showing the propagation vector at angle θ relative to the magnetic field

In the $\vec{k}$-co-ordinate system Eq. (10) becomes

$$\begin{vmatrix} s^2 + 4\pi s\sigma_{11} & 4\pi s\sigma_{12} & 4\pi s\sigma_{13} \\ 4\pi s\sigma_{21} & s^2 + c^2k^2 + 4\pi s\sigma_{22} & 4\pi s\sigma_{23} \\ 4\pi s\sigma_{31} & 4\pi s\sigma_{32} & s^2 + c^2k^2 + 4\pi s\sigma_{33} \end{vmatrix} = 0 \qquad (13)$$

As discussed in [3], $4\pi s\sigma_{ii} = 0(\omega_p^2)$ and $s^2 + c^2k^2 = 0(\omega_p^2)$. Thus the 11 element of Eq. (13) is of order c^2k^2 while all other elements are of order ω_p^2. We are interested only in $c^2k^2 \gg \omega_p^2$. Thus neglecting $\omega_p^2/c^2k^2 \ll 1$, Eq. (13) becomes

$$(s^2 + 4\pi s\sigma_{11})\,[(s^2 + c^2k^2 + 4\pi s\sigma_{22})\,(s^2 + c^2k^2 + 4\pi s\sigma_{33}) - (4\pi s)^2\sigma_{23}\sigma_{32}] = 0. \qquad (14)$$

The term in square brackets is associated with the transverse waves while $s^2 + 4\pi s\sigma_{11}$ is associated with longitudinal waves. The dispersion relation for transverse waves is thus

$$(s^2 + c^2K^2 + 4\pi s\sigma_{22})\,(s^2 + c^2k^2 + 4\pi s\sigma_{33}) - (4\pi s)^2\,\sigma_{23}\sigma_{32} = 0 \qquad (15)$$

and thus $s = \pm\, ick + 0(\omega_p^2)$.

Since $4\pi s\sigma_{ij} = 0(\omega_p^2)$ we can obtain $4\pi s\sigma_{ij}$ correct to first order in (ω_p^2/c^2k^2) by using $s = s_0 = -ik$ wherever s appears in $4\pi s\sigma_{ij}$. Eq. (15) can then be solved for s correct to first order in (ω_p^2/c^2k^2). We obtain

$$(s^2 + c^2k^2) = -2\pi s_0 \left[(\sigma_{22} + \sigma_{33}) \pm \sqrt{(\sigma_{22} - \sigma_{33})^2 + 4\sigma_{23}\sigma_{32}}\right].$$

Now $\alpha_\pm = 2\gamma_\pm/c$ and thus

$$\alpha_\pm = \frac{2\pi}{c} \operatorname{Re} \left[(\sigma_{22} + \sigma_{33}) \pm \sqrt{(\sigma_{22} - \sigma_{33})^2 + 4\sigma_{23}\sigma_{32}}\right] \tag{16}$$

where $\sigma_{ij} = \sigma_{ij}(s_0)$.

The integrals defining $\overleftrightarrow{\sigma}$ are extremely complex numerically and we will not carry them out here. However the dependence of $\overleftrightarrow{\sigma}$ on the parameters, β_e and B is easy to obtain.

From Eq. (16) we have

$$\alpha \propto \sigma \propto \frac{ne^2}{\Omega}\, g\left(\frac{\omega}{\Omega}, \frac{KT_e}{mc^2}\right) \tag{17}$$

where $g(m, T_e)$ is numerically obtained. As a function of the harmonic number, $m = \omega/\Omega$, and the electron temperature, T_e. In what follows here the notation $g(m, T_e)$ will be used for any function of the variables m and T_e.

The condition $\alpha(\omega^*)L = i$ is thus of the form

$$\alpha(m^*, T_e)L = \frac{ne^2}{\Omega} L\, g(m^*, T_e) \tag{18}$$

$$= \beta_e BL\, g(m^*, T_e) = 1.$$

Solving this for m* thus yields the fact that m* is a function of T_e and the combination $\beta_e BL$

$$m^* = m^*(T_e, \beta_e BL). \tag{19}$$

Numerically for electron temperatures of interest, i.e., approximately 50 keV

$$m^* \cong 0.5\sqrt{\beta_e BL}. \tag{20}$$

Thus we are able fairly easily to determine the parameter dependence of the critical frequency and thus the cyclotron radiation loss/unit area, $R = W_+ + W_-$

$$R = 2\pi \int_0^{\omega^*} I_{RJ}\, d\omega = \frac{KT}{12\pi^2c^2}\, \Omega^3 m^{*3}. \tag{21}$$

III. THERMONUCLEAR ENERGY PRODUCTION

To determine the net thermonuclear energy production in a thermonuclear reactor requires some essentially "economic" assumption.

For a D-D reactor we first note that approximately half of the D-D reactions lead to the production of T and approximately half lead to the production of He^3. The T, because of the large D-T cross-section reacts immediately while the He^3, since the D-He^3 cross-section is only moderate, will not react immediately. For continuous operation, an equilibrium amount of He^3 will be present. This will also happen for pulsed operation if the He^3 is separated from the exhaust products and re-injected.

The rate at which thermonuclear energy is released is given by

$$\begin{aligned} Q_{DD} &= \frac{1}{2} n_D^2[\Sigma_1(E_1 + E_3) + \Sigma_2 E_2] + n_D n_{He^3} \Sigma_4 E_4 \\ &= \frac{1}{2} n_D^2[\Sigma_1(E_1 + E_3) + \Sigma_2(E_2 + E_4)] \end{aligned} \tag{22}$$

where $1/2\, n_D^2 \Sigma_1$ is the D-D reaction rate to produce T, E_1 is the energy released in charged particles by this reaction, E_3 is the energy released in charged particles by the almost instantaneous subsequent D-T reaction, and E_2 and E_4 are the energies released in charged particles by the D-D reaction producing He^3 and by the D-He^3 reaction respectively.

We shall assume that a certain fraction, f, of the total energy loss, L, from the machine, i.e., bremsstrahlung, cyclotron radiation, and neutron energy, is returned to the machine and further we shall assume that it is all returned to the ions. This seems quite possible.

In the simplest case where f = 0 the energy balance for the ions yields

$$Q_{DD} f_g - Q_{ie} = 0 \tag{23}$$

where f_g is the fraction of Q_{DD} which is given directly to the ions, Q_{ie} is the energy transfer/vol from ions to electrons by elastic collisions.

Since Q_{DD} and Q_{ie} are both proportional to n^2, and since Q_{DD} is a function only of T_i and Q_{ie} is a function of T_e and T_i, Eq. (23) yields the equilibrium ion temperature, T_i, for a given electron temperature, T_e.

For the electrons the energy balance equation is

$$Q_{DD}(1 - f_g) + Q_{ie} - SR/V - Q_B = 0 \tag{24}$$

where S is the surface area of the reactor, V is the reactor volume, i.e., L = V/S, and Q_B is the bremsstrahlung radiation loss.

Since from Eq. (23) we can obtain T_i for a given T_e, Eq. (24) can be written as

$$R = Ln^2 g(T_e). \tag{25}$$

Equating this to the R obtained from Eq. (21) yields

$$Ln^2 = \Omega^3 m^{*3} g(T_e) \tag{26}$$

which can be written as

$$\beta_e = \frac{m^{*3}}{\beta_e BL} g(T_e). \tag{27}$$

This is the critical equation for the reactor. The function $g(T_e)$ has a rather sharp minimum for $T_e = 50$ keV and so to minimize L one should choose to operate the reactor in this range of T_e. Using Eq. (20) for m^{*3} we thus obtain

$$L = \frac{0.25\ g^2(T_e)}{\beta_e^3 B}. \tag{28}$$

We note the very strong dependence of L on β_e and thus from the point of view of cyclotron radiation, we would like to keep β_e as high as possible. On the other hand, in many confinement schemes the total $\beta = \beta_e + \beta_i$ plays a critical role in stability considerations and there is often a critical β which must not be exceeded if stability is to be maintained. Thus we are limited to rather small β.

To illustrate the results, we shall calculate L for a slab, $T_e = 50$ keV, $B = 10^5$ G, a total β of 0.10 and 0.40, f = 1/3, 0 and (case I) including the He^3 reactions and (case II) omitting the He^3 reactions.

It thus appears that for $\beta = 0.40$ that the critical size is quite reasonable for all cases except II 0, and that even for $\beta = 0.10$ case $I_{1/3}$ yields a reasonable critical size without a reflector. If reflectors are used at the plasma surface the critical L is reduced by a factor of 1-R, where R is the reflectivity of the surface. It seems quite possible to build reflectors with a reflectivity R of between 0.90 and 0.99. For R = 0.90, the critical

TABLE I

ILLUSTRATION OF RESULTS

	$\beta = 0.10$	$\beta = 0.40$
Case I	$L_{1/3} =$ 141 cm L_0 = 1 068 cm	2.20 cm 16.7 cm
Case II	$L_{1/3} =$ 776 cm L_0 = 20 500 cm	12.1 cm 320 cm

size is reduced by a factor of 10, and this appears to be more than adequate for all cases except II 0, which would require a reflectivity of 0.99 to yield a reasonable critical size. The critical sizes for cylinders are larger by a factor of about 2.5.

ACKNOWLEDGEMENTS

The pioneer work in this field was done by TRUBNIKOV [1, 2]. The derivation given in the present paper follows the later work of DRUMMOND and ROSENBLUTH [3].

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II

APPLICATIONS

MAGNETOHYDRODYNAMIC GENERATORS

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Magnetohydrodynamic (MHD) generators [1, 2] may be regarded as an attempt to find practical applications of magnetic forces on ionized gases at the lowest possible temperatures. A sketch of one elementary MHD generator configuration is shown in Fig. 1. Heated gas flows through the channel across magnetic field lines which are perpendicular to the plane of the drawing. A vertical EMF is induced which causes current to flow between the electrodes and through the load. Thus, as the gas expands through the channel, it does work on the magnetic field and produces electric power.

The primary energy source for heating the gas may be either ordinary combustion or fission. For central power station using combustion the efficiency of a conventional steam cycle is limited by the upper temperature at which rotating machinery can operate. MHD offers the advantage that energy can be extracted from the gas at temperatures above those which can be withstood by solids. The chamber walls may always be cooled below the gas temperature, as is done in a conventional rocket. Suggested designs use an MHD generator for the upper part of the temperature range, i.e. from the combustion temperature down to the temperature at which the conductivity becomes too low for effective interaction with the magnetic field. At this temperature there is still significant heat energy in the gas so that a steam cycle is used to extract the remaining energy.

For the case of a fission reactor as the prime energy source it seems possible to develop solid fuel elements which can have a long life time at temperatures greater than those at which rotating machinery can operate. Thus the MHD generator can again raise the upper cycle temperature. The temperatures in this case are lower than for combustion; however, somewhat higher gas conductivities can be achieved at any given temperature since it is possible to choose a working gas such as argon which has a much lower elastic collision cross-section for electrons than combustion products do. An interesting possibility, which will be discussed a little more fully below, is the question of achieving greater than equilibrium ionization in the gas. This would allow higher electrical conductivities at the same gas temperature and thus decrease the temperature requirements on the fuel elements.

It has also been suggested that the solid temperature limit might be avoided entirely if the fissionable material were used in a gaseous state. While this is certainly a highly speculative suggestion, the plasma physics problems related to such a device would still be much simpler than those associated with fusion. The gas would be primarily contained by cooled walls and would simply expand through a magnetic field to extract energy.

A rather gross picture of progress in the development of MHD generators can be obtained from the following history. In 1959 ROSA [3] was able to obtain 11 kW of electrical power from a small generator channel using a gas heated by an electric arc. The efficiency of this channel was extremely low due to wall losses; however, the performance was in agreement with theoretical calculations. Since wall losses increase roughly as the wall area but MHD power extracted goes as the volume of the channel, the efficiency should increase with increasing size and high efficiency is only expected at large sizes. In 1962 LOUIS et al. [4] were able to produce 1.5 MW of electrical power from a generator channel using combustion gases. Again operation in agreement with theoretical prediction was achieved not only in terms of gross power output but also for pressure, voltage and current distribution within the channel. This experiment still did not produce net power since the power required for the magnetic field was greater than the power produced in the channel. During 1964 a self-excited (i.e. the power for the magnetic field was provided by the generator itself) generetor operation has been achieved with a gross output of 11 MW by MATTSON et al. [5].

For simplicity in design all of the above experiments were run only for short times of 3 min or less. Concurrently work on the development of both insulating walls and electrodes with long endurance times has been carried out. So far endurance times in excess of one week have been achieved and further development looks promising [1].

The above brief summary of the technical development suggests that the prospects for solution of the scientific and development problems are very promising. Evaluation of the economic potential therefore has been considered appropriate. Full scale power plants using combustion are expected to obtain efficiencies greater than 50% as compared to the maximum of 42% which has been achieved after many years of development of steam cycles. Overall cost estimates appear to be competitive with steam cycles and, because of the increased efficiency, are particularly attractive in the high-fuel-cost areas of the world. It should be borne in mind that fuel and capital costs are usually comparable and thus order-of-magnitude decreases in power costs would seem improbable with any system.

In addition to direct competition with steam cycles for central station power sources, the MHD generator appears attractive for cases when short duration pulses of high power are of interest. In this situation efficiency is not of prime importance, and thus the MHD generator can be used without the steam cycle for the lower temperature range. The capital costs for the MHD channel are considerably lower than those of turbines, so considerable cost reduction seems possible. The prospects in conjunction with fission are still hard to estimate.

Let us turn now to a discussion of some of the physics associated with MHD generators. It should be apparent from the success of the experiments mentioned above that the basic physical principles are much more clearly understood in this case than they are in the higher temperature plasmas which are of interest for fusion and in space. In what follows we shall discuss some aspects of the basic plasma physics with particular emphasis on those areas in which problems still remain. In a very gross sense these problems arise when the product of electron cyclotron frequency and mean free time exceeds unity by some factor. This might be regarded as the be-

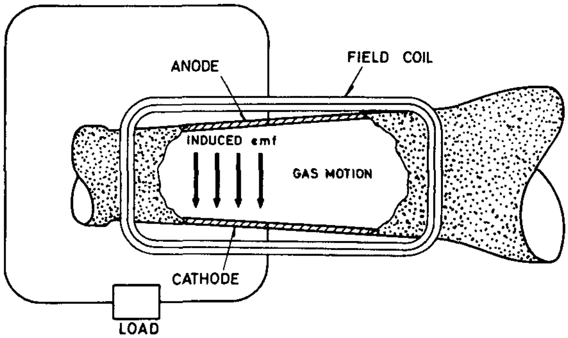

Fig. 1

Line diagram showing principal elements of the basic DC MHD-generator configuration

ginning of a collision-free regime in that magnetic effects have become comparable with collisional effects. It is, however, not clear whether the difficulties are associated with inherent instabilities or merely with great sensitivity to small non-uniformities in the flow.

The basic operation of a channel such as the one sketched in Fig. 1 can be analysed by using the scalar Ohm's law:

$$J = \sigma\left(E + \frac{vB}{c}\right) = \sigma(1-\eta)\frac{vB}{c}, \tag{1}$$

where we have written the electric field which results from the voltage across the load as $-\eta vB$. η may be regarded as an electrical efficiency since it is the ratio of the power delivered to the load $(\vec{E}\cdot\vec{J})$ to the work done by the gas $\vec{v}\cdot(\vec{J}\times\vec{B}/c) = \vec{J}\cdot(\vec{v}\times\vec{B}/c)$. Since the gas conductivity, and therefore the gas currents, are low the magnetic field may be regarded as that determined by the external field coil and is not affected by the gas currents. The velocity of the flow in the channel can vary however; for a rough estimate of the operation, we will assume it a constant. In order to extract a significant fraction of the energy the total magnetic force along the length L of the channel must be comparable to the pressure:

$$\frac{JBL}{c} = \frac{\sigma(1-\eta)vB^2L}{c^2} \approx p. \tag{2}$$

Since wall losses increase with L, high conductivities, flow velocities and magnetic field strengths are desirable. The product σv has a maximum for a given stagnation temperature since it is zero at zero velocity and at high velocities. (The gas temperature, and therefore the conductivity, are sharply reduced since the sum of the thermal and kinetic flow energies remains constant.) The optimum product generally occurs in the neighbourhood of a flow speed comparable to the speed of sound.

Electrical conductivity at low temperature is attained by the addition of small amounts of seed elements such as potassium or caesium which have low ionization potentials. At equilibrium the degree of ionization can be calculated from the Saha equation and is typically less than 0.1%. The conductivity can then be calculated in terms of collision cross-sections for the various species in the gas. For seeded combustion gases conductivities of 3 mhos/m at 2400°K and 100 mhos/m at 3000°K have been obtained and are in agreement with prediction [6]. At a conductivity of 100 mhos/m, a velocity of 10^5 cm/s, a magnetic field of 3×10^4 G and $\eta = 0.75$, a pressure change of about one atmosphere is obtained in a length of about one metre, Eq.(2).

The desirability of high magnetic field strengths indicated by Eq.(2) leads to interest in values of the product of electron cyclotron frequency Ω and mean free time τ greater than unity. In this case the Hall effect becomes significant and the tensor form of Ohm's law must be used. If one considers only cases with $\Omega\tau < 10$ one may assume that the ion and neutral velocities are equal and Ohm's law takes the form

$$\vec{J} = \sigma\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right) + \vec{J}\times\vec{\Omega}\,\tau, \tag{3}$$

where $\vec{\Omega}$ has the magnitude of the cyclotron frequency and the direction of the magnetic field. If we now require that the current flow only across the channel and has no component in the flow direction Eq.(3) shows that the current across the channel is still given by Eq.(1) but an electric field in the flow direction is required whose magnitude is

$$E_{\text{Hall}} = -\Omega\tau(1-\eta)vB. \tag{4}$$

For values of $\Omega\tau$ in the neighbourhood of unity and bearing in mind that the channel is larger than it is wide, this field results in potentials along the channel which are significantly larger than those across the channel. If the electrodes were made of continuous conductors as indicated in the sketch in Fig. 1, they would short-circuit this potential, resulting in a decrease in current across the channel. In actual practice, therefore, electrodes are made of strips which go across the channel but are electrically isolated from one another in the flow direction. Each anode-cathode pair has its own load. Some local shorting of the axial field still exists over a region whose size is of the order of the width of the individual electrode strips.

The existence of this axial field leads to another possible generator configuration, which is commonly called a Hall current generator as opposed to the configuration in Fig. 1 which is referred to as a conduction current generator. In a Hall generator the load shown in Fig. 1 is short-circuited and additional electrodes are inserted at the upstream and downstream ends of the channel. These electrodes are then connected through the load. The generator then works on the axial potential developed by the Hall effect. A disc geometry is sometimes convenient for a Hall generator in order to avoid electrode effects. In this case a radial flow across an axial magnetic field is used. The required shorting in the $\vec{v}\times\vec{B}$ direction is then achieved by symmetry and does not require electrodes. The load is connected between electrodes that are separated radially.

Considerable improvement in generator performance could be achieved if the electrical conductivity could be raised above thermodynamic equilibrium values. This could be accomplished if the electron temperature were raised above the gas temperature. The degree of ionization would then tend to be in equilibrium with the electron temperature rather than with the gas temperature since ionization and recombination are determined predominantly by electron-atom collisions. A crude estimate of the conditions required can be obtained by matching the rate at which electrons are heated by joule dissipation to the rate at which they loose energy to atoms by collisions:

$$\frac{J^2}{\sigma} = N_e \frac{m_e}{m_i} k \frac{(T_e - T_n)}{\tau} . \tag{5}$$

The energy loss has been written assuming elastic collisions, i.e. the energy transfer per collision is equal to the mass ratio between electrons and ions. This is only applicable in a monatomic gas. For a diatomic gas or combustion products the energy transfer is much more rapid due to excitation of rotational and vibrational degrees of freedom. As a result raising the electron temperature significantly only seems feasible in a monatomic gas. Using Eq.(1) and the kinetic theory expression for σ, Eq.(5) can be rewritten

$$\frac{T_e - T_n}{T_n} \approx \frac{m_n v^2}{kT_n} (1-\eta)^2 (\Omega\tau)^2 . \tag{6}$$

The ratio of flow energy per particle $m_n v^2$ to thermal energy kT_n cannot greatly exceed unity without decreasing the temperature of the neutrals excessively. Eq.(6) therefore requires that $\Omega\tau$ be somewhat above unity if noticeable departures of electron temperature from neutral temperature are to be achieved at reasonably high values of electrical efficiency η.

Recent experiments [7] suggest that significant non-equilibrium conductivity has been achieved although the interpretation is not entirely unambiguous. They used seeded argon gas heated in a graphite heater to about 2000°K flowing through the disc geometry indicated above. They were then able to obtain a voltage/current curve by varying the load. If the conductivity in the channel were independent of the current flowing the voltage/current curve should be linear, corresponding to an ordinary load line with the generator - a fixed circuit parameter. However, the observed load line was curved, corresponding to a higher conductivity when the current was highest, as expected. The slope of the curves would indicate increases in conductivity up to a factor of ten. However, oscillations which were observed to occur at the high conductivities may invalidate the quantitative estimate.

The performance of generator channels sometimes appears to be degraded [3] at large values of $\Omega\tau$. Whether this is due to instabilities or to extreme sensitivity to non-uniformities already present in the flow is not clear. It was observed by ROSA [3] that non-uniformities in the plane defined by the magnetic field and the electric field (as seen in a coordinate system moving with the gas) could lead to a factor of two change in joule dissipation when the fractional change in conductivity was of the order of $(\Omega\tau)^{-2}$. Thus, at high $\Omega\tau$, extreme uniformity is required to obtain predicted performance. It is interesting to note that a similar process was suggested under considerably different conditions as a partial explanation of anomalous diffusion in a low density discharge by ROSE and YOSHIKAWA [8].

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INTRODUCTION TO CONTROLLED THERMONUCLEAR RESEARCH

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I. INTRODUCTION

The greatest stimulus to the study of plasmas has been given by the hope of controlling the release of nuclear energy by the fusion of light nuclei. Both light and heavy nuclei are rich in energy, as compared to those of intermediate mass, and it is energetically as possible to tap the potential energy available in light nuclei by fusing them as to tap that of heavy nuclei by splitting them. Furthermore, the maximum cross-sections for the exothermic reactions between the isotopes of hydrogen are high:

$$D + D \rightarrow He^3 + n + 4.03 \text{ MeV}$$

$$D + D \rightarrow T + p + 3.27 \text{ MeV},$$

$\sigma_{DD} \simeq 0.2$ barns at 1 MeV.

$$D + T \rightarrow He^4 + n + 17.6 \text{ MeV},$$

$\sigma_{DT} \simeq 4$ barns at 100 keV.

Nuclear fusion has two advantages over fission as an energy source: firstly, the fuel is universally available and effectively inexhaustible, since D forms 10^{-4} of natural hydrogen and its separation presents no serious problems; secondly, a fusion reactor should be safe. Although fairly large amounts of energy would be stored in such a device, the hazard presented by long-life radioactive and chemically poisonous fission fragments is absent and a D-D reactor could be made completely free of radioactive risk.

A thermonuclear reactor based on D-T is somewhat less attractive, since it would burn lithium as well as deuterium and supplies of the former, while plentiful, are not so much more so than are those of uranium. At the same time, some neutron multiplying system is needed (which could be beryllium), tritium would need to be produced, and the neutron engineering might be complicated. On the other hand, the required charge of radioactive material in a thermonuclear reactor could scarcely exceed a few grams, and the hazards would be slight.

This paper will discuss in a simple, approximate, semi-quantitative, "back-of-envelope" way some of the requirements for a thermonuclear reactor. Many of th points are dealt with in greater detail by other authors, (see Bibliography)

II. ENERGETIC CONSIDERATIONS

The low energy cross-section for interaction between electrically charged particles is determined, in form, by the barrier penetration factor,

$$\exp - \frac{\sqrt{2m_r}}{\hbar} \int_{r=0}^{r_0} \left(\frac{Z_1 Z_2}{r} - E \right)^{\frac{1}{2}} dr \simeq \exp - \frac{\sqrt{2m_r}}{\hbar} \frac{Z_1 Z_2 e^2}{\sqrt{E_r}} \frac{\pi}{2} . \tag{1}$$

For D, this becomes $\exp(44.2\ E^{1/2})$ where E = deuteron energy in keV and the total cross-section becomes

$$\sigma \simeq AE^{-1} \exp(BE^{-1/2}). \tag{2}$$

For D–D, $A = 1.8 \times 10^{-22}\ \mathrm{cm^2/kV}$ and for D–T, $A = 2.2 \times 10^{-21}\ \mathrm{cm^2/kV}$.

The Coulomb scattering cross-section is

$$\pi \left(\frac{e_1 e_2}{2m_r g^2} \right)^2 \log \Lambda \simeq \pi (Z_1 Z_2)^2 \left(\frac{e^2}{mv^2} \right)^2 \log \Lambda$$

$$\simeq (Z_1 Z_2)^2 10^{-20} E^{-2} \log \Lambda \simeq (Z_1 Z_2)^2 4 \times 10^{-19} E_r^{-2}, \tag{3}$$

and for relative energies < 500 keV, elastic cross-section exceeds the inelastic cross-section. The larger value of the elastic cross-section means that there is no hope of obtaining significant amounts of energy either by sending beams into a cold target, or by letting two highly ordered beams collide. In either case, the dominant process is the disorganizing of the energy in the beams. This need not imply that the distributions are completely Maxwellian, since relaxation times are long.

For disordered systems, the particle distribution function may, however, be approximated by a Maxwellian, and the thermonuclear reaction rate calculated, by forming the average at $\langle \sigma v \rangle$. The integral required can be reduced

$$I = (\pi\beta)^{-3/2} \int A(\exp - B/\sqrt{\mathscr{E}})(\exp - \beta\mathscr{E})\mathscr{E}\, d\mathscr{E}, \tag{4}$$

which may be approximated by using the method of steepest descents, i.e. writing

$$\int_0^\infty \exp - f(x)\, dx = \int_0^\infty \exp - [f(x) + \tfrac{1}{2}(x - x_0)^2 f''(x_0) + \ldots]\, dx \tag{5}$$

$$\simeq \sqrt{2\pi} [f'']^{1/2} \exp - f(x_0), \tag{6}$$

where $f'(x_0) = 0$. With $f = B\mathscr{E}^{-1/2} + \beta\mathscr{E}$, $f' = -\frac{1}{2}B\mathscr{E}^{-3/2} + \beta$, $f'' = \frac{3}{4}B\mathscr{E}^{-5/2}$ and $\mathscr{E}_0 = B/2\beta$, $f(\mathscr{E}_0) = [2^{1/3} + 2^{-2/3}]\,B^{2/3}\beta^{1/3}$, $f'' = (\frac{4}{3})^{1/2}\,2^{-5/6}B^{1/3}\beta^{-5/6}$. Consequently

$$\langle \sigma v \rangle \simeq aT^{-2/3} \exp - bT^{-1/3}. \tag{7}$$

Using the value of the cross-section and of the energy released by the nuclear reactions, we can calculate the rate at which nuclear energy is released,

$$\left.\frac{d\mathscr{E}}{dt}\right|_{DD} \simeq 1.3 \times 10^{-26} n^2 T^{-2/3} \exp - 18.8T^{-1/3} \ \mathrm{W/cm^3} \tag{8}$$

$$\left.\frac{d\mathscr{E}}{dt}\right|_{TT} \simeq 10^{-23} n^2 T^{-2/3} \exp - 19.9\, T^{-1/3} \ \mathrm{W/cm^3}, \tag{9}$$

where n is in cm^{-3} and T is in keV.

Because of the exponential dependence on temperature here, the power release is small until $T \cong 1$ keV. At such high temperatures, the possible densities will be limited by the maximum permissible pressure. (At $n = 10^{16}$ and $T = 1$ keV, the pressure $\approx 10^5$ atmospheres. Under such conditions, the power release is 0.1 W/cm^3 for D–D and 10 W/cm^3 for D–T.)

To be useful, the nuclear energy released must exceed that needed to heat the plasma, for although the energy stored in the plasma is recoverable at the end of a cycle, it must be returned to the hot gas, and can be circulated only with some finite efficiency η; hence the nuclear energy released in a cycle must exceed $\sim (1 - \eta)3NkT$. This in turn implies that the fractional burn-up during a cycle, dn/n, must be of order $3kT/\mathscr{E}_N$ where $\mathscr{E}_N$ is the energy released per reaction. With $T \simeq 1$ keV, $\mathscr{E} \simeq 4$ MeV, $dn/n \simeq 10\%$. Since the fractional burn-up

$$\frac{dn}{n} \simeq n\langle \sigma v \rangle \tau = n\langle \sigma v(T) \rangle \tau > \frac{3kT}{\mathscr{E}_N}(1 - \eta) \tag{10}$$

this yields a constraint on the product $n\tau$ as a function of temperature.

$$n\tau \simeq 4(1 - \eta)10^{10}\, T^{5/3} \exp 18.8T^{-1/3} \ \text{for D–D}, \tag{11}$$

and

$$n\tau \simeq 5(1 - \eta)10^{7}\, T^{5/3} \exp 19.9T^{-1/3} \ \text{for D–T}. \tag{12}$$

At such high temperatures, the plasma must loose energy by radiation, as well as by the loss of particles. In a hot solid radiation is $\propto T^4$, but a diffuse plasma is transparent. Indeed, the radiation mean-free-path, which at high frequency is determined by the collision between photons and electrons, with a total cross-section of the order of $(e^2/mc^2) \simeq 10^{-25}\ cm^2$, must be approximately 10^9 cm. At low frequencies, i.e. when $\omega < \omega_p$ or $\omega \simeq \Omega_-$, the propagation becomes more complex, but these regions are not important for high frequency losses, i.e. those which are determined entirely by the radiation source. (Here ω_p is the plasma frequency and Ω_- the electron gyro-frequency.)

An inevitable radiation loss is bremsstrahlung, the radiation released when an electron is accelerated in the field of an ion. This is, as we have shown, of the order of

$$\frac{8}{3}\pi n_+ n_- \left(\frac{e^2}{mc^2}\right)^2 \times \left(\frac{e^2}{\hbar c}\right) mc^2 \bar{v}_\theta \simeq 5.3 \times 10^{-31} n^2 \sqrt{T} \ \mathrm{W/cm^2}. \tag{13}$$

At $n = 10^{16}$, $T = 1$ keV this is about 53 W/cm², which is much greater than the rate of release of nuclear energy. However, since the latter increases rapidly with temperature, the radiation loss is eventually exceeded by the nuclear energy release. This happens at about 40 keV for D–D, and about 4 keV for D–T. At these temperatures the values of $n\tau$ become:

$n\tau \gg 3 \times 10^{15}$ s cm^{-3} for D–D,

$n\tau \simeq 1.3 \times 10^{14}$ s cm^{-3} for D–T.

To ensure the long ion life time needed, the plasma must be confined in some way. In nature confinement is by a gravitational field, but in the laboratory either inertial effects, or electromagnetic fields, are required. Of these, inertial confinement is suitable for large sudden release of energy; for a controlled release electromagnetic forces must be used. The forces required are considerable, and the quasi-neutrality of the plasma renders electric fields relatively ineffective. On the other hand, a magnetic field, $\vec{B}$, acting on a current density, $\vec{j}$ gives rise to a force density $\vec{j} \times \vec{B}$ and can balance a pressure of approximately $B^2/8\pi$. For $B \simeq 50$ kG this is a pressure of about 10^8 dyn/cm² $\simeq$ 100 atm.

When a plasma is confined by a magnetic field a further radiation loss occurs, for electrons are accelerated continuously by the magnetic field. The energy radiated

$$P = \frac{2n}{3}\frac{e^2}{c^3} a^2 = \frac{2}{3} n \frac{e^2}{c^3}\left(\frac{e}{mc}\right)^2 B^2 v_\perp^2$$

$$\simeq \frac{4n}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{v_\theta^2}{c^2} B^2 c. \tag{14}$$

The ratio of this to the bremsstrahlung loss is

$$\frac{1}{4\pi}\frac{2}{3}\frac{B^2}{nmc^2}\frac{v_\theta}{c}\left(\frac{\hbar c}{e}\right) \simeq \frac{4}{3}\left(\frac{v_\theta}{c}\right)^3 \left[\frac{B^2}{8\pi n (m v_\theta^2)}\right] \frac{\hbar c}{e^2}$$

$$\simeq \frac{4}{3}\frac{137}{\beta}\left(\frac{T}{500}\right)^{3/2} \simeq 0.05 \frac{T^{3/2}}{\beta}. \tag{15}$$

For the 4 keV temperature needed for T–D the ratio is $\simeq 0.4/\beta$, but at the 40 keV temperature needed to produce a net gain in energy from D–D, the magnetic bremsstrahlung is approximately nine times the normal bremsstrahlung. The radiation emitted here is not in a transmitting band, since it appears at the electron-cyclotron frequency, but detailed calculations show that enough radiation does escape, to leave the result not seriously altered.

Energy is radiated by electrons, but sometimes the energy put into the plasma may be given to the ions. The elastic cross-section, however, is fairly small, and there may be a considerable difference between ion and electron temperature. The transfer of energy between ions at a temperature T_+ and electrons at T_- is given by (cf. Spitzer)

$$\frac{d\mathscr{E}}{dt} \simeq 4(2\pi)^{1/2} \ln\Lambda\, n^2 \frac{m_-}{m_+}\left(\frac{e^2}{m_- c^2}\right)^2 c\left[\frac{kT_+}{m_+c^2}+\frac{kT_-}{m_-c^2}\right]^{-\frac{3}{2}}[kT_+ - kT_-]$$

$$\simeq 4(2\pi)^{1/2} \ln\Lambda\, n^2 \frac{m_-}{m_+}\left(\frac{e^2}{mc^2}\right)^2 c\left(\frac{kT_-}{m_- c^2}\right)^{-\frac{3}{2}}(kT_+ - kT_-), \tag{16}$$

if $T_+/T_- \ll m_+/m_-$.

If the electron temperature is determined by the magnetic radiation, and the ion temperature is fixed, in a steady state T_- is determined by balancing the magnetic radiation loss to the rate of energy transfer from the ions.

$$\frac{2}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{nv_\theta^2}{c^2} B^2 c = \frac{2}{3}\frac{nkT_-}{mc^2}\left(\frac{e^2}{mc^2}\right)^2 B^2 c \tag{17}$$

$$= \frac{16}{3}\pi \frac{(nkT_-)^2}{B - mc^2}\left(\frac{e^2}{mc^2}\right)^2 c$$

$$= 4(2\pi)^{1/2} \ln\Lambda\, n^2 \frac{m_-}{m_+}\left(\frac{e^2}{mc^2}\right)^2 c\left(\frac{m_- c^2}{kT_-}\right)^{\frac{3}{2}} (kT_+ - kT_-) \tag{18}$$

and if $T_- \ll T_+$

$$T_- = 500\left(\frac{3}{2}\frac{\ln\Lambda}{(2\pi)^{1/2}}\frac{m_-}{m_+}\frac{T_+}{500}\beta_-\right)^{\frac{2}{7}} \tag{19}$$

$$\simeq 10(T_+\beta_-)^{2/7}$$

$$\simeq 50\,\beta_+^{2/5}\ \text{keV (when } \ln\Lambda = 10).$$

The rate of loss of energy from the ions is then approximately

$$\frac{d\mathscr{E}}{dt} = \frac{2}{3}\frac{50}{500}\beta_+^{2/5} 8\pi n^2 kT_+ \frac{B^2}{8\pi n^2 kT_+} \simeq 6\times 10^{-30} n^2 T_+ \beta_+^{-3/5}\ \text{W/cm}^3. \tag{20}$$

The actual loss in a thick plasma is significantly reduced by self-screening (cf. Drummond).

Thus, although magnetic bremsstrahlung increases the radiation loss, in a low-β plasma the electron temperature is kept low, but although the radiation increases not at $T_+^{1/2}$ but as T_+, the losses need not be too severe.

III. THE GEOMETRY OF MAGNETIC CONFINEMENT

1. Magnetohydrostatic approximation

In a plasma the collision frequency is given by

$$\nu \simeq 4\pi nc \left(\frac{e^2}{mc^2}\right)^2 \left(\frac{mc^2}{mv_\theta^2}\right)^{\frac{3}{2}} \log \Lambda \simeq 2\times 10^{-9}\, nT^{-3/2}, \tag{21}$$

and for $n \simeq 10^{16}$, $T \simeq 10$ keV this is of order 10^6 s^{-1}; hence for systems lasting for about 1 s, the distribution of particle velocities will be roughly isotropic and a magnetohydrodynamic description is appropriate. The equilibrium can then be described by the balance between electromagnetic and hydrostatic forces, i.e.

$$\vec{\nabla} p = \vec{j} \times \vec{B}. \tag{22}$$

The immediate consequences of this are

$$\vec{j}\cdot\vec{\nabla} p = \vec{B}\cdot\vec{\nabla} p = 0, \tag{23}$$

hence current and magnetic field both lie in surfaces of constant pressure. These are the magnetic surfaces. If the system is to be confining, then the surfaces in which $\vec{B}$ lies must be closed and nested. The requirement that $\vec{\nabla}\cdot\vec{B} = 0$ then demands either that these surfaces have singular points, or that they be at least as complicated, topologically, as toroids. The magnetic surfaces can be characterized by the flux threading them, and hence may be described by the flux φ, as well as by the pressure p. On a toroidal surface it is often useful to characterize the field lines by the number of turns made about the small circle for one turn about the large one. This topological quantity is a constant on any given surface. Expressed as an angle it is called the rotational transform. Help in finding equilibria is provided by an analogy with hydrodynamics, for the magnetic hydrostatic equation may be written

$$\vec{\nabla}\left(p + \frac{1}{8\pi} B^2\right) = \frac{1}{4\pi}(\vec{B}\cdot\vec{\nabla})\vec{B}, \tag{24}$$

with the condition

$$\vec{\nabla}\cdot\vec{B} = 0. \tag{25}$$

The equation for steady flow of an incompressible fluid of uniform density reads

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p, \tag{26}$$

$$\vec{\nabla} \cdot \vec{v} = 0. \tag{27}$$

hence by identifying $\vec{v}$ with $B/(4\pi\rho)^{1/2}$, and $p + B^2/8\pi$ with $-p$, a complete analogy is formed between incompressible steady hydrodynamics and magnetohydrostatics. From this analogy many useful results follow; for example, corresponding to cavitating flow there exists a magnetic surface confining an unmagnetized plasma, pressure balance being secured by a surface current. Corresponding to the vortex ring, is a toroidal configuration of equilibrium in which a torus of plasma is confined by an external field, or by the field produced by currents flowing in the plasma. Corresponding to the Hill spherical vortex is a spherical plasmoid distribution held in a spherical bulge in an otherwise uniform magnetic field.

If a field is to contain a plasma, clearly

$$\vec{\nabla}\times\vec{\nabla}\left(p+\frac{B^2}{8\pi}\right) = \frac{1}{4\pi}\vec{\nabla}\times(\vec{B}\cdot\vec{\nabla})\vec{B} = \vec{B}\cdot\vec{\nabla}\vec{j} - (\vec{j}\cdot\vec{\nabla})\vec{B} = 0, \tag{28}$$

a relation which cannot be satisfied by a purely azimuthal field on a toroidal surface.

The magnetohydrostatic equation may be solved to express the current in terms of the pressure of the magnetic fields as

$$\vec{j} = \frac{1}{B^2}(\vec{B}\times\vec{\nabla}p) + \lambda\vec{B}, \tag{29}$$

whereupon the continuity condition on the current demands

$$B\frac{\partial\lambda}{\partial x_{\parallel}} = -\vec{\nabla}\cdot\frac{(\vec{B}\times\vec{\nabla}p)}{B^2} = \frac{2}{B^3}\vec{\nabla}|B|\cdot(\vec{B}\times\vec{\nabla}p). \tag{30}$$

Clearly λ must be a single valued function of space, hence

$$\oint dx_{\parallel}\frac{\vec{\nabla}|B|\cdot(\vec{B}\times\vec{\nabla}p)}{B^4} = 0 \tag{31}$$

is a constraint on the pressure gradient.

This subject is, however, treated in some detail in another paper by Taylor.

2. Low-β systems

If the temperature is high and the density low, the distribution of particle velocity need not be isotropic; but the helical nature of the orbits requires that it be characterized by the component of velocity perpendicular and parallel to the magnetic field. It follows that the stress tensor, rather than being isotropic and characterized by a scalar p, is a tensor with two independent components $p_{\parallel}$, $p_{\perp}$;

$$\overleftrightarrow{p} = p_{\perp}\overleftrightarrow{1} + (p_{\parallel} - p_{\perp})\vec{b}\vec{b}. \tag{32}$$

The equilibrium then becomes

$$\vec{j}\times\vec{B} = \vec{\nabla}\cdot\overleftrightarrow{p} = \vec{\nabla}_{\perp}p_{\perp} + (p_{\parallel} - p_{\perp})\vec{n}/R, \tag{33}$$

$$0 = \frac{\partial}{\partial x_{\parallel}}p_{\parallel} - \frac{1}{B}\frac{\partial B}{\partial x_{\parallel}}(p_{\parallel} - p_{\perp}). \tag{34}$$

In these low pressure conditions, the anisotropy in the pressure can maintain a pressure gradient along a field line in a direction of decreasing $|B|$, and magnetic field lines can leave the plasma. There are still, however, magnetic surfaces within the plasma.

A more detailed understanding of this type of confinement can be obtained by studying the motion of individual particles. In a steady state, the kinetic energy $\mathscr{E} = \frac{1}{2}m(v_{\parallel}^2 + v_{\perp}^2)$ is a constant of motion, as is the magnetic moment

$$\mu = \tfrac{1}{2}mv_{\perp}^2/B; \tag{35}$$

hence $v_{\parallel}$ must satisfy

$$\tfrac{1}{2}mv_{\parallel}^2 = \mathscr{E} - \mu B(x), \tag{36}$$

and the magnetic field acts as a potential keeping particles with finite μ near the regions of minimum B. This is the principal of mirror confinement. If at minimum-B $B = B(0)$, then $v_{\parallel} = 0$ at $\mathscr{E} - \mu B(x) = 0$, but we can define μ and B in terms of the angle θ of the orbit with respect to the magnetic field on the minimum-B surface:

$$\mu = \frac{\mathscr{E}\sin^2\theta}{B_0}, \tag{37}$$

hence the turning point for a particle is defined by x_0, where

$$1 - \sin^2\theta\frac{B(x_0)}{B(0)} = 0. \tag{38}$$

If B rises to some maximum value B_{max} we can define the ratio $B_{max}/B_{min} = R$ the mirror ratio, which is greater than unity; all these particles will be trapped for which $\sin^2\theta \geq R^{-1}$. The cone of lost particles ($\sin^2\theta \leq R^{-1}$) is described as the loss cone, and is filled only by collision processes.

3. Stability

It is not enough to produce a configuration of equilibrium; if it is to be preserved it must be stable against small disturbances. To understand the stability of complex systems one may notice that disturbances must occur at constant total energy. The energy consists of both potential and kinetic energy, and if there exist displacements which will cause the potential energy to decrease, then kinetic energy can increase and motion away from the static equilibrium is possible. The potential energy in a fluid system is

$$\mathscr{E} = \int d^3\tau \left[\frac{B^2}{8\pi} + \frac{p}{\gamma - 1}\right], \tag{39}$$

the sum of the magnetic and the thermal energy of the gas. In a perfectly conducting fluid, where

$$\vec{E} + \frac{1}{c}\vec{v} \times \vec{B} = 0, \tag{40}$$

a reasonable model for a plasma, the magnetic flux must be conserved on all motions.

If a displacement in the fluid is made, and flux conserved, we can then ask whether or not the energy is increased. An important type of displacement is one in which flux tubes are just interchanged. An important quantity then is the volume associated with a tube of flux. Since a flux tube is defined by field lines so that the flux $BA = \varphi$ is constant, as one moves along a line of force the volume of a flux tube φ is

$$\int dlA = \varphi \int \frac{dl}{B}, \tag{41}$$

and the energy

$$\int d\varphi \int dl \left[\frac{B}{8\pi} + \frac{p}{B(\gamma - 1)}\right]. \tag{42}$$

Suppose we perform an interchange for which the volume of the two flux tubes is the same, but the flux is not. Then the change in energy will be

$$\begin{aligned}\delta\mathscr{E} &= \int dl_2 A_2 \left[\left(\frac{B_1'^2}{8\pi} + \frac{p_1'}{\gamma - 1}\right) - \left(\frac{B_2^2}{8\pi} + \frac{p_2}{\gamma - 1}\right)\right] \\ &\quad + \int dl_1 A_1 \left[\left(\frac{B_2'^2}{8\pi} + \frac{p_2'}{\gamma-1}\right) - \left(\frac{B_1^2}{8\pi} + \frac{p_1}{\gamma-1}\right)\right] \qquad (43) \\ &= \frac{1}{8\pi}\left\{\int dl_2 A_2 \left[\frac{\varphi_1^2}{A_2^2} - \frac{\varphi_2^2}{A_2^2}\right] + \int dl_1 A_1 \left[\frac{\varphi_2^2}{A_1^2} - \frac{\varphi_1^2}{A_1^2}\right]\right\} + 0(p) \\ &= \frac{1}{8\pi}(\varphi_1^2 - \varphi_2^2)\left[\int \frac{dl_2}{A_2} - \int \frac{dl_1}{A_1}\right] + 0(p). \qquad (44)\end{aligned}$$

But $\int A_2\,dl_2 = \int A_1\,dl_1$, since the volume, and hence the thermal energy, is conserved. Thus

$$\delta\mathscr{E} = \frac{1}{8\pi}(\varphi_1^2 - \varphi_2^2)\int\frac{dl_1}{A_1}\left[\left(\frac{dl_2}{dl_1}\right)^2 - 1\right]$$

$$= \frac{1}{8\pi}(\varphi_1^2 - \varphi_2^2)\left\langle\left(\frac{dl_2}{dl_1}\right)^2 - 1\right\rangle \; ; \qquad (45)$$

hence if $\varphi_2 > \varphi_1$ and $dl_2/dl_1 > 1$, the system is unstable. Since the plasma is confined by the field, B usually increases outward, and if the length of a field line also increases, i.e. its centre of curvature lies inside the plasma, the system is unstable. If, on the other hand, the magnetic field lines curve away from the plasma surface, the system is stable.

If, on the other hand, the field strength is high compared to the thermal energy, i.e. $\beta = 8\pi p/B^2 \ll 1$, then the field is nearly a vacuum field and is already in a state of minimum energy. The only decrease in energy, then, must be obtained at the price of the thermal energy of the gas. Suppose then, an interchange occurs between two tubes of equal flux. If this happens adiabatically, so that $p_1' v_1'^{\gamma} = p_1 v_1^{\gamma}$, i.e. $p_1' = p_1(v_1/v_2)^{\gamma}$;

$$\delta\mathscr{E} = \frac{1}{\gamma - 1}(p_1' v_2 + p_2' v_1 - p_1 v_1 - p_2 v_2) \qquad (46)$$

$$= \frac{1}{\gamma - 1}[p_1 v_1^{\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma}) + p_2 v_2^{\gamma}(v_1^{1-\gamma} - v_2^{1-\gamma})]$$

$$= \frac{(v_2^{1-\gamma} - v_1^{1-\gamma})}{\gamma - 1}(p_1 v_1^{\gamma} - p_2 v_2^{\gamma})$$

$$= v^{-\gamma}\,\delta v\,\delta(pv^{\gamma}) \qquad (47)$$

$$= \frac{\delta v}{v}[v\delta p + \gamma p\,\delta v] \qquad (48)$$

Now the plasma surface $p > 0$ and this reduces to $\delta\mathscr{E} \simeq \delta p\,\delta v$; but δv is the gradient with respect to the flux surface ψ of the flux tube volume $v = \varphi\int dl/B$. Hence, for instability

$$\frac{\partial p}{\partial\psi}\frac{\partial}{\partial\psi}\oint\frac{dl}{B} < 0, \qquad (49)$$

but since $\partial p/\partial\psi < 0$ near the surface stability requires that $\partial/\partial\psi \oint dl/B > 0$, and $\int dl/B$ must decrease outward for stability.

To defeat the interchange instability, one may introduce in a toroidal system a shear in the magnetic field. In that case, the winding number or rotational transform may vary with flux surface; hence neighbouring flux tubes may be topologically inequivalent and flux tubes on neighbouring surfaces cannot be interchanged without being broken. This means that an ideal hydrodynamic configuration may be stabilized by magnetic shear.

To produce magnetic shear, however, larger currents are required and the dangerous instabilities are localized about the flux surface on which the perturbation does not vary along the magnetic field. Attempts to demonstrate stability in systems which have sheared fields have been unsuccessful, and it is easy to see how their failure can be understood by considering the effects of finite resistivity. The actual current equation is

$$\eta\vec{j} = \eta\vec{\nabla}\times\vec{B} = \sigma(\vec{E} + \frac{1}{c}\,\vec{v}\times\vec{B}), \tag{50}$$

and the left hand side is negligible if $\eta/L \ll v/c$. The scale length L entering here may be determined by the perturbation itself and, in fact, there exist oscillations for which the wavelength $l \gg \eta c/v$. For such oscillations the effect of finite resistivity is important and the flux is not accurately preserved. Motions are then possible, in which field lines break and rejoin, and in a highly sheared plasma these are important and lead to new instabilities.

A crucial experiment on plasma instability was the hard core, or unpinch experiment, in which a theoretically stable configuration was produced by excluding a plasma from the centre of a cylinder by the field of an axial current flowing in a rigid conductor. This configuration, stable for a perfectly conducting plasma proved, in fact, to be unstable. This instability arises because magnetic flux tubes can be cut in a plasma of finite conductivity. In systems stabilized by shear produced by currents flowing in the plasma, basket-weave devices, magnetic energy can be released by the cutting of flux tubes and so-called tearing instabilities grow rapidly.

Since shear stabilization has been shown unsatisfactory, one must try systems which do not depend on topological prohibitions, but in which $\int dl/B$ increases on leaving the plasma surface. In such systems, the centre of curvature of field lines lies outside the plasma, hence any confining surface

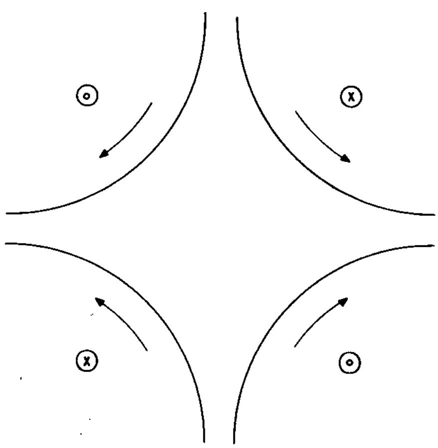

Fig. 1

A cusp-field configuration.

must have singular points. The simplest of such systems is the cusp, in which the field is produced by two opposing current loops, see Fig. 1. At the singular points, the real fluid properties of the plasma result in losses, and there is a leak of size $(r_L^- r_L^+)^{1/2}L$, where the $r_L^\pm$ are electron and ion Larmor radii and L is the length of the ring cusp. In a large enough system this loss is tolerable. In many high density experiments, the plasma is cool and the loss is determined by resistive diffusion which causes the leak to widen at a rate determined by conductivity.

In a mirror, plasma is confined because the magnetic field increases as one goes along field lines away from the plasma. It is unstably confined because the magnetic field decreases as one goes away from the plasma across the magnetic field. In the cusp the magnetic field increases in all directions as one leaves the plasma, hence $\int dl/B$ decreases outward and the plasma is stably confined in a region of minimum B.

In a cusp, the central magnetic field is zero, hence the magnetic moment, μ, which is approximately constant only in strong magnetic fields, no longer defines a potential $\mu B(x)$ in which particles move; and unlike a mirror, the motion along the field line is not limited.

This defect, however, can be overcome by putting in an axial conductor carrying a current along the axis of the cusp, whereupon a toroidal region appears in which B is a minimum, but still large so that particle motion is adiabatic and particles are confined on the field line by increasing field strength.

The minimum B configurations can be exploited if the pressure is made a function of the magnetic field strength. Then, for example, transverse current alone is divergence free for

$$\vec{\nabla}\cdot\frac{1}{B^2}(\vec{B}\times\vec{\nabla}p) = -\frac{2}{B^3}\vec{\nabla}|B|\cdot(\vec{B}\times\vec{\nabla}p) = -\frac{2}{B^3}\frac{\partial p}{\partial B}\vec{\nabla}|B|\cdot\vec{B}\times\vec{\nabla}|B|$$

$$\simeq \vec{B}\cdot\vec{\nabla}|B|\times\vec{\nabla}|B| = 0. \tag{51}$$

The minimum B configurations have been elegantly treated by J. B. Taylor in his paper (these Proceedings).

Many configurations of this type have been studied and stable confinement has been achieved in the hybrid mirror by M. Ioffe. At present, these seem the 'white hope' of thermonuclear research, but since experiments are limited to very low values of β, much more knowledge is required before the high-β systems needed for energy producing systems are in sight.

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EXPERIMENTS ON PLASMA

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I. METHODS OF MEASUREMENT

1. Total currents

The total currents, either in circuits or in the bulk of plasma are measured by a Rogowski coil (sometimes called a magnetic potentiometer or current transformer). This consists of a long solenoid, usually of fine wire, wrapped on a flexible tube, see Fig. 1, which is then curved to form

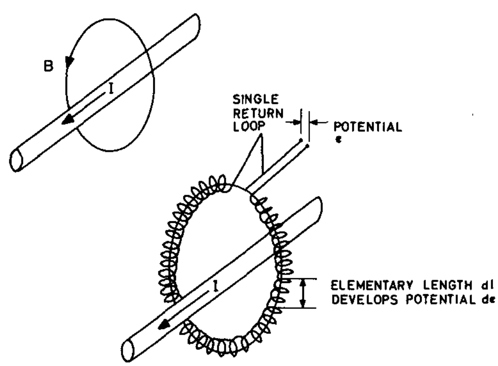

Fig. 1

A Rogowski coil.

a closed loop enclosing the current to be measured. Let there be n turns per centimetre length, the turns being of area A.

If the current produces a magnetic field, whose value at some point coincident with a section of the coil is B, then if this field is varying in time, the EMF de induced in a short length dl of the coil is

$$de = nA\frac{\partial B}{\partial t}dl.$$

Thus integrating around the whole loop, the total EMF (e) generated is

$$e = \oint nA\frac{\partial B}{\partial t}dl = nA\frac{\partial}{\partial t}\oint Bdl.$$

But $\oint B dl = 4\pi \times$ total current looped. Hence

$$e = 4\pi n A \frac{dI}{dt}.$$

This EMF is usually fed on to an oscilloscope through an integrating circuit (resistance in series with a capacitor), giving a potential V on the terminals such that

$$V = \frac{1}{RC}\int e dt, \text{ thus } V \propto I.$$

Provided the major radius of the loop greatly exceeds the minor radius of the turns of wire (so that B can be regarded as constant across the area A) the actual path of the loop does not matter. Essential refinements and precautions are:

(a) an electrostatic screen (not completely closed of course, otherwise the magnetic field would not penetrate)

(b) one long thread of wire is taken back along the path of the major loop to balance out any flux linked by this loop.

2. Local current density

The local current density is deduced from local magnetic field measurements, made with very small pick-up coils consisting of many turns of fine wire ("magnetic probes"). Here again only time varying magnetic fields generate an EMF, but the largest signals come from the most rapidly varying fields. Thus these small coils are especially useful in identifying and measuring rapidly fluctuating local fields in pulsed discharges (e.g. pinch discharges). The EMF generated is proportional to $\partial B/\partial t$; here again an integrating circuit is usually used to give a signal directly proportional to B. The time constant CR is at least ten times longer than the duration of the fluctuation being examined. Much longer time constants reduce the amplitude of the final signal.

Desirable properties in a magnetic probe are a high sensitivity and a fast time response: these are to some extent contradictory requirements. A high sensitivity is achieved by winding many turns, but this increases the inductance of the coil and lowers the frequency at which a fluctuating magnetic field will be in resonance with the natural frequency of the coil, set by its own inductance and capacity of the associated circuits. Useful compromises are however possible, and probes with frequency responses up to, say, 5 Mc/s are common: some have been used up to nearly 50 Mc/s. The flattest response is achieved by making the circuit critically damped. A typical circuit is shown in Fig. 2. It can be stated that this circuit has the flattest response (i.e. is critically damped) when

$$R_{total} = \left(\frac{L}{2(C + C_s)}\right)^{\frac{1}{2}}.$$

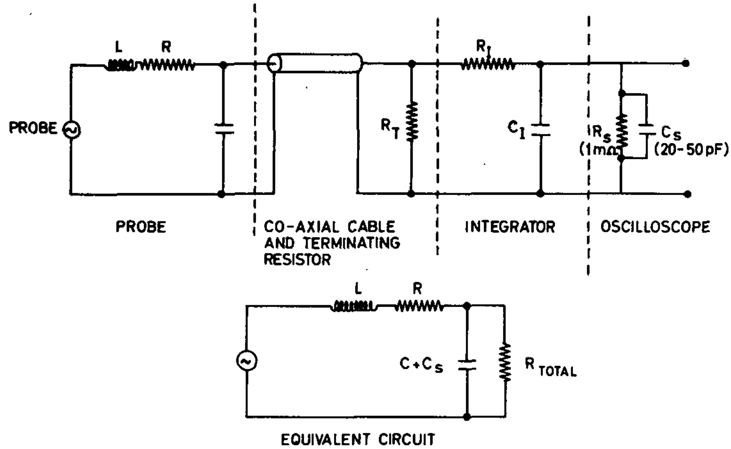

Fig. 2

Circuit and equivalent circuit of magnetic probe*.

Typical values are as follows: $L = 1{-}5\,\mu H$, $R = 1{-}30\ \Omega$, $C = 20 - 50\,pF$, and $C_s = 35\,pF$.

Take for example L as $5\ \mu H$, $C = 40$ pF; in this case $R_{total} \simeq 183$ ohms. Most co-axial cables have characteristic impedances in the range $100 - 500\ \Omega$ so generally critical damping and proper termination of the cable (to avoid reflections) is possible.

An electrostatic screen is also necessary to ensure that only magnetic effects are measured. Magnetic probes are often used in thin stainless steel tubes: the frequency response is then determined by the penetration time of the magnetic field through this tube.

The local current density $\vec{j}$ is deduced from the magnetic traces, using the component of

$$\vec{\nabla} \times \vec{B} = 4\pi\vec{j}.$$

Thus one frequently needs to measure a gradient of the magnetic field: for this purpose a tube is poked into the plasma carrying a spaced array of these probes.

3. Electric field

The electric field in the plasma can be measured by inserting into the plasma small metal plates, each connected through a very high impedance to some point at common potential (usually earth).

A plate immersed in a plasma in this way is said to be approximately floating – i.e. it takes up a potential which is close to the local potential

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* e.g. see SEGRÉ, S.E. and ALLEN, J.E., Magnetic probes of high frequency response, J. sci. Instrum. 37 (1960) 369.

in the plasma. Usually it becomes a few volts (ΔV) negative with respect to the plasma because of the greater mobility of electrons – these charge up the plate negatively until an ambipolar potential is set up which equalizes the flow of ions and electrons. (If, however, the ions have greater random velocity than the electrons this ambipolar potential will be reversed.)

Such electric field measurements are accurate only if this ambipolar potential ΔV is much less than the potential difference between the probes, because it cannot be assumed that ΔV is the same for both probes. (This would be true only if there were no spatial variations in electron or ion temperature: it is clearly not possible to assume this in general.)

4. Electron density and temperature

The electron density and temperature can, using a probe technique, be deduced (making certain assumptions) by measuring the current drawn to a metal plate immersed in the plasma, and varying the potential on the probe. The theory and technique is due to LANGMUIR [1].

4.1. Floating probe. Suppose that a potential V_s (the "sheath" potential) is set up between the plasma and the surface of the probe. Then on the simplest assumption the flux of ions to the probe is $\frac{1}{4}n_i v_i$ per cm^2 and the flux of electrons is

$$\frac{1}{4} n_e v_e \exp\left(-\frac{eV_s}{kT_e}\right).$$

The assumptions implicit here are that the electrons have a Maxwellian distribution corresponding to a temperature T_e, and that no ionization occurs within the sheath.

Equating these two fluxes gives

$$V_s = \frac{kT_e}{e} \log \frac{v_e}{v_i}.$$

At first sight it would appear that v_e and v_i are the random velocities of electrons and ions respectively: v_e certainly is the random velocity of the electrons, but v_i is somewhat larger than the random velocity of the ions, because the electric field penetrates from the sheath some distance into the plasma and accelerates the ions. The ratio v_e/v_i in most discharges in light gases is such that V_s is typically 3 to 5 times kT_e/e, i.e. 3 to 5 times the mean electron energy.

4.2. Fixed potential probe. Suppose that now the probe is not allowed to float, but connected to a source of potential V with respect to the plasma, and the current drawn is measured. Both electron and ion currents will vary: however the electron current will vary very rapidly, being proportional to $\exp(-eV/kT_e)$, whereas the ion current varies only very slowly. Thus if $\log j$ is plotted against V, a straight line results with a slope of $-e/kT_e$ from which the electron temperature can be deduced.

Now if the probe is made more positive, so that it reaches the same potential as the plasma, then both ions and electrons diffuse freely to it:

$$j_i = \frac{1}{4} n_i e v_i$$

$$j_e = \frac{1}{4} n_e e v_e.$$

If the probe is made slightly more positive then, assuming the ion temperature to be rather small, all ions are repelled and a space-charge limited current of electrons is collected. The log j against V curve shows a sharp change of slope (in practice not all that sharp, but extrapolation allows the point to be reasonably well defined). The current density at which this happens is given by $\frac{1}{4} n_e e v_e$; since v_e (the electron thermal speed) has already been found n_e can be determined.

Consider the reverse situation, when all electrons are repelled and a saturation ion current is collected. Then for a plane probe

$$j_i = \frac{1}{4\pi}\left(\frac{2e}{M}\right)^{\frac{1}{2}} \frac{V^{\frac{3}{2}}}{d^2} = \frac{1}{4} n_i e v_i .$$

Disregarding for the moment actual numerical values, we should expect for a given plasma $d \propto V^{3/2}$, and for V = const., $d \propto 1/\sqrt{n_i}$. Both have been verified (provided V is sufficiently strongly negative, and provided the size of the probe, and the thickness of the sheath is less than λ_e or λ_i, the mean free path of the electrons or ions). v_i deduced from this saturation current is however absurdly high to be ascribed to the random motion of the ions. Bohm has shown that, due to acceleration of the ions v_i is closer to $(2kT_e/M)^{1/2}$ than to $(2kT_i/M)^{1/2}$.

II. PROBES IN A MAGNETIC FIELD

Probes placed perpendicular to B are unaffected (except that they measure only $T_{\parallel}$). Probes collecting across magnetic field lines are affected due mainly to the curvature of the electron paths, the ion paths being much less curved.

If the electron orbit (radius ρ_e) is much bigger than the thickness d of the sheath, then it can be shown [2] that the probe current is

$$j_e = \frac{1}{4} n_e e v_e \left(\frac{\pi}{\omega\tau}\right) \exp\left(-\frac{eV}{kT_e}\right),$$

where $\omega = eB/mc$ (the gyro frequency) and τ is the mean collision time for electrons. The electron flux to the probe is thus much reduced, the ion flux much less so (since $\omega\tau_{\text{ions}} \ll \omega\tau_{\text{electrons}}$): thus the potential across the sheath is much reduced also.

A plot of $\log j_e$ against V should still give a straight line enabling T_e to be deduced. (For example if $p = 10^{-3}$ torr, $T_e = 2 \times 10^4$ °K, B = 240 G, $\omega\tau = 10^3$ for electrons.)

At the other extreme suppose $\rho_e \ll d$ and the electrons make so many collisions within the sheath that their motion is controlled by collisional diffusion in the presence of the electric field of the sheath. Detailed theory shows that the sheath potential is reduced to

$$V_s = \frac{kT_e}{e} \log \frac{v_e}{v_i} + \log \frac{16}{9} \frac{\lambda_e}{1 + (\omega\tau)^2} \left(\frac{eV_s}{kT_e} \right).$$

The main disadvantage of the simple Langmuir probe is that a large electron current can be drawn from the plasma, thereby seriously changing the very properties being measured. Also the probe has to be connected to some external potential source. This can be overcome by using a double probe.

1. Double probe

Two equal area probes (JOHNSON and MALTER [3]) are inserted into plasma as close together as possible, consistent with their sheaths not overlapping. A potential V_d is applied between them, and there is no connection to any external absolute potential. Now they take up potentials V_1 and V_2 with respect to the plasma, one moving nearer to, the other further from plasma potential - the first collects more electrons, the other correspondingly fewer, the net current taken from the plasma being zero. If V_d is sufficiently large all the electrons that were collected on one probe are repelled, and that probe collects a saturation ion current. (Reversal of V_d merely reverses the roles of the two probes.) Saturation ion current occurs at points A and B shown in Fig. 3. The total current from the plasma to the two probes is zero: therefore

$$\begin{aligned} \Sigma j_{ions} &= (j_i)_1 + (j_i)_2 = (j_e)_1 + (j_e)_2 \\ &= (j_{e0})_1 \exp(-eV_1/kT_e) + (j_{e0})_2 \exp(-eV_2/kT_e), \end{aligned}$$

where $(j_{e0})_1$ means the electron current to the probe 1 when V = 0, etc. Putting $V_d = V_1 - V_2$, then

$$\Sigma j_{ions} = (j_{e0})_2 \exp\left[-e\frac{(V_2 + V_d)}{kT_e} \right] + (j_{e0})_2 \exp\left[-eV_2/kT_e \right].$$

Therefore

$$\frac{\Sigma j_i}{j_e} - 1 = \exp(-eV_d/kT_e)$$

and hence by plotting $\log \left[\frac{\Sigma j_i}{j_{e2}} - 1 \right]$, we can obtain $-eV_d/kT_e$. Σj_i and j_{e2}

are shown on the graph. j_{e2} is the sum of the electron current to probe 2 with $V_d = 0$ (i.e. j_{e02}) and the extra electron current repelled from probe 1 and collected by probe 2.

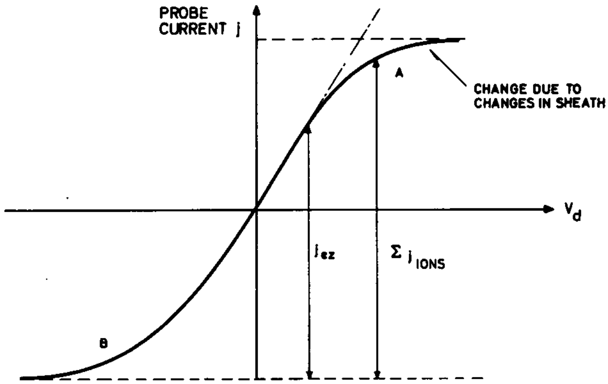

Fig. 3

Current collected by the double probe.

The electron density cannot be deduced from double probe traces, but since the saturation ion current is measured we may take the theoretical flux of ions from the plasma into the sheath:

$$j_i = Cn_i ev_i$$

$$= Cn_i e(2kT_e/M)^{\frac{1}{2}} \quad ,$$

where C is a constant which, according to various authors, varies from 0.4 to 1.0.

A fundamental disadvantage of the double probe method as compared with the single Langmuir probe is the very small fraction of the electron energy distribution that is sampled. When the probes are unbiased and floating each receives only the electrons able to overcome the sheath potential i.e. the fraction $\exp(-eV_s/kT_e)$. We have already shown that eV_s/kT_e is about 3 to 5, which means that fewer than one tenth of the electrons - those at the tail of the distribution - reach the probe surface. Now when a potential V_d is applied these energetic electrons are repelled from one probe until, when V_d is large enough, they are all repelled. It can easily be shown, since the maximum electron current cannot exceed the saturation ion current, that the entire probe trace arises by adjustment of the flow of energetic electrons only.

To overcome this it is possible to use probes of unequal areas, A_1 and A_2. Then

$$A_1 j_i - A_1 j_{e0} \exp(-eV_1/kT_e) + A_2 j_i - A_2 j_{e0} \exp(-eV_2/kT_e) = 0. \qquad (1)$$

To sample the whole distribution we need to make one sheath disappear, i.e.

the total electron current to the large probe at floating potential must equal the saturation ion current to the small probe when it is biased to plasma potential. (The saturation ion current to one probe is

$$A j_i = ACn_i e \left(\frac{2kT_e}{M}\right)^{\frac{1}{2}} .)$$

The electron current to the other probe in the presence of a sheath of potential V_s is

$$A_2 j_e = A_2 n_e e v_e \exp(-eV_s/kT_e)$$

when $A_1 = A_2$ and, assuming $n_i \approx n_e$, then clearly the saturation ion current is $(m/M)^{1/2}$ times the electron current.

To make the sheath potential negligible and sample the whole electron distribution:

$$A_1 j_i = A_2 n_e e v_e ,$$

i.e.

$$\frac{A_1}{\sqrt{M}} = \frac{A_2}{\sqrt{m}}$$

or

$$\frac{A_1}{A_2} = \sqrt{\frac{M}{m}} .$$

More precisely, from Eq. (1), if A_1 is the large probe and A_2 the small probe, we can assume that, when the small probe is saturated with electron current ($V_2 = 0$), and also nearly all electrons are repelled from the big probe, $j_{e0} \exp(-eV_1/kT_e)$ is negligible. Thus

$$A_1 j_i + A_2(j_i - j_{e0}) = 0 ,$$

and therefore

$$\frac{A_1}{A_2} = \frac{j_{e0} - j_i}{j_i} ,$$

which, using the above substitutions, is approximately $(M/m)^{1/2}$.

Any ratio of areas greater than this allows the whole electron energy distribution to be sampled. Now the plot $\log j_e$ against V is distorted and T_e can be found only for small variations in V close to the plasma potential.

As a rough guide we can construct the following table, comparing the Debye length λ_D (which is roughly equal to the thickness of the sheath) with the Larmor radius of the electrons. The number given is the ratio ρ_e/λ_D.

For values above the line electron collisions in the sheath are important.

TABLE I

VALUES OF ρ_e/λ_D FOR VARIOUS VALUES OF MAGNETIC FIELD H AND ELECTRON DENSITY n_e

$$\text{Larmor radius } \rho_e = \frac{m_e}{eH}\left(\frac{8\,k\,T_e}{\pi\,m_e}\right)^{1/2}$$

$$\text{Debye distance } \lambda_D = 6.9\left(\frac{T_e}{n_e}\right)^{1/2}$$

n_e (cm^{-3})	H (G) 100	200	300	400	500	1000	1500	2000
10^8	0.51	0.255	0.17	0.13	0.102	0.051	0.034	0.0255
10^9	1.61	0.805	0.54	0.4	0.32	0.161	0.107	0.0805
10^{10}	5.1	2.55	1.7	1.3	1.02	0.51	0.34	0.255
10^{11}	16.1	8.05	5.4	4	3.2	1.61	1.07	0.805
10^{12}	51	25.5	17	13	10.2	5.1	3.4	2.55
10^{13}	161	80.5	54	40	32	16.1	10.7	8.05
10^{14}	510	255	170	130	102	51	34	25.5
10^{15}	1610	805	540	400	320	161	107	80.5
10^{16}	5100	2550	1700	1300	1020	510	340	255

III. RANGE OF USE OF PROBES

The lowest density is given by the physical size. For example the Debye length ($\simeq$ sheath thickness) for a plasma of density $10^8/cm^3$ and electron temperature 20 eV is about 3-5 mm. Thus a probe would have to be about 3 - 5 cm in linear dimensions to approximate to a plane probe. Thus densities of 10^8- $10^9/cm^3$ are the usual lower limits. The upper limit is set by the tendency of the probe surface to become the cathode of an arc when subjected to intense bombardment. This usually occurs when $n \gtrsim 10^{14}/cm^2$. Arcs form less readily in short pulsed discharges ($\sim$ a few microseconds).

IV. CONSTRUCTION OF PROBES

Choice of Materials. The probe materials should have a high work function, and a high melting point is also desirable. Platinum is satisfactory, whereas tungsten has proved unsatisfactory at high densities.

Choice of Size. The size depends on the plasma to be examined; it should be as small as possible consistent with its linear dimensions being appreciably greater than the sheath thickness. When unequal area probes are used the area of the larger one should not exceed 1 cm^2.

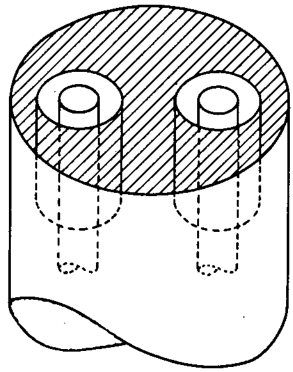

Fig. 4

Construction of equal area double probe.

Mechanical Design. Arcs tend to form at the junction between plasma and insulator. A gap must therefore be left, see Fig. 4. The probes must be set close together to avoid picking up any stray electric field. The design and use of such probes has been summarized by JONES and SAUNDERS [4].

V. MEASUREMENTS ON MOVING PLASMA

Plasma guns are a convenient and widely used source of energetic plasma. Such plasmas have directed velocity v, in addition to their thermal motion. Let us assume that such a plasma is guided by a magnetic field: then we should like to know the density, directed velocity, electron and ion temperatures, and composition of the plasma. The following techniques are used.

1. Ion collector

Since ions and electrons travel with the same directed velocity, electrons have much less energy and can be easily repelled by the bias potential, see Fig. 5. The flux of ions through the hole is nAv, where v is the directed velocity of the ions.

2. Electrostatic particle-energy analyser

The electrostatic particle-energy analyser is usually to be preferred to a magnetic-momentum analyser since it does not require collimated beams nor precise magnetic fields. The design of the instrument (see Fig. 6)

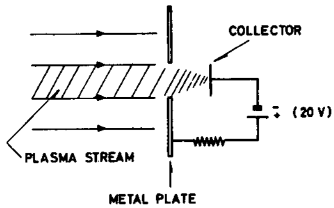

Fig. 5

Biased ion-collector.

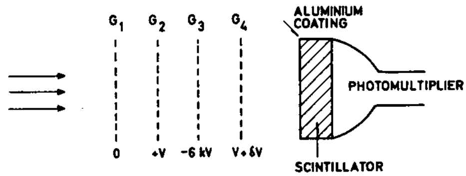

Fig. 6

Energy selective ion analyser.

described here is due to MASON [5]. Ions of energy between V and $V + \delta V$ are repelled from G_4, strike the back of G_3 and emit secondary electrons which are accelerated through about 6 kV to penetrate the aluminium film and activate the scintillator.

The main ion is that of the gas used to fill the gun: correlation of energy with time of flight identifies the ion.

3. Diamagnetic loop

A single loop of wire (electrostatically screened) placed around the guiding field has an EMF generated in it as the plasma passes because the plasma pressure pushes out the magnetic flux. Let us assume that the area of the loop is A and the static magnetic field is B_0, reduced to B_1 when the plasma passes. Then the change of flux $\phi = A(B_0 - B_1)$. But $B_0^2/8\pi - B_1^2/8\pi = p = \beta B_0^2/8\pi$, by the definition of β. Now if β is small these equations reduce to

$$\beta \approx 2\frac{\Delta B}{B_0},$$

and

$$e = \frac{\partial\phi}{\partial t} = \frac{A\partial B}{\partial t},$$

where e is the EMF generated.

If this is fed via an integrating circuit to the measuring oscilloscope, a signal V appears, where

$$V = \frac{1}{CR}\int e\,dt = \frac{A}{CR}\Delta B .$$

Hence β can be derived.

If the loop is much bigger than the cross-section of the plasma, the latter being of area A_1, then the change in flux due to the passage of the plasma is

$$\Delta\phi = A_1 B_0 \left(1 - \frac{1}{(1-\beta)^{1/2}}\right).$$

β can now be derived only if the radial distribution of the plasma is known. Since β measures, in this instance, $p_\perp$ we derive $nk(T_{e\perp} + T_{i\perp})$: n is known from the ion collector measurements.

Now some indication of T_e can be obtained provided $kT_e \gg \frac{1}{2}mv^2$, where v is the directed velocity. This is frequently true (for example $T_e \approx 20$ eV, $\frac{1}{2}mv^2 \approx 0.5$ eV are typical figures). Then the ion collector biased to different small potentials will collect different currents, the differences being due to collection of electrons in the tail of the distribution. Thus using standard double probe theory T_e along the field lines can be found. Usually $T_{e\perp}$ and $T_{e\parallel}$ are not significantly different, due to the short electron-electron collision time.

VI. EXPERIMENTS ON MAGNETIC WELLS

The techniques of measurement described in the earlier sections have been used in studying the behaviour of plasma in a hybrid trap ("magnetic well") consisting of a conventional magnetic mirror with a hexapole cusp field imposed orthogonally [6].

Plasma is produced in a coaxial gun, the performance of the gun being monitored by measuring the current waveform using a Rogowski coil. The ejected plasma is guided along a static magnetic field which converges gently from 450 G at the gun to 4 kG at a point 3.5 m distant: it remains constant over a length of about two metres, and thereafter diverges symmetrically. Pulsed coils produce mirror fields (an additional 3.6 kG) at points disposed 70 cm apart about the middle of the central region. Plasma is trapped by first energizing the far mirror, firing the gun and then energizing the nearer mirror before the reflected plasma can escape.

The properties of the injected plasma are measured with ion probes, particle detectors, and diamagnetic loops: the results are:

Mean directed energy $\simeq$ 1 keV
$n \simeq 3.10^{12}$ ions/cm^3.

Perpendicular energy (of ions) $\simeq$ 60 eV (at gun)
$T_e \simeq 12$ eV.

The density of the plasma trapped in a simple mirror, as measured by a microwave interferometer (λ = 3 cm), is 4×10^{10} per cm^3. Its lifetime is measured by the flux of neutral particles resulting from charge exchange. These are converted back into ions by a water vapour cell, and the resulting ion flux in a given energy interval (1.6 keV ± 200 eV) measured by the particle detector. The lifetime is 50 - 80 μs: this is of course the lifetime of energetic ions in the magnetic mirror. Double probes are used to detect the presence of plasma. They are disposed close to the walls and arranged both parallel to the field lines and azimuthally around the circumference. They show the growth of a rotating flute which drives the plasma into the walls, and makes one complete revolution in 64 μs.

In subsequent experiments a hexapole stabilizing field was established within 50 μs during the trapping process. Although some plasma was inevitably spilt into the walls, the density of that remaining was comparable with that in the simple mirror, but it decayed much more smoothly with a a $1/e$ decay time of 250 to 400 μs. All trace of the rotating flute disappeared, and plasma was observed by probes only where the lines of force of the hexapole field cut the walls.

These experiments show that the plasma contained in a magnetic well is much more stable than that contained in a simple magnetic mirror.

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PLASMA DIAGNOSTICS BASED ON REFRACTIVITY

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The idea of using a beam of light as a plasma probe occurs as soon as the need arises to diagnose a discharge without introducing large perturbations. In an attempt to classify the phenomena involved when an externally produced beam of light crosses a plasma, one has to distinguish between absorption, transmission and reflection.

Let us disregard the absorption which would lead us away from the main purpose of this paper. The elementary mechanism underlying the transmission of radiation being that of the scattering of a photon by a free electron, the different aspects of refraction, diffusion and diffraction with which transmission manifests itself are characterized by the behaviour of the phase cancellation relationship of the scattered photons. In this discussion we limit ourselves to the refractivity and this implies that the spatial density distribution of plasma - and hence the refractive index - does not undergo abrupt variations within distances of many light wave-lengths.

1. REFRACTIVITY OF PLASMA

Quite a good deal of work has been done to date on this subject but we mention here only those items which are strictly related to our diagnostic aim. Considering the plasma as a mixture of electrons, ions and residual gas atoms, the refractivity $n-1$ of the mixture can be expressed by

$$n - 1 = (n-1)_{\text{atoms}} + (n-1)_{\text{electrons}} = \sum_i K_i N_i , \tag{1.1}$$

K_i being the specific refractivity and N_i the number density of the i-th component (atoms in the various excited states, ions, electrons) of the mixture. The contribution given by the atoms of the residual gas in various states of excitation can be of some importance in the case of low-energy discharge or, in any discharge, during the breakdown and the afterglow.

Both classical and quantum theories yield the well-known formula for refractivity

$$n - 1 = \frac{2e^2}{m} \sum_l N_l \sum_K \frac{f_{lk}}{\omega_{lK}^2 - \omega^2} , \tag{1.2}$$

which holds, provided that $\omega \neq \omega_{lK}$. Here ω_{lK} is the angular frequency of the line arising from the jump between the K and l states and f_{lK} is the corresponding oscillator strength. This can be calculated by means of quantum mechanics, degeneracy being taken into account. In applying the dispersion formula the difficulty arises of including the transitions to the continuum, as well as to the discrete energy states. This sometimes turns out to be of some importance [1].

The population of the energy levels of the atoms depends upon the discharge: the refractivity of a discharge in thermal equilibrium was first studied by KRAMERS [2] but no information can be given for other cases. Generally one could avoid the difficulty arising from the lack of information on the population of the excited levels by simply using a probe light beam of much lower frequency than that of the resonance lines. However, owing to the very large effects to be expected in a wave-length region where anomalous dispersion occurs, it seems reasonable to believe that the study of low energy discharges - the so-called "gaseous electronics" - could benefit from this rather neglected field of research [3, 4].

TABLE I

REFRACTIVITIES OF GASES (0°C, 760 torr)
REFERRED TO CAUCHY'S FORMULA

	A	B(cm²)
He	3.48×10^{-5}	0.08×10^{-14}
Ne	6.66×10^{-5}	0.16×10^{-14}
A	27.97×10^{-5}	1.56×10^{-14}
Kr	41.89×10^{-5}	2.92×10^{-14}
Xe	68.23×10^{-5}	6.92×10^{-14}
H	13.58×10^{-5}	1.02×10^{-14}
N	29.06×10^{-5}	2.24×10^{-14}
Hg	87.8×10^{-5}	19.8×10^{-14}

If we consider only wave-lengths far enough from the resonance lines which, for the most usual gases are given in Table I, we can develop Eq. (1.2) in a series of powers of λ^{-1} which, stopped at λ^{-2}, gives Cauchy's formula

$$n - 1 = A + \frac{B}{\lambda^2} \tag{1.3}$$

in terms of which refractivities (or polarizabilities) of most gases are tabulated [5]. In so doing one disregards the contribution of excited states. This is still a rather questionable point because, whilst on the one hand polarizability is proportional to the fourth power of the mean radius of the outermost electron [6], on the other, the number density of excited atoms remains rather low because of the rather low temperature or because of the decrease of neutral atoms available in the case of hot plasmas.

A worse situation occurs in considering the ion refractivity. In this case it seems hard to admit a low contribution of excited ions, at least in the vicinity of their resonance levels. In this case Cauchy's formula is meaningless. Methods are known for estimating the order of magnitude of ion polarizabilities [7, 8]. Generally speaking it turns out that the refractivity of ions can be considered - apart from resonances - to be the same as that of the corresponding atoms, but of course a better knowledge of this field would be helpful. These considerations naturally do not apply to proton and deuteron gases, whose refractivity can be obtained by using the formula for an electron gas.

Let us now turn to the electron gas refractivity. A very simple approach to its evaluation is obtained [7] by dropping out the frequencies of bound states ω_{IK} :

$$n-1=-\frac{2\pi e^2}{m}\frac{N_e}{\omega^2}, \tag{1.4}$$

or

$$n-1=-4.46\times 10^{-14}\,N_e\lambda^2, \tag{1.4'}$$

$$(N_e \text{ in cm}^{-3},\ \lambda \text{ in cm}).$$

For protons one has:

$$n-1=-2.42\times 10^{-17}\,N_e\lambda^2, \tag{1.4''}$$

and for deuterons

$$n-1=1.21\times 10^{-17}\,N_e\lambda^2. \tag{1.4'''}$$

These formulae hold well only for a gas of free charged particles for frequencies at which they can be considered independent.

In the plasma approach to the problem different assumptions can be made and, accordingly, one finds in the literature a large number of dispersion laws [9, 10, 11], the one quoted in Eq. (1.4) being the limiting case. We refer here to a rather general one [12], which reads

$$n^2-1=\frac{-\omega_p^2}{\omega^2}\,\frac{1\mp\frac{\omega_L}{\omega}+\frac{i\nu}{\omega}}{\left(1\mp\frac{\omega_L}{\omega}\right)^2+\left(\frac{\nu}{\omega}\right)^2}. \tag{1.5}$$

It takes into account the presence of a magnetic field parallel to the light path by means of the Larmor frequency ω_L and the electron collision frequency ν.

The $\pm$ signs take into account the two circularly polarized waves which propagate with different phase velocities (Faraday effect).

This formula coincides with that of BURKHARDT and SCHLÜTER [13] if one disregards the effect of the magnetic-field:

$$n^2 - 1 = -\frac{\omega_p^2}{\omega^2}\,\frac{1}{1 - i\nu/\omega}\,. \tag{1.6}$$

The collision frequency ν can be evaluated by means of

$$\nu = \frac{N_e^2\, 16\pi^3\, N_i\, e^4}{3\sqrt{3}\, h\omega\, (2\pi kT)^{\frac{1}{2}}}$$

and the result is that in the optical region, even with the largest range of parameters, collisions cannot have any practical effect on the refractivity.

Therefore Eq. (1.6) reduces to the well-known expression

$$n = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}} \tag{1.7}$$

and, to a first approximation

$$n - 1 - \frac{1}{2}\frac{\omega_p^2}{\omega^2} = -\frac{2\pi e^2}{m}\,\frac{N_e}{\omega^2}, \tag{1.8}$$

(cf. Eq. (1.4), since

$$\omega_p = \left(\frac{4\pi N_e e^2}{m}\right)^{\frac{1}{2}}. \tag{1.9}$$

Summarizing, the refractivity of a plasma is the result of the following contributions: electron gas; ion gas; atoms, excited or not; molecules; collisions; and the magnetic field.

Thus, in planning an experiment based on refractivity, a good rule is to evaluate the relative importance of these contributions [14]. Nothing can be said a priori for a general discharge, whereas in a high temperature plasma in hydrogen or deuterium the electron gas contribution predominates.

2. REVIEW OF METHODS AND TECHNIQUES IN REFRACTIVITY DIAGNOSIS

Let us consider a plasma contained in a vessel of suitable shape in order to allow electromagnetic waves, at least in the UV - IR wave-length interval, to cross the plasma without being deflected by irregularities in the transparent walls. The problem is that of observing what happens to the rays of the beam of light sent through the plasma and predicting what can be inferred from these observations about the plasma itself. Here as a first approach the light illuminating the density inhomogeneities of the plasma is taken to consist of rays whose course is determined by Fermat's law. Let us now consider an individual light ray which in the absence of any disturbance would

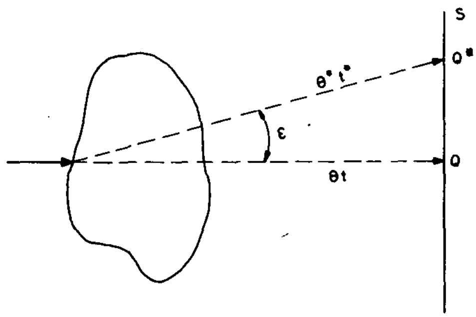

Fig. 1

Pattern of a light ray through a density inhomogeneity

have reached the registering screen S at the point Q from the direction θ at the time t, but which actually reaches it at a point Q* from the direction θ^* and at time t* (Fig. 1). The insertion of appropriate optical equipment into the light path will furnish on S a record of one of the following:

The phase lag $\tau = t^* - t$;
The deflection $\epsilon = \theta^* - \theta$;
The displacement $d = Q^* - Q$;

or a function of two or all of them. The task of the experiment is:

(a) To reconstruct from records τ, ϵ, d;
(b) To then deduce the values of n(x, y, z); and
(c) To calculate local values of the density from the law which relates the density and the refractivity.

An interferometer is a device which is able to record time lags, a schlieren is that which records deflections, and a shadowgraph records displacements.

Since a change in travelling time means a change in refractive index and hence, by (1.3) and (1.4) a change in atom or electron density, an interferometric measurement gives the value of the mean density of particles along the light path. As may be anticipated, the schlieren method depends upon the first derivative while the shadow method depends upon the second derivative of the refractive index.

Although these methods apply in principle to any density distribution - a review of the method for evaluating the results being given, for instance, in [15, 16, 17, 18] - we here refer only to a two-dimensional n(x, y) and consider a light beam travelling in the z-direction.

In the literature the subject of how a beam of light is deflected in the presence of a spatial change of the refractive index has been extensively discussed. We quote here the book by MASCART [19], part of which is devoted to the study of light patterns in the atmosphere, the article by WOLTER [20] and the book by TATARSKY [21].

The following considerations are based on the hypothesis that the total amount of deflection suffered by a light beam is small. We only need to remember that the trajectory of a thin beam of light which, in a given point of the propagating medium (plasma) makes an angle β with the direction of the density gradient, has a radius of curvature R given by

$$\frac{1}{R} \approx \frac{\sin\beta}{n} \vec{\nabla} n. \tag{2.1}$$

When a ray enters the discharge tube in a region where there is a constant gradient perpendicular to the ray, it suffers a deviation along an arc of a circle. Thus the trajectory remains roughly straight since the bending is an effect of the second order caused by the density distribution. In the mean-time the velocity of propagation of the light ray changes as it enters the vessel and the same occurs for the related phase τ as compared to that of a ray travelling in vacuo. This gives an instructive explanation of the difference in behaviour between the interferometer and schlieren methods; in both cases the rays are bent but, whilst in the schlieren this is a required feature, in the case of the interferometer this is something which is only a nuisance.

The shadowgraph shows the difference in displacement suffered by one ray with respect to that of an adjacent one, so that the net record will be zero if the displacements are equal, that is if the transverse refractive index gradient (and consequently the transverse density gradient) is constant (Fig. 2).

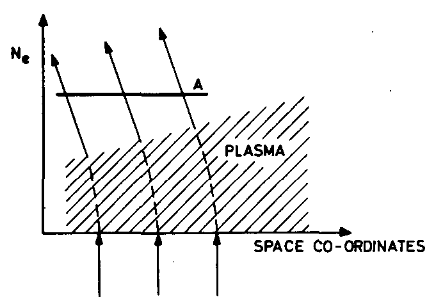

Fig. 2

Pattern of a light ray through a constant transverse density gradient.

A characteristic feature of the shadowgraph method is that a ripple in the space density distribution focuses a set of rays (Fig. 3) so that on a screen interposed for instance at A, the presence of a ripple is marked by a bright spot. With reference to the applications we quote here the measurement of the thickness of shock waves and the mapping of turbulent regions.

It is clear that generally it is preferable to measure a quantity directly instead of obtaining it by integration, and thus schlieren and shadow methods are to be supplemented by the interferometric method. But each of these methods has its own field of application where it is particularly suitable: interferometry in cases where the refractivity is slowly varying in space, the schlieren when the value of gradients is needed, and the shadowgraph when rapid (even if small) changes in refractivity occur (as ripples) and are of interest.

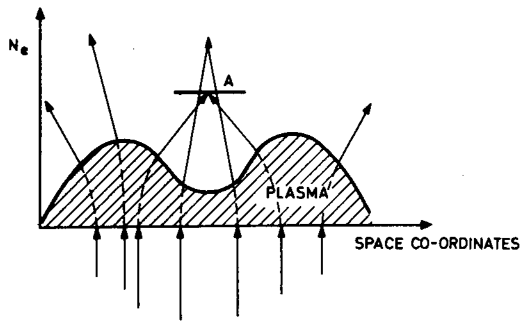

Fig. 3
The way the shadowgraph method works

3. SOME LIMITS TO THE METHODS

Each of the methods already mentioned requires an appropriate optical arrangement which will be examined in the following paragraphs; these arrangements have, however, some features in common which can be described immediately. First of all, the exploring rays which are sent through the discharge chamber are generally parallel. This requires a light source (extended or not) at the focus of a lens or mirror. The receiver, for instance, a photographic plate, is intended to take a picture of the cross-section of the tube. Therefore, on the side of the receiver there is an objective which focuses the mid-plane of the discharge or part of it, for example, a diametrical slab, after the rays have passed the particular device (beam splitters, knife edges, rulings and so on) which distinguish the method. In the case of the shadowgraph the objective is missing and the representation is left to the parallel rays which now must emerge from a point source. Let us consider the following optical arrangement in order to show some characteristic limits of the methods. It does not detect any particular effect, but it is the common part of the devices we are going to study (Fig. 4).

A beam of parallel light is sent by the extended source through the discharge tube by means of the lens L_1. The objective L_2 images the mid-plane of the discharge on the photographic plate π. Owing to the finite size of the source the rays entering the discharge tube form, at each point, a bundle of rays with an angle Γ, given by

$$\Gamma \simeq \operatorname{tg}\Gamma = \frac{\delta}{f_1}. \qquad (3.1)$$

Clearly on the geometrical optics approximation all the information belonging to the cone a b c d is recorded at the point P^*, the image of P. Two image points P_1^* and P_2^* record completely different information provided that they are images produced by two non-overlapping cones. As a consequence of this, along the diameter of the discharge tube only $D/(\Gamma L/2)$ pieces of information are completely independent of each other.

A slightly more careful evaluation [22] leads to the conclusion that as a first approximation the above quantity might be increased to $D/(\Gamma L/6)$.

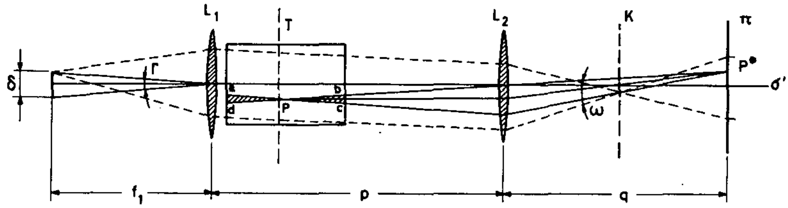

Fig. 4

Optical arrangement

But in principle we cannot rule out the possibility that some localized plasma disturbances inside the cone (shock wave, turbulence) produce some diffracted light. Let us therefore suppose that an inhomogeneity of size d is present along the pattern of the rays; the order of magnitude of the diffraction angle is $\Gamma' = \lambda/d$. The cone containing "mixed" information is $\Gamma'' = \Gamma' + \Gamma$ and in the worst situation, when the disturbance is near the surface where the rays enter into the tube, the size of the base of the cone of diffracted light on the other surface of the tube is $L\lambda/d$. If we want to have a record of the inhomogeneity d we need a Γ'' such that

$$\frac{\Gamma'' L}{6} = \frac{L\lambda}{d} = d,$$

and thus

$$\Gamma = 5\left(\frac{\lambda}{L}\right)^{\frac{1}{2}}. \tag{3.2}$$

As a consequence of the above discussion we find that one could make use of two different criteria for designing the optical equipment according to whether one wants to collect information on very small regions (diffraction limited) or whether one is satisfied with the geometrical optics approximation. In the first case Γ is established by the formula in Eq. (3.2). This has to be taken into consideration in deciding the parameters of the optical device, where one chooses that set of components which achieves the highest value of some "quality" of the instrument, for example the resolution or the "speed" (which is proportional to the light flux per unit area of the receiver). It is immediately seen that an increase of Γ diminishes the spatial resolution but increases the speed. The latter could also be improved independently by diminishing the size of the image, but at a certain point one hits the finite resolving power of the receiver itself (e.g. about 20 lines/mm on a photographic plate). Generally one comes to some compromise between the conflicting requirements of the instrument. In the field of time-resolved plasma diagnostics a very strong requirement is that of the amount of light needed in order to obtain pictures with a very short exposure time. In order to obtain useful photographs with a photographic density of about one the necessary illumination flux generally quoted [23] is 0.2 erg/cm^2 of blue light. It is rather easy to supply such an amount of light either in steady conditions or in a pulsed flash. In fact, even in the worst of these cases, when the required duration of the flash is of the order of 10^{-7} s and the dimensions are very small (e.g. a circular hole of 0.5-mm diam.) many methods have successfully developed. But a very different problem occurs if the requisite amount of light has to be at longer wave-lengths. Here one faces at the same time both the problem of the source and of the receiver; however a set of methods, though not so useful as before, can be used [24, 25, 26, 27, 28]. The situation worsens if one requires a narrow wave-length interval for interferometric purposes, owing to the condition

$$n \Delta \lambda = \lambda/2, \tag{3.3}$$

which gives the highest number n of fringes which can be produced in an interval $\Delta\lambda$ around the central wave-length λ. In this case, which of course can still be solved within certain limits, one is forced to make the most sparing use of the light at one's disposal. Let us evaluate the orders of magnitude of the parameters involved with the problem of the light source.

Let us call Σ the sensitivity of a photographic plate, defining $1/\Sigma$ as the luminous energy per cm^2 necessary to get a photographic density of one, and let E be the illumination of the plate, that is the light energy per second per unit area reaching the plate.

If R takes into account the reciprocity failure of the plate and τ is the exposure time, then

$$ER\tau \geq \frac{1}{\Sigma} \tag{3.4}$$

is the required condition in order to get a photographic density which is not less than unity. By definition

$$E = \frac{d\phi}{d\sigma'},$$

where the flux ϕ is given by Lambert's law:

$$\phi = \frac{\pi^2}{4} B \delta'^2 \sin^2\frac{\omega}{2}.$$

Here σ and σ' are corresponding surfaces in the image and object planes, that is at π and T respectively, B is the brightness of the source, and δ' is the size of the image of the source at T (referring to Fig. 4). Since

$$\delta' = \delta \frac{f_2}{f_1} \qquad \text{and} \qquad \sigma = \pi f_2^2 \,\mathrm{tg}^2 \frac{\omega}{2},$$

calling G the magnification (G = q/P), we get, for small values of ω:

$$E = \frac{\pi}{4} \frac{B\Gamma^2}{G^2} \quad . \tag{3.5}$$

By means of Eqs. (3.4) and (3.5) we get

$$\frac{\pi}{4}\frac{B\Gamma^2}{G^2} R\tau \geq \frac{1}{\Sigma}, \tag{3.6}$$

where the ratio Γ/q has to be decided in order to suit the requirements of the experiment. Let n_r be the number of resolved lines per centimetre of the photographic plate and let us assume that for each piece of information we assign K lines on the plate, that is, the point P* is represented by K lines.

The resolving power of the optical arrangement is

$$N^{-1} = \frac{1}{3}\frac{\Gamma L}{2} \tag{3.7}$$

at the discharge tube and therefore $GN^{-1} = G\Gamma L/6$ at the plate.

Matching the two resolving powers gives

$$\frac{G}{N} = \frac{G\Gamma L}{6} = \frac{K}{n_r}. \tag{3.8}$$

Making use of this, Eq. (3.6) becomes

$$BR\,\Sigma\, n_r^2 \geq \frac{N^4L^2K^2}{\tau}\frac{1}{9\pi}. \tag{3.9}$$

If one wants to choose a value of Γ satisfying Eq. (3.2), since by Eqs. (3.2), (3.7) and (3.9) one has

$$N = \frac{6}{5}\frac{1}{(\lambda L)^{\frac{1}{2}}}, \tag{3.10}$$

the following condition results:

$$BR\,\Sigma\, n_r^2 \geq \left(\frac{6}{5}\right)^4 \frac{1}{9\pi}\frac{K^2}{\tau}\frac{1}{\lambda^2} \tag{3.11}$$

which, being related by the above discussion to the resolution limit given by the wave optics, is much harder to fulfil than Eq. (3.9). Equations (3.9) and (3.11) are equations to be satisfied between the parameters of the light source and the photographic plate (B, n_r, Σ, R) in order to record within an exposure time τ a picture with a photographic density of unity. A similar relationship can be written for photocathode devices. The survey of light sources and receivers in terms of these constants is a matter for more specialized literature; some references will be briefly discussed here. Disregarding stationary sources, photographic flashes and related commercially available sources, the following rough classification can be given for very fast, high brightness sources:

(a) Free sparks in air or in xenon;

(b) Guided discharges (Wood's magnesium slabs, capillary discharges);

(c) End-on viewed discharge channels; and
(d) Exploding wires.

All of these make use of high voltage, fairly low capacitance condensers; the duration of spark light and its brightness depends upon the inductance of the circuit and special care must be taken to reduce this. Probably the minimum time of duration of such light sources is due to the decay time of the excitation. The most advanced spark light sources are generally obtained by matching the impedance of the source to the spark gap, sometimes a low impedance transmission line being used. Also barium titanate transmission lines have been employed with considerable advantage.

When referring to data on the brightness of their sources most authors use the photometric unit system and since this is related to the emitted spectrum it is often difficult to obtain a useful figure. Fortunately one of the most interesting light sources which simply makes use of an extremely fast condenser and of a very low inductance circuit, due to FISHER [29], is calibrated in MKS units. He quotes brightness of $B = 10^4$ W/cm^2 sr at $\lambda = 5500$ Å during a time $\tau = 2 \times 10^{-8}$ s.

Very different from the above mentioned sources, the ruby laser light source constitues the most singular light source ever known. Its brightness strongly depends upon exactly how it is excited, but it is yet too soon to be in a position to predict ultimate values or limits to its performance. Owing to the exceedingly small angular spread (less than 20'') its beam can be very precisely focused to give a point source. The brightness can be estimated by the formula

$$B = \frac{16}{\pi^2} \frac{f^2}{\delta^2 \phi^2} P, \tag{3.12}$$

where the symbols are explained in Fig. 5 and P is the power in watts. In the case of a coherent light source δ is of the order of the Airy disc.

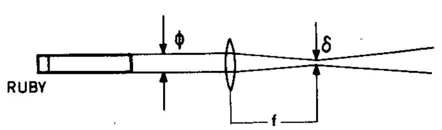

Fig. 5

Quantities appearing in the evaluation of the ruby laser performances

Using the usual figures, for a medium-powered monopulsed ruby laser the brightness given by Eq. (3.12) ranges from 10^7 to 10^{11} W/cm^2 sr.

We wish now to make a comparison between Fischer's source and a mid-powered ruby laser light source from the point of view of Eq. (3.11), assuming the following typical case:

$R \simeq 1$

$\Sigma \simeq \frac{1}{3}$ erg^{-1} cm^2 (for instance polaroid 3000 ASA)

$n_r \simeq 300$ lines/cm

$K \simeq 10$

$B \simeq 1.7 \times 10^4$ W/cm^2 sr (Fisher's source $\lambda \simeq 5500$ Å)

$B \simeq 10^9$ W/cm^2 sr (ruby laser $\lambda \simeq 6940$ Å)

$\tau \sim 2 \times 10^{-8}$ s (Fisher's source)

$\tau \sim 5 \times 10^{-8}$ s (ruby laser)

We get, for the right hand side of (3.11)

$$\left(\frac{6}{5}\right)^4 \frac{1}{9\pi} \frac{K^2}{\tau} \frac{1}{\lambda^2} = \begin{cases} 1.2 \times 10^{17} \ \mathrm{s^{-1}\,cm^{-2}} & \text{(Fisher's source)} \\ 3 \times 10^{16} \ \mathrm{s^{-1}\,cm^{-2}} & \text{(ruby laser)}, \end{cases}$$

and for the left side of (3.11):

$$BR\,\Sigma\, n_r^2 = \begin{cases} 5 \times 10^{15}\,\mathrm{s^{-1}\,cm^{-2}} & \text{(Fisher's source)} \\ 3 \times 10^{20}\,\mathrm{s^{-1}\,cm^{-2}} & \text{(ruby laser)}. \end{cases}$$

We see that a ruby laser, even of a modest performance, is able to give pictures of the best resolution, i.e. up to the limit set by wave optics, whereas one of the best known conventional light sources cannot reach this goal. About the colour composition of the light source, taking into account Eqs.(1.3) and (1.4) one realizes that the refractivity of atoms is larger in the violet region, whereas electrons can be better measured by using red-infrared light and that, in order to be able to distinguish between electrons and atoms, both colours are necessary, so that a pair of equations of the kind (1.1) can be written, one for each colour. For each colour, monochromaticity is not strictly needed. Also from this point of view the ruby laser turns out to be the best suited, owing to the well-known characteristic (due to the high power level) of being able to excite the second harmonic wave-length, $\lambda = 3471$ Å, when pulsing a suitable crystal (e.g. ADP) [30, 31, 32, 33].

In so doing, that is, making use of the fundamental and of the second harmonic at the same time, one gets both colours needed for analysing a plasma in the presence of neutral gas. On the receiver side one can simply make use of the fact that the two radiations, by the mechanism of generation, are polarized at 90°.

By using Eqs.(1.4') and (1.3') in connection with the values given from Table I, for the sensitivity, evaluated as

$$\sigma_{\substack{e\\a}} = \left|\frac{n-1}{N_{\substack{e\\a}}}\right|$$

one obtains the following values:

For electrons

$$\sigma_e = 2.15 \times 10^{-22}$$

at $\lambda = 6943$ Å and four times less at $\lambda = 3471$ Å.

For atoms

$$\sigma_{H2} \simeq \sigma_H \simeq 4.5 \times 10^{-24}$$

independent of the wave-length all along the visible spectrum [22].

Thus $$\frac{\sigma_e}{\sigma_a} = \begin{cases} 48 & \text{(on the red)} \\ 12 & \text{(in the violet).} \end{cases}$$

This means that refractivity is always favourable to the electron density measurements such that they can be measured even in discharges where very low ionization is present.

4. INTERFEROMETRIC MEASUREMENTS

The interferometric method is a very old established technique for measuring densities of any transparent material and I report here only those items which are more strictly related to plasma.

The common idea underlying all interferometric work is that of producing fringes by interference of two coherent beams of light.

The coherency is obtained by splitting the beam of light (e.g. by means of a semi-reflecting mirror) or by dividing the wave front issuing from a point source of light by using a mirror or a prism. Once these coherent beams of light are obtained they can have quite different histories. For example, one can travel in the air the other one being inside the discharge tube. If not exactly parallel, these two beams cross somewhere, in a real or virtual position, and interference fringes appear there.

Assuming, for simplicity, that indices of refraction n_1 and n_2 are constant over each path, whose geometrical lengths are l_1 and l_2 respectively, the places where the maxima of fringes occur satisfy the condition

$$n_1 l_1 - n_2 l_2 = N\lambda, \tag{4.1}$$

whereas for the minima of the fringes the above difference must be an odd multiple: $(2N+1)\,\lambda/2$. A change in refractivity of the medium of one beam carries a change in the place where Eq. (4.1) is satisfied, that is a shift in the fringe system in one or the other direction depending on whether the refractivity increases or decreases. The fringe shift expressed in number of fringes is proportional to the variation in the optical path

$$s = \frac{\Delta s}{\delta} = \frac{L\Delta(n-1)}{\lambda}, \tag{4.2}$$

where L is the thickness of the plasma crossed by the light and $\Delta(n-1)$ is the variation in refractivity. In a discharge tube the change of refractivity due to the introduction of the plasma is

$$\Delta(n-1) = (n-1)_{Plasma} - (n-1)_{Gas} \; . \tag{4.3}$$

The fringes are said to be localized at the place where the two beams intersect. The usual way of using the interferometer with a plasma is to let one beam travel along the axis of the discharge tube, the other one crossing the former virtually at the mid plane of the tube itself. In this case there is a correspondence between the points of the mid plane P of the discharge tube and the fringes. To each point on P there corresponds a particular point on one fringe. A displacement of the fringe following a change in refractivity changes the correspondence.

Permanent correspondence between the points of P and the plane at which the fringes are formed is easily established by forming the image of a wire grating on the plane P. Thus on the photographic plate which records the fringes the pattern of the wire grating is superimposed. The evaluation of densities (fringe shifts) is carried out for each point of the plane P simply by measuring the change in phase between the two fringe patterns taken before and during the discharge at the relevant points. This implies that the photometric shape of the fringe must be accurately known, but this requirement is very difficult to meet for various practical reasons. One should perhaps be satisfied with a measurement of the fringe shift made at one point per fringe. The most sensitive points on a fringe are of course those where the change in photographic density is the fastest – in practice they are the only useful points of a fringe. Therefore we see that each fringe can give only the value of density regarding two points in the plane P. Thus the number of the measured points in the mid plane of the discharge tube in the direction of the fringe shift is twice the number of fringes; in the direction perpendicular to this, the spatial resolution is that allowed by the optical device. Unfortunately this is not useful for recording by means of a rotating mirror camera.

It is not the purpose of this article to review all the instruments. We limit ourselves here to some features of those which have been most used. The Jamin interferometer is relatively easy to use but suffers from two great drawbacks: the fringe localization is at infinity and the two beams travel very close to one another. In the Michelson interferometer, because of the double transit, the localization occurs at one end of the discharge tube and the sensitivity is twice that of a single transit interferometer. The Mach-Zehnder interferometer is in most use today owing to its very peculiar characteristic of being able to give fringes which can be localized in any plane along the discharge tube.

A very extensive literature has been written on the Mach-Zehnder, its history, properties and applications [34, 35, 36, 37, 38, 39, 40, 41, 42, 14].

As far as we are concerned here the effect of the instrument can be described as that of producing on a plane M, considered as the locus of an infinite number of light sources, an image M' slightly tilted with respect to the former. Each pair of light sources, like points P and P', represent

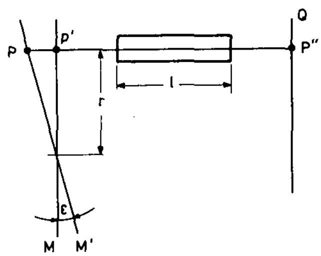

Fig. 6

Optical circuit equivalent to the Mach Zehnder interferometer

a coherent pair of sources (see Fig.6). Q is a plane where interference fringes result. In the absence of the discharge tube, fringes of order K and wave-length λ are related to the ambient refractive index n_0 and the geometric quantities ϵ and r by the relation:

$$K\lambda = n_0 \epsilon r. \tag{4.4}$$

Disregarding the possibility of having a three-dimensional distribution of refractivity as requiring too long a mathematical treatment to be presented here, we confine our discussion to the problems arising from a two-dimensional density distribution. In this case the light is sent along the third axis.

A one-to-one correspondence between the geometrical space inside the tube and the plane of the fringes being assumed, variation in density in a given region of the discharge tube is noticed by displacement of the corresponding fringes. Considering the interferometric procedure, let us assume for a moment that the variation in density is constant across the discharge tube. The condition of interference, Eq.(4.4), in the case where a tube l in length containing a gas of refractive index n_1 is placed in the path P' P'', is

$$l(n_0 - n_1) - \epsilon r n_0 = K\lambda.$$

Therefore

$$r = \frac{l(n_0 - n_1) - K\lambda}{\epsilon n_0}. \tag{4.5}$$

Calling n_1^* the refractive index of the subsequently formed plasma and r^* the corresponding distance according to Eq. (4.5), we find the fringe displacement due to the plasma

$$\Delta r = r^* - r = \frac{l}{\epsilon n_0}(n_1 - n_1^*). \tag{4.6}$$

Going from the k^{th} to the $(k+1)^{th}$ fringe one travels the thickness of one fringe and from Eq.(4.5) this turns out to be

$$\delta = r_K - r_{K+1} = \frac{\lambda}{\epsilon n_0}. \tag{4.7}$$

From Eqs(4.6) and (4.7) we obtain in the usual formula (cf. 4.2)

$$\frac{l(n_1 - n_1^*)}{\lambda} = \frac{\Delta r}{\delta}. \tag{4.8}$$

But if we now suppose that the density (or refractive index) varies across the tube and repeat the above calculation, supposing that

$$n_1^*(r + \Delta r) = n_1(r) + \frac{dn_1^*}{dr}\Delta r \tag{4.9}$$

but still with the hypothesis of negligible bending of the rays, we obtain for the fringe shift

$$\Delta r = \frac{l(n_1 - n_1^*)}{l(dn_1^*/dr) + \epsilon n_0} \tag{4.10}$$

and for the fringe thickness

$$\delta^* = \frac{\lambda}{\epsilon n_0 + l(dn_1^*/dr)r^*}. \tag{4.11}$$

From Eq. (4.10) and for Eq. (4.11) we thus obtain

$$s = \frac{\Delta r}{\delta^*} = \frac{l(n_1 - n_1^*)}{\lambda} \frac{\epsilon n_0 + l(dn_1^*/dr)r^*}{\epsilon n_0 + l(dn_1^*/dr)r}. \tag{4.12}$$

This enables us to use the same method of measuring $n_1^* - n_1$, even in the presence of constant gradients, provided that the fringe thickness is that of the fringes of the plasma interferogram and the density (and hence refractivity) gradient is unchanged in the space $r^* \div r$.

From Eqs. (4.7) and (4.11) one obtains a method for measuring gradients, which can be used as a check of the results:

$$l\left(\frac{dn_1^*}{dr}\right)_{r^*} = \lambda\left(\frac{1}{\delta^*} - \frac{1}{\delta}\right). \tag{4.13}$$

But if the gradient produces a large variation across the field of one fringe so that one can no longer grant the hypothesis underlying Eq. (4.12) the usefulness of interferometry is reduced. In order to discuss this point let us distinguish between extended and point light sources. In the latter case the following facts occur:

(a) A bent ray travels across regions of different refractivity, this giving rise to an additional phase change whose value can only be estimated on the basis of some assumption.

IGENBERG [14] has evaluated this phase shift $\Delta\varphi$ on a first but largely valid approximation, using the hypothesis of a constant value of density (or refractivity) gradient. The result is

$$\Delta\varphi = \left(\frac{1}{n}\frac{dn}{dr}\right)^2 \frac{l^3}{\lambda}\frac{1}{12}. \tag{4.14}$$

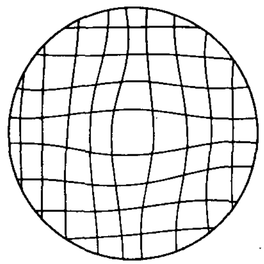

Fig. 7

Image of wire mask as seen through a discharge

(b) Still as a consequence of the bending, the magnification of the image of the cross-section of the tube is no longer constant across the field itself as shown in Fig. 7. Let us now go to the extended source. Using a point source the fringes of monochromatic light can be observed inside all the region where the beams intersect; so the focal depth of the fringe images is as long as this region.

Displacement of the point source across the focal plane of the entering lens produces rotation of the planes of the fringes. Thus each point of an extended source of diameter d in the focus f of the collimator gives a set of fringes whose plane is rotated with respect to that belonging to the on-axis light point. This rotation entails a maximum value of $\epsilon' = d/f$. As a consequence of this, the focal depth is reduced; it can be seen that now the focal depth is

$$T \simeq \frac{1.22\lambda}{\epsilon\epsilon'} = \frac{1.22\lambda f}{\epsilon d}, \tag{4.15}$$

ϵ being the usual angle between the two beams.

The bending of rays adds a rotation α to ϵ' and consequently T decreases to

$$T \simeq \frac{1.22\lambda}{\epsilon(\epsilon' + \alpha)} . \tag{4.16}$$

The evaluation of α can be made by integrating the equations of the trajectory of a light ray on the basis of some assumption about the density distribution (e.g. constant gradient).

But this is not the only drawback which occurs in the presence of gradients. A beam of light issuing from an extended light source makes cones with their apex at P (referring to Fig. 4) inside the discharge tube. Rays belonging to a given cone can experience different patterns and different phase shifts because of the possible different gradients along their trajectories. But since the cross-section T is focused on the recording plate, all rays belonging to the cone contribute to the same image point.

But the superposition of different phases gives as the result a blurring or diminution of contrast of fringes and may even cause their disappearance.

Summarizing, we notice that the difficulties arising from the presence of gradients can be reduced by the use of a point source (e.g. from a laser) but of course one cannot go beyond the limit given by Eq. (4.14).

Turning to the practical use of the interferometer, we now deal with its potentiality, which can be evaluated in terms of its sensitivity and the amount of information it is able to measure. Let us first consider the measurement of uniform density. The amount of information which can be measured potentially is given by the inverse of the least fringe shift which can be detected or, supposing that the fringe is not altered in shape during the shift, the number of parts in which a fringe can be meaningfully divided by the receiver (e.g. photographic plate plus microdensitometer). If the thickness δ^* of the fringe can be analysed by steps of size Δl the number of pieces of information contained in one fringe shift is $\delta^*/\Delta l$ and the sensitivity is

$$s_{min} = \frac{\Delta l}{\delta^*} \tag{4.17}$$

whence, because of Eqs. (1.4') and (4.2)

$$s = -4.46 \times 10^{-14} N_e \lambda l, \tag{4.18}$$

we can evaluate $(N_e l)_{min}$.

Where the density is not uniform and $\mathscr{N}$ fringes are needed, the total information is $\mathscr{N}(\delta^*/\Delta l)$ or Eq. (4.17) $\mathscr{N}/s_{min}$. Now $\Delta l/\delta^*$ depends upon the manner in which the receiving system is used; two typical cases can be proposed according to whether the light for recording the fringes is energy (or time) limited or not.

If d is the dimension of the used part of the photographic plate,

$$d = \mathscr{N}\delta^*, \tag{4.19}$$

thus

$$\frac{d}{\Delta l} = \frac{\mathscr{N}}{s_{min}} . \tag{4.20}$$

In the first of the two cases mentioned above d is limited by the energy and in principle could be estimated on the basis of the condition in Eq. (3.9). But let us assume for the sake of brevity that $d \simeq 5$ cm, $\Delta l \simeq 1/200$ cm as a typical case. By means of Eq. (4.18), Eq. (4.20) becomes

$$5 \times \frac{1}{200} = \frac{\mathscr{N}}{4.46 \times 10^{-14} \lambda \ (N_e l)_{min}} \tag{4.21}$$

that is, at $\lambda = 7000$ Å the relationship for the minimum perceptible plasma density is

$$(N_e l)_{min} = 3.2 N \times 10^{14} . \tag{4.22}$$

In doing this we have assumed, as is true in practice, that the limitation in the energy of the light source consequently limits the size of the plate.

In the other case, the size of the fringes can be made larger and their evaluation is only limited by the microphotometer, the number of fringes being irrelevant. Therefore in Eq. (4.17) one could introduce for $\delta^*/\Delta l$ a value which, for a high-quality microphotometer, is of the order of 10^4 obtaining, at $\lambda = 7000$ Å

$$(N_e l)_{min} = 3.2 \times 10^{13}.$$

Apart from the requirements on the optical devices, it seems to be very doubtful if one could reach such high sensitivity because, with conventional sources, the necessary amount of light can only be delivered in fairly long exposure times and even in steady plasma the fluctuations in density could diminish the contrast of the fringes.

Practical values of $\Delta l/\delta^*$ quoted in the literature range between 1/10 and 1/100 and only a very accurate measurement obtained by KENNEDY and cited in [43] reaches 1/1000. Satisfactory measurements of density distributions within low density shock wave fronts up to 1/500 fringe have been obtained by very sophisticated methods [44, 45].

A few words and references on other kinds of interferometers that may be useful in plasma physics. The diffraction grating interferometer based on the Ronchi method of optical testing is capable of giving fringes whose displacements are proportional either to the density [46] or to the density gradient [47], depending upon the set-up used.

In plasma research the region surrounding the discharge tube is often occupied by coils, vacuum facilities, etc. and cannot therefore be used for the reference beam needed in interferometry. Clearly one possibility is that of using the so-called "series interferometer", the first application of which was in 1950 by Saunders and is mentioned in [48, 49] in more general papers by POST. It consists of three partial mirrors arranged one behind the other with approximately equal optical separation. Apart from the advantage of exhibiting a very simple mechanical mount, because of its multiple-beam character, it is able to produce sharpened fringes (roughly δ-shaped) without any stringent restriction on mirror separation and monochromatic purity. This allows observations to be made in the 1/1000 fringe range [50]. A questionable point could be that of having a strong localization of its fringes. If the light source is a laser, because of its very high coherence length, one mirror could be removed [51].

5. THE SCHLIEREN METHOD

The word schliere is often used to indicate inhomogeneous regions in optical glasses which appear as streaks. Here, as in many other cases of a similar nature, the underlying phenomenon involves small changes of the refractive index of the transparent material, so that direct visual observation of the refractive index is difficult. Schlieren is the name given to optical methods which are based on the refractive index change (or gradient) and are able to show such inhomogeneities. Extensive use is being made of this method in its various forms [52, 20], but here it is only necessary to recall their general outlines, emphasis being placed on their use for plasma problems.

Regarding a discharge tube, whose geometrical disposition is shown in Fig. 8, it can be shown easily that the deflection suffered by the beam of light travelling through a refractivity gradient $\partial n/\partial x$, $\partial n/\partial y$ is, for small angles,

$$\epsilon_x = \frac{1}{n_0}\int_0^l \frac{\partial n}{\partial x}\, dz; \qquad \epsilon_y = \frac{1}{n_0}\int_0^l \frac{\partial n}{\partial y}\, dz, \tag{5.1}$$

n_0 being the ambient refractivity.

The schlieren techniques are able to detect these small angles. To describe these methods we choose Topler's method, which translates this deviation into a variation of luminous intensity. The set-up is given in Fig. 8.

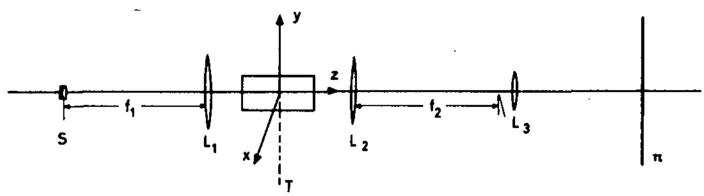

Fig. 8

Optical arrangement and quantities of a Toepler's schlieren

The light source is an extended one, but still of small size - its dimensions are estimated later in this paper. The discharge tube is crossed by a set of parallel beams, as in Fig. 4. The image of the source is obtained at K, L_1 and L_2 being placed at their focal distances from S and K respectively. A third lens forms the image of the test section T on the screen π. The same result can be obtained by using two concave mirrors in the places of the first two lenses. If now part of the image of the light source is intercepted by a partial screen (knife edge) placed at K, the illumination of the screen diminishes, each pencil of light being subjected to the action of the knife edge to the same extent; each point on the screen is darkened to the same degree. Let us now suppose that the knife edge is placed along the x direction. When optical disturbance is present in the test section at a given luminous pencil P, the latter suffers a deviation given by ϵ_x, ϵ_y. At the knife edge the deviation of this bundle of rays is given by

$$\Delta b = \epsilon_x f; \qquad \Delta a = \epsilon_y f. \tag{5.2}$$

The deviation Δb displaces the image of the source along the knife edge and thus does not produce any change in the amount of light transmitted past the knife edge. The contrary happens for the Δa deviation, which produces on the image point of P on the screen, a change in intensity which is proportional to $\epsilon_y f$. In this connection two parameters of the method are of interest – the sensitivity and the range.

In order to get the first we must have an expression for the illumination E of the screen. In the same way as in section 3, but making use of a rectangular-shaped light source, we have the following expression:

$$E = B\frac{(f+r)^2 a\, b \cos^2 \omega/2}{t^2 f^2}, \tag{5.3}$$

where B is the brightness of the light source and a and b are the dimensions of the part of the image formed by rays which pass the plane of the knife. The other quantities are shown in Fig. 4. This expression is derived from the definition of illumination $E = d\phi/d\sigma$. Lambert's law $\phi = \pi B\, a\, b \sin^2 \omega/2$, the magnification $\sigma'/\sigma = [t/(f+r)]^2$ and the geometrical relation $\sigma = \pi f^2 tg^2\omega/2$. For small angles and with $\Gamma \ll f$ we have:

$$E \simeq B\frac{a\, b}{t^2}. \tag{5.4}$$

The optical disturbance at P changes the illumination of its image by

$$\Delta E = B\frac{b}{t^2}\Delta a \tag{5.5}$$

with Δa given by (5.2).

Thus for the relative sensitivity $\Delta E/E$ the following relationship holds:

$$\frac{\Delta E}{E} = \frac{\epsilon_y f}{a} = \frac{\Delta a}{a}. \tag{5.6}$$

This gives the constant to be expected between background intensity of image of non-perturbed points and intensity of perturbed points.

One could also define a sensitivity per unit deflection:

$$S = \frac{d(\Delta E/E)}{d\epsilon_y} = \frac{f}{a}. \tag{5.7}$$

With reference to the range, let us suppose, as is usually the case, that the light source is imaged half on the knife and half on the free part. Then the possible range of displacement is 2a and the corresponding angle $\overline{\epsilon}$ is

$$\overline{\epsilon} = \frac{2a}{f}. \tag{5.8}$$

As a consequence the sensitivity, after establishing a range that is the size of the source, turns out to be independent of the properties of the optical system, and the product of sensitivity and range is a constant:

$$S\overline{\epsilon} = 2. \tag{5.9}$$

This description based on the geometrical optics of the Toepler schlieren arrangement is seen to be not completely valid when one considers the limitations due to wave optics. The first difference is that the useful part of the optical device is limited in extent, since the image of the source on the knife is altered by the diffraction with an indetermination Δ^* of the order of $\lambda f/D$, D being the inner diameter of the discharge tube. This may be expressed in terms of a spurious effect of deviation of magnitude

$$\epsilon_u = \frac{\lambda}{D}, \tag{5.10}$$

so that

$$\Delta^* = \epsilon_u f. \tag{5.11}$$

This is present even in the absence of perturbations and adds a blurring halo of size

$$a^* = \frac{\Delta^*}{2} \tag{5.12}$$

to the contours of the image of the light source.

The same occurs if, instead of being limited by the finite size of the tube the plane wave is limited to a size D^* by some variation in refractivity. Consequently, if the highest sensitivity is to be reached one cannot go to sizes of a smaller than a^*. Therefore the minimum observable deviation is, by Eq. (5.6)

$$\epsilon^*_{\min} = \frac{\lambda}{2D^*}\left(\frac{\Delta E}{E}\right)_{\min}, \tag{5.13}$$

and depends on the spatial resolution (through D^*) and the minimum variation of the illumination resolvable $(\Delta E/E)_{\min}$. Indicating $N = D/D^*$ the spatial resolution, we obtain from Eqs. (5.10) and (5.13)

$$\epsilon_{\min} = \tfrac{1}{2}\epsilon_u N \eta , \tag{5.14}$$

where $\eta = (\Delta E/E)_{\min}$ is a characteristic feature of the receiver (photographic plate, image converter used as shutter or intensifier or both). It is a measure of the smallest percentage of brightness which can be perceived by the receiving device and is generally a function of the brightness itself and thus of the quality of the source.

Now $\epsilon_{\min}$ can be written in terms of the minimum resolvable plasma gradient. Supposing that the plasma density is uniform in the z-direction we get from Eq. (5.1)

$$\epsilon_y = \frac{1}{n_0} 1 \vec{\nabla} n, \tag{5.15}$$

and from Eq. (1.4')

$$\epsilon_y = -4.46\times10^{-14}\lambda^2 \; l\vec{\nabla}N_e \tag{5.16}$$

with the usual units.

Thus the minimum resolvable gradient of electron (= plasma) density is, from Eqs. (5.14) and (5.16)

$$\vec{\nabla}N_e = \frac{\epsilon_u N\eta}{2\times 4.46\times 10^{-14}\;\lambda^2\; l}\,. \tag{5.17}$$

It might be worth while pointing out that this result is an aspect of the uncertainty principle for optics which is contained in Eq. (5.10).

A much better method for examining the theory of the schlieren effect can be followed by using Abbe's procedure for the theory of images.

This has been done by Zernike and one of the results was the phase contrast method which can be thought as an improvement of the usual schlieren method. The starting point lies in the statement that the distribution of the light just leaving the discharge tube and that at the plane of the knife edge are complex Fourier transforms of each other. The effects of the knife edge, or whichever screening device is employed, or of a phase plate in the case of the phase contrast method, are described by dropping out or modifying the corresponding terms in the transform. In so doing one reaches a unified representation of the various different forms of the schlieren method. By working out some particular case mathematically a set of relationships is obtained which must be satisfied in order to optimize the quality of the information received. For the Toepler system this has been done by H. J. SHAFER [53]. We quote here some of his conclusions.

A large aperture gives a high (aperture)/(disturbance-size) ratio with consequent high contrast and density;

A large light source will give high density and low contrast;

The optimum-size light source is one whose geometrical image in the focal plane of the objective is equal to the width of the Airy disc of the objective; and

The minimum-size light source is one whose half width is equal to the distance from the central maxima of the diffraction pattern of the disturbance under study to the optical axis in the focal plane of the objective.

Hitherto we have dealt with the Toepler arrangement. It is the only one capable of giving information in a pictorial, intuitive form, but the results of various experiments made on plasma suggest that this arrangement cannot be used as a quantitative tool. Of the various practical reasons for the above assertion it will be sufficient to remark that in most cases, wherever the ionization process is developing, it is necessary to have, for each instant of interest, two photographs taken at different wave-lengths (violet and red) to be able to distinguish between atoms and electrons. Therefore a dichroic beam-splitter is required and a pair of interferential filters. Generally for fast discharges requiring very short exposure times it is necessary either to make use of very fast photographic plates or of image-converter intensifiers [54]. In most cases these receivers do not reach a range of two decades of linear response and the overall accuracy is very poor.

A slight improvement can be obtained, still using photographic densitometry, by including in the field of the test object a glass wedge or a lens of known characteristics (standard schlieren) and thus making comparisons and interpolations between the observed darkenings of the plate. This is the hypothesis that the deflection produced by a thin wedge is equivalent to that of a thick layer if

$$(1/n_0)\int_0^l (\partial n/\partial x)dz$$

is the same in the two cases. It can be shown that this holds exactly only within the limit of geometrical optics.

In deriving Eq. (5.17) we assumed that the field of the discharge tube was divided into parts of extension D* supposing tacitly that inside this field the value of the density gradient was constant.

This allowed for the hypothesis that the plane of the knife edge was the focal plane of the lens L_2 (Fig. 8), whence Eq. (5.10) and the subsequent development. But this could not be so, for example, where the plasma has a bell-shaped density distribution. In this case the plasma behaves like a lens and one is no longer allowed to take a constant value for f. Here the knife edge is no longer at the right setting and Fresnel diffraction fringes occur, as is demonstrated in Fig. 9 [20]. This effect of course can be seen particularly in the case where, with the aim of increasing sensitivity, one makes use of very small a - Eq. (5.7) - as in the case of ruby laser point sources. A theta-pinch schlieren photograph showing this effect is given in Fig. 10. It can be demonstrated [51] that, in the case of a point source of light, in order that the change in light intensity due to this de-focusing effect be a small percentage c of the change due to true deflection, the following relationships must hold between the second and the first derivative of the density distribution:

$$cD^* \left|\frac{d^2N_e}{dx^2}\right| \leq \left|\frac{dN_e}{dx}\right|. \tag{5.18}$$

This is generally a more stringent condition on $(\nabla N_e l)_{min}$ than Eq. (5.17).

It is therefore essential to look for other ways of recording the deflections experienced by a beam of light in the presence of density gradients. The most spontaneous method is due to LAMM [55] and is mentioned in [56]. The optical arrangement is that of Fig. 11 a point light source being necessary. Near the entrance window a transparent scale is added. The lens gives an image of the scale on the plate. In the absence of disturbance the image of the scale is reproduced without any distortion. An optical di-

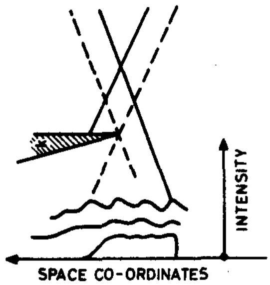

Fig. 9

Light distribution of an out-of-focus schlieren arrangement

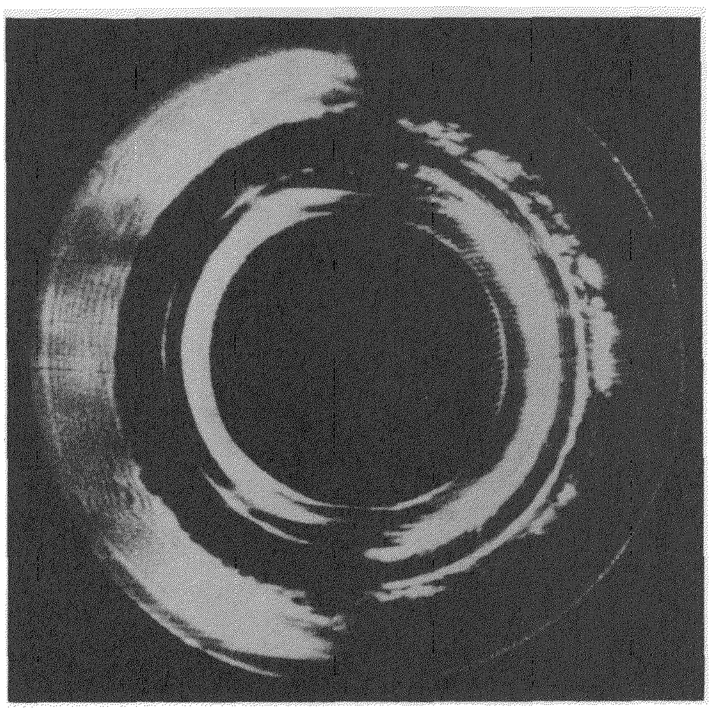

Fig. 10

A schlieren Toepler picture taken with a ruby laser during a θ-pinch experiment showing the out-of-focus effect referred to in the text

sturbance bends the pattern of the ray inside the tube and the image of the scale becomes distorted. In Fig. 11 the disturbance displaces the point A' image of A to A'' which, in turn, in the absence of disturbance, is the image of A_1. The displacement $AA_1 = \Delta_1$ can be written in terms of ϵ:

$$\Delta_1 = a\,\epsilon, \tag{5.19}$$

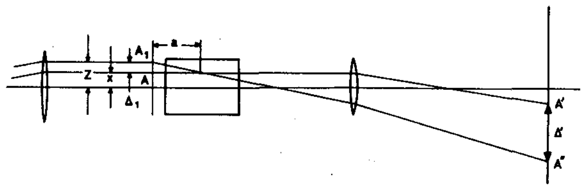

Fig. 11

Lamm's optical arrangement

and that of $A'A'' = \Delta'$ is evaluated in terms of Δ_1 and of the magnification G:

$$\Delta' = G\Delta_1 . \tag{5.20}$$

Since from

$$\epsilon = \frac{1}{n_0}\int_0^1 \frac{\partial n}{\partial x}\,dz \simeq \frac{1}{n_0}\, 1\vec{\nabla}n \simeq 1\vec{\nabla}n, \tag{5.21}$$

$$\epsilon = 1\,\vec{\nabla}n, \tag{5.22}$$

we have

$$\vec{\nabla}n = \frac{\epsilon}{1} = \frac{\Delta_1}{a1} = \frac{\Delta'}{aG1} . \tag{5.23}$$

This displacement occurs at a value of x given by

$$x = z - \Delta_1 = z - \frac{\Delta'}{G} = \frac{z' - \Delta'}{G} . \tag{5.24}$$

It can be seen that the relationship holds exactly, provided that the trajectory is a parabola, because only a parabola among the elementary curves has the property that the tangent at c crosses parallel to the optical axis of the device at BP = BC/2 (Fig. 12). This requires that the ray travels inside a constant gradient density field.

This method is very accurate but requires a lengthy evaluation of the recording plate.

If the discharge has axial symmetry measurements can be limited to those made along one diameter. In this case the focal shift of the line with respect to the straight diametrical line gives only the azimuthal component; if the scale is dotted, one also obtains the radial component by measuring the spacing between the projections of the dots on the straight line. A different method [57] of recording has been studied by the author and co-workers. The optical arrangement is shown in Fig. 13. Here W is a point source of light (a ruby laser light beam focused by means of a lens whose focus is at W). The light is made parallel by means of a collimator, C. The image

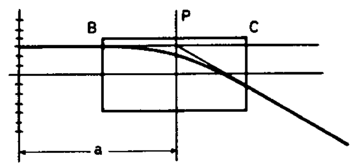

Fig. 12

The bending of the ray referred to in the text

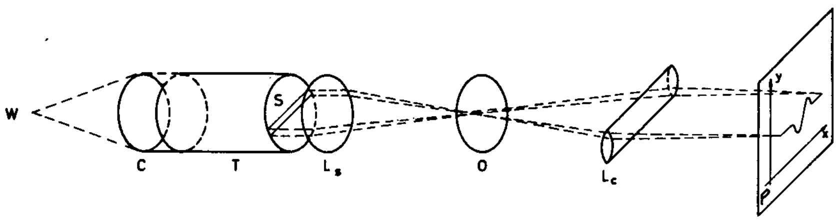

Fig. 13

Optical arrangement of a direct reading schlieren system

of the source, formed by L, is altered by means of a cylindrical lens L into a straight line (in the absence of disturbances) focused at π. The objective O focuses at π an image of the slit S which is placed in front of the discharge tube in order to restrict the observed region to a thin slab through the diameter of the tube. The lens L_c has no focusing effect on light in the x-direction so that each co-ordinate corresponds to a given point of the tube diameter. In the presence of an optical disturbance the angular deviation is recorded on π with a displacement s such that

$$s = G\epsilon f_s, \tag{5.25}$$

f_s being the focal length of L_s and G the magnification of the cylindrical lens.

The effect of the objective O can be neglected. From Eqs. (5.15) and (5.16) it follows that

$$\frac{\vec{\nabla} n}{n_0} = \frac{j}{G l f_s}, \tag{5.26}$$

and

$$\vec{\nabla} N_e = \frac{j}{G f\ l \lambda^2 \times 4.46 \times 10^{-14}}. \tag{5.27}$$

It can be seen that $\vec{\nabla} N_e$ is the azimuthal component of the refractivity gradient. To measure at the same time both radial and azimuthal components at each point of the slab of the discharge tube defined by the slit a beam splitter and two cylindrical lenses are used. Their axes are perpendicular to each other and are inclined at 45° with respect to the slit S. In this arrangement the radial component causes displacements equal but in opposite directions in the two records, whereas the displacements resulting from the azimuthal component are equal and in the same direction. In the photographs the records are taken together with a zero line. For each value of r the azimuthal and radial components are thus obtained simply by summing or subtracting the corresponding displacements of the two records. A substantial improvement in the quality of the pictures can be obtained by adding the Minimumstrahlbezeichnung method [20].

All schlieren systems employing a parallel beam of light are unable to distinguish between the density gradients occurring at various positions along the light path. In many cases it would be desirable to obtain an image in which density gradients in a given plane determine the image obtained. A schlieren system with multiple sources and corresponding knife-edge has sharp-focusing properties. It is based on the principle that for planes out of focus, superposition of the images blurs out the effects of density gradients not in the focal plane.

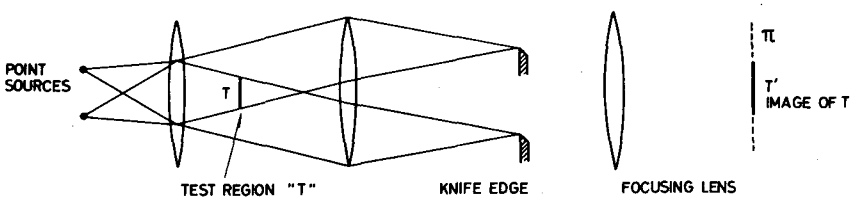

Fig. 14

The principle of the focus schlieren

Shock waves in a nozzle have been successfully studied with such a device.

In plasma research a great deal of work could be done with such an arrangement and the study of tearing modes and of end effects of the discharge tubes could greatly benefit.

It is now worth recording some more methods which appear to have rather high performances in the case of plasma diagnostics. One is that called interferential strioscopy, based on the theory given by Francon. By this method the entering window δ (Fig. 4) and the knife edge are replaced by two Savart plates or Wollaston prisms and a couple of polarizers.

Fringes are obtained on π as a result of interference between two adjacent beams. These fringes are thus sensitive to density gradients. The advantage of this method over the classical Toepler's is that of easily getting figures by means of the fringe shift; a disadvantage could be that of giving smaller spatial resolution, as in every fringe recording.

Another method which could to be some interest is the one based on an observation by Gayhart and Prescott and theoretically described by TEMPLE [33].

According to these authors, interference fringes are observed in the schlieren system, which makes possible a quantitative evaluation of the starting constant needed when one wants to obtain a density profile by integrating the schlieren working equations.

6. THE SHADOWGRAPH

Referring back to the presentation of section 2, the shadowgraph is a method which uses only the inhomogeneity of the plasma and no further optical equipment to obtain a record. The information is given by the mapping Q→ Q* and is of an implicit nature. A complex analysis would be required in order to go back from a shadowgraphic picture to the density distribution of the inhomogeneities.

Up to now very little use has been made of the shadowgraph in plasma physics, the reason being the strong requirement on point light sources, as is clear from the following. This method is quoted here and deserves particular mention because of the peculiar (potential) quality of the available information.

The underlying idea having been already presented in para. 2, we need only mention here the most important features with application to plasma diagnostics.

Referring to the Fig. 15(a), let us introduce a rectangular system (x y)($\mathcal{S}$) in the plane π of the discharge tube. In the absence of disturbances the beam

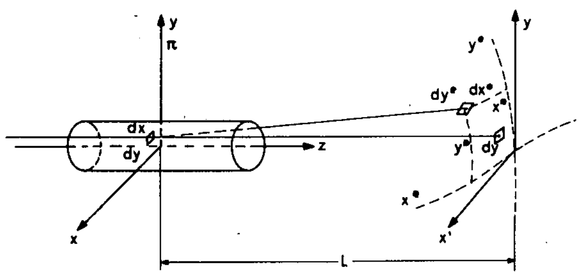

Fig. 15(a)

Quantities related to the deflection of a beam

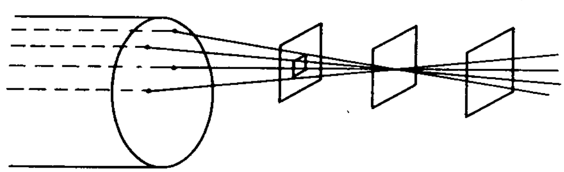

Fig. 15(b)

Focusing effect of a shadowgraph

of parallel light issuing from S will project another system of rectangular co-ordinates (x' y') ($\mathcal{S}$') equal to the former on the photographic plate π, whose distance from π is L. Disturbances in the plasma change the co-ordinate system $\mathcal{S}$' in shape and position. Let us call (x*, y*) ($\mathcal{S}$*) the new co-ordinate system superposed on x'y'. The correspondance between $\mathcal{S}$ and $\mathcal{S}$* is given by a relationship of the form

$$x^* = \phi_1 (x, y) \tag{6.1}$$

$$y^* = \phi_2 (x, y), \tag{6.2}$$

where ϕ_1 and ϕ_2 depend on the disturbance.

Let $I(x, y)$ be the light intensity distribution at π in the absence of disturbances in the plasma, and $I^*(x^*, y^*)$ the corresponding one at π when disturbances are present. Then we have

$$I^* \, dx^* \, dy^* = \sum I \, dx \, dy, \tag{6.3}$$

where $\sum$ is extended to all area elements $dx\,dy$ which contribute to I^* by collapsing into $dx^* \, dy^*$. Since

$$dx^* \, dy^* = \frac{dx\,dy}{\partial(\phi_1, \phi_2)/\partial(x, y)} \tag{6.4}$$

Eq. (6.3) gives

$$I^* = \frac{\sum I}{\partial(\phi_1, \phi_2)/\partial(x, y)}. \tag{6.5}$$

In general all these relationships are useless if one does not stipulate the very strong limitation that the displacements must be infinitesimal both inside the tube and along the entire path from the tube as far as the plate. This is verified in practice by testing at different distances L", L', L' and checking that a crossing of rays as shown in Fig. 15(b) does occur. Under this hypothesis

$$x^* = x + \Delta_x (x, y), \tag{6.6}$$

and

$$y^* = y + \Delta_y (x, y), \tag{6.7}$$

where Δ_x, Δ_y are small quantities. Since

$$J = \begin{vmatrix} \dfrac{\partial x^*}{\partial x^*} & \dfrac{\partial x^*}{dy} \\[2ex] \dfrac{\partial y^*}{\partial x} & \dfrac{\partial y^*}{dy} \end{vmatrix} = \begin{vmatrix} 1 + \dfrac{\partial \Delta x}{\partial x} & \dfrac{\partial \Delta x}{\partial y} \\[2ex] \dfrac{\partial \Delta y}{\partial x} & 1 + \dfrac{\partial \Delta y}{\partial y} \end{vmatrix} \simeq 1 + \frac{\partial \Delta x}{\partial x} + \frac{\partial \Delta y}{\partial y} \tag{6.8}$$

and

$$\begin{cases} \Delta_x = L \operatorname{tg} \epsilon_x \simeq L \epsilon_x \\ \Delta_y = L \operatorname{tg} \epsilon_y \simeq L \epsilon_y \end{cases}, \tag{6.9}$$

substitution into Eq. (6.5) gives the result

$$\frac{I(x,y) - I^*(x,y)}{I^*(x,y)} \simeq L \left(\frac{\partial \epsilon_x}{\partial x} + \frac{\partial \epsilon_y}{\partial y}\right). \tag{6.10}$$

Now, since

$$\frac{I,(x,y) - I^*(x,y)}{I^*(x,y)} \simeq \frac{I(x,y) - I^*(x,t)}{I(x,y)} = \frac{\Delta I}{I}, \tag{6.11}$$

using Eqs. (5.1) and (6.10) we get, finally,

$$\frac{\Delta I}{I} \simeq \frac{L}{n_0}\int_0^l \left(\frac{\partial^2 n}{\partial x^2} + \frac{\partial^2 n}{\partial y^2}\right) dz, \tag{6.12}$$

or

$$\frac{\Delta I}{I} \simeq -4.46 \times 10^{-14} L \lambda^2 \int_0^l \left(\frac{\partial^2 N_e}{\partial x^2} + \frac{\partial^2 N_e}{\partial y^2}\right) dz. \tag{6.13}$$

This is the working equation for the shadowgraph, in the same sense that Eqs. (4.18) and (5.16) are for interferometry and the schlieren method respectively. In this case the sensitivity – once the length of the discharge tube l is given – is proportional to L. Once the measurement of $\Delta I/I$ has been made point-by-point across the shadowgraph field it is possible, in principle, to go back to the number density by making a double integration. Apart from the possible errors, the two integration constants involved must be obtained from other experiments, one from a schlieren method and the other from an interferometric measurement. This is a disadvantage of the method but it is not its purpose.

Before reviewing the possible application, let us first consider some limitations of the method. By its very nature a shadow picture involves uncertainty in the contour lines due to Fresnel diffraction. The size Δx of the minimum resolvable region can be evaluated by means of the uncertainty principle*:

$$\Delta x \geq (\lambda L)^{\frac{1}{2}},$$

so that, for example, for $\lambda = 6943$ Å (ruby laser) and $L = 1$ m, $\Delta x \simeq 0.83$ mm.

This applies not only to the contours of the discharge tube, a picture of this phenomenon being given in Fig. 16, but also to those images arising from the focusing effect of the plasma distribution, as seen in Fig. 17, which shows a shadowgraph taken at the maximum compression of a θ pinch at 0.1 torr [58].

In this latter case one can no longer speak about pure Fresnel diffraction, but one is still concerned with the uncertainty principle. Of course, one

* WOLTER [20] p. 587.

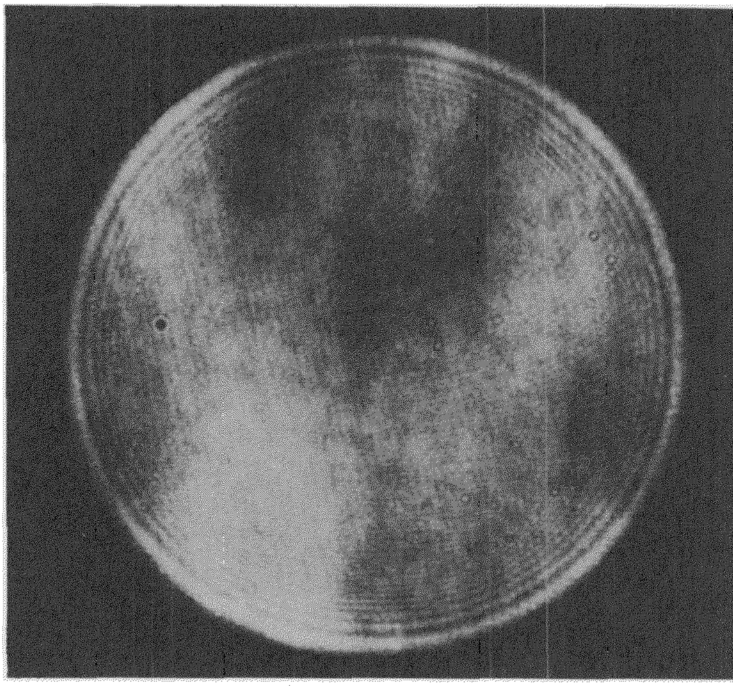

Fig. 16

Diffraction effect of a laser beam

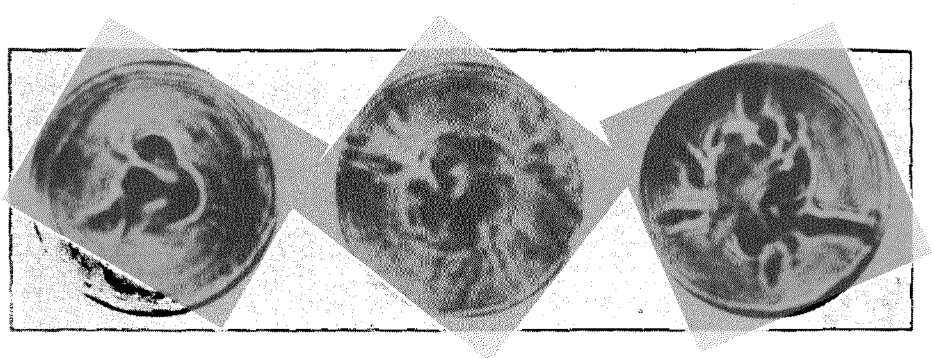

Fig. 17

Shadowgraphs of a pinch experiment of instants near the maximum compression

way of reducing this inconvenience is to reduce either λ or L as far as is compatible with the necessity of distinguishing between atoms and electrons (in the case of λ), and with the required sensitivity (in the case of L).

Reducing L has the drawback of very much increasing the amount of (unwanted) plasma light on the recorder; this can be avoided by using (Fig. 18) the optical arrangement where T' is the image of T made by the lens L'.

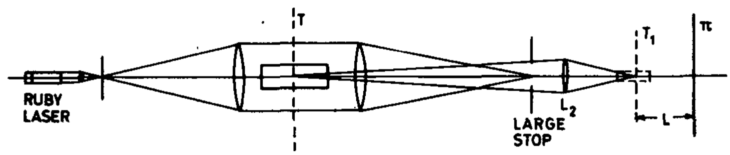

Fig. 18

A single improvement in the shadowgraph technique

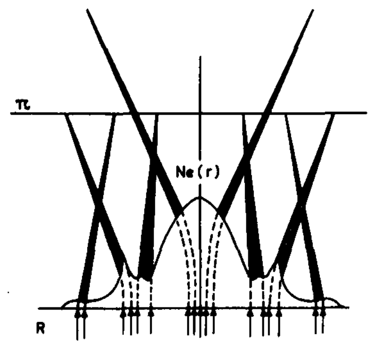

Fig. 19

The effect of changing the topology of a shadowgraph picture

Another feature has to be considered, especially when the shadowgraph is applied to gas discharges. The beam of rays is not only "focused" by the presence of small density ripples, but it is also deflected by the mean density distribution acting as a lens or wedge.

This effect is capable of changing the topology of the representation, as indicated in Fig. 19.

Applications of the shadowgraph are all related to the above-mentioned property of being sensitive to sudden changes in the refractive index. In plasma physics these occur in the case of shock-waves, instabilities and other "peculiar events", like the turbulent behaviour of plasma. But this is up to now only a list of applications which are possible in principle, because, in spite of the relative ease of this method, especially when use is made of a monopulsed ruby laser arranged to give a point source, only a few exploratory experiments by the author and his co-worker are known to the author.

In aerodynamics extensive use has been made of shadowgraphs in the study of shock-waves (thickness, shape of the shock front and its velocity, Mach-number). All this can be translated into the field of plasma physics without any great effort.

The possible application of the shadowgraph method to the analysis of turbulent density fluctuations was first discussed by KOVASZNAY and co-workers [27, 59, 60, 61]; see also Bibliography (UBEROI, U.S.).

In fact they have shown that, with the assumption of homogeneity and isotropy, the three-dimensional spectrum of the turbulent density fluctuations is a Fourier-Bessel transform of the measured correlation function of the shadow picture. This is only a rather limited application of a wider treatment, due to the same authors, of the problem of obtaining statistical properties, like correlation and spectra of random fields, from measurements obtained using averaging methods.

Unfortunately these random fields are supposed to be statistically homogeneous and isotropic.

It is unlikely that these hypotheses can be accepted in the case of plasma and thus the method cannot be directly translated from aerodynamics to plasma physics without substantial revision.

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SOME RELATED PHENOMENA IN PLASMAS, IN SOLIDS AND IN GASES

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The purpose of this paper is to acquaint you with some examples of plasma phenomena in solids which are related to those you have been studying in fusion research and to show how noting the relationship has extended plasma research and enlarged its area of application.

1. THE SCREW INSTABILITY

In 1958 two apparently unrelated discoveries were made, one by LEHNERT [1] in Sweden on electrical discharges in gases and the other by IVANOV and RYVKIN [2] in the USSR on electrical conduction in a semiconductor. The basic circuits used in these two experiments are shown side-by-side in Fig. 1(a) and (b). Though the circuits are quite similar the kinds of observations made were quite different as shown in Fig. 2 (a) and (b). Lehnert observed the static electric field necessary to maintain the discharge as a function of the static magnetic field. Ivanov and Ryvkin observed a small oscillating electric field across the sample. These experiments were important because of the unexpected and striking phenomena observed (the sudden increase in E_0 <u>versus</u> B in one case, and the oscillations in the other case) and because of their utter simplicity.

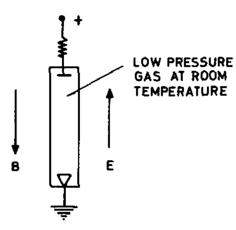


Fig. 1(a)

Basic circuit of Lehnert's experiment

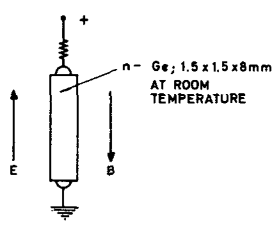
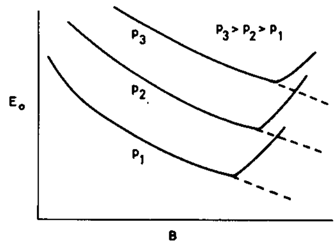


Fig. 1(b)

Basic circuit of Ivanov and Ryvkin's experiment [2]



a)

Fig. 2(a)

Nature of Lehnert's observations (p represents gas pressure)

E_ω ∿∿∿∿ ‡ - 0.1 v/cm : 10kc/sec

b)

Fig. 2(b)

Nature of Ivanov and Ryvkin's observations [2]

Let us examine Lehnert's experiment first. Why should E_0 decrease as B increases? In a magnetic field, electron and ion orbits are curled into spirals along the magnetic field lines. Collisions with neutral gas atoms cause sudden displacements of the axis of these orbits. The average displacement is about equal to the radius of the orbit and so decreases as the magnetic field increases. The frequency of collisions is proportional to the neutral gas pressure, p. The ions and electrons execute a random walk motion from the volume of the plasma out to the walls where they recombine. Thus at large magnetic fields or low pressure recombination is slower, so the rate at which the electric field must cause new ion pairs to be produced in order to sustain the discharge is smaller. The equations governing this steady state situation are

$$\underset{\text{(particle balance)}}{\vec{\nabla} \cdot (n\vec{v}_\pm) = \nu_i\, n}\,, \tag{1.1}$$

$$\underset{\text{(momentum balance)}}{\vec{v}_\pm = \pm\mu_\pm(\vec{E} + \vec{v}_\pm \times \vec{B})} - \frac{D_\pm}{n}\vec{\nabla} n\,, \tag{1.2}$$

$$\underset{\text{(energy balance)}}{e\vec{E}(\vec{v}_+ - \vec{v}_-) = K_+T_+\nu_+ + K_-T_-\nu_-}\,, \tag{1.3}$$

where n is the concentration of electrons (assumed approximately equal to the concentration of ions), $\vec{v}_{\pm}$ is the macroscopic velocity of ions/electrons, $\mu_{\pm}$, $D_{\pm}$ are the mobility and diffusion constants of ions/electrons, $\vec{E}$, $\vec{B}$ are the macroscopic electric and magnetic field strengths, $K_{\pm}$ is the fraction of the kinetic energy of an ion/electron given up in an inelastic collision, $T_{\pm}$ is the mean kinetic energy of an ion/electron and $\nu_{\pm}$ is the frequency of inelastic collisions of an ion/electron. The solution for n is of the form $J_0(2.4\ r/R)$, R being the radius of the tube. The dashed lines in Fig. 2 (a) show that this steady state theory predicts the continued decrease of E_0 with B.

Because of the importance of these two experiments, they were each repeated at several laboratories throughout the world. For instance, in 1960 Paulikas and Pyle at the University of California confirmed the Lehnert discovery and were carefully extending the range of their measurements. To certify the uniformity of the discharges they were using, they employed a streak camera to observe a section of the column. To their dismay, they found that the discharges were far from homogeneous at large magnetic field; the discharges seemed to twist and writhe despite continuing efforts to control them.

The following year KADOMTZEV and NEDOSPASOV [3] published their theory of this state. They added small time-dependent terms to the functions n and the electric potential, V, both of the form

$$J_1\left(\beta \frac{r}{R}\right) \exp i\,(kz + \theta - \omega t + \delta), \tag{1.4}$$

where k and ω are unknown constants, and θ is the angle around the tube, and t is time. They also added a term $\partial n/\partial t$ to the left side of Eq. (1.1). The result of their calculation was a complicated dispersion relation with many paramenters:

$$\omega = \omega(k; B, E, p, R). \tag{1.5}$$

A typical form for the imaginary part of ω versus k for fixed values of the parameters is given in Fig. 3. The system becomes unstable when the parameters are adjusted so that the peak of the curve in Fig. 3 pokes above the k-axis. Thus the critical conditions for marginal stability are

$$\mathrm{Im}\,\omega = 0 = \frac{\partial}{\partial k}\,\mathrm{Im}\,\omega. \tag{1.6}$$

From the two Eqs. (1.6) the value of k and E at criticality can be determined as functions of the remaining parameters, B, p and R. A typical plot of the resulting critical value of E versus B for fixed values of p and R is given in Fig. 4. Also plotted on Fig. 4 is the corresponding electric field, E_0, required by the previous theory to maintain a static discharge. The critical value of magnetic field strength, B, at which these curves cross is determined as a function of p and R by the Kadomtzev-Nedospasov instability theory. It agreed quite well with the Lehnert experiments and with many others that were subsequently conducted.

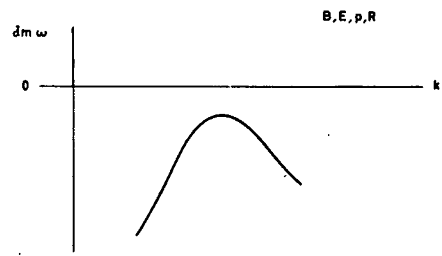


Fig. 3

A typical plot of the imaginary part of ωVsk for fixed B, E, p, R from the Kadomtsev-Nedospasov [3] dispersion law, Eq. (1. 5)

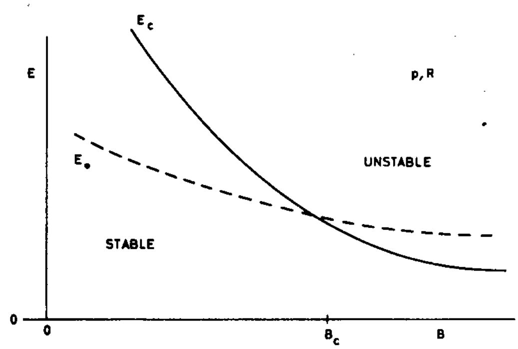


Fig. 4

Plots of critical axial electric field strength [3], E_C, and the required field, E_0, for maintainance of a static discharge versus magnetic field strength, B

When PAULIKAS and PYLE [4] became aware of the instability theory they measured the frequencies of rotation of their discharges and found fair agreement with the real part of the Kadomtzev-Nedospasov dispersion law at the critical conditions. Because the rotating column is like the shape of the threads of a machine screw, this has become known as the helical or screw instability.

Thus we see that later theory and observation was showing the gaseous plasma experiment to have oscillations like those observed in the semi-conductor experiment. Another clue to the connection between these phenomena was the demonstration in 1960 by LARABEE and STEELE [5] that a necessary condition for these oscillations to occur in semiconductors is the existence of an electron-hole plasma within the solid. An oscillator based upon the phenomenon in solids they named the oscillistor. GLICKSMAN [6] recognized the similarity of the phenomenon in solid state plasmas to that in gaseous plasmas and in 1961 adapted the Kadomtzev-Nedospasov theory to electron-hole plasmas in insulators. Observations by ANCKER-JOHNSON [7] of E_0 versus B electron-hole plasmas injected into semi-conductors showed a striking similarity to Lehnert's measurements in gases.

The only dissident experiments at this time were those conducted by R.R. Johnson. He found excellent agreement [8] with the old static theory

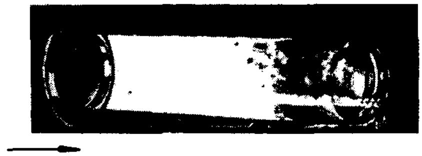


Fig. 5

Drawing of discharge tube used by R. R. Johnson [8]

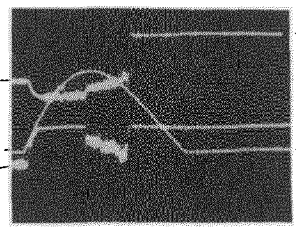


Fig. 6

Oscillogram of data obtained by R. R. Johnson [8] using discharge tube of Fig. 5. Both disk and ring currents are measured downward from base lines shown at the right

of plasma conduction in a magnetic field for values of the field as much as three to four times the critical field required for outbreak of the screw instability. In order to test the classical theory more severely, he used a discharge tube with anode separated into inner disk and concentric ring electrode as shown in Fig. 5. A typical oscillogram of the data he obtained with this tube is shown in Fig. 6. After the discharge through the tube had begun (as shown by the values of disk and ring current at the left side of Fig. 6) the applied magnetic field was made to increase slowly from zero. Very quickly the disk current increased and the ring current decreased showing confinement of the plasma column which persisted until the maximum field strength was reached. The data obtained in this way during the rising portion of the magnetic field variation are plotted in Fig. 7. The solid curve is JOHNSON and JERDE's extension [8] of quiescent plasma diffusion-conduction theory in a magnetic field. The value of magnetic field at which 98% confinement (as measured by the ring current) was reached is plotted against gas pressure in Fig. 8. As you can see, the data show somewhat more confinement than predicted by Johnson and Jerde's extension of classical static diffusion-conduction theory. This is in spite of the fact that the Kadomtzev-Nedospasov theory predicts instability over most of the range of these data.

The Kadomtzev-Nedospasov theory might be wrong. Their assumed radial factor in the form of the time-dependent perturbation (1.4) seemed suspect. JOHNSON and JERDE [9] considerably improved upon the mathematical basis of the theory but over the range of the existing experiments

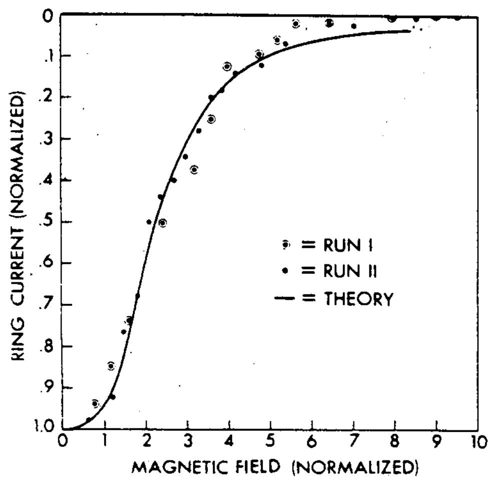


Fig. 7

Plot of confinement data obtained by R. R. Johnson [8] from (expanded) oscillograms of ring current (Fig. 5) in slowly rising magnetic field. The solid curve is from his and Jerde's extension of static plasma diffusion-conduction theory [8]

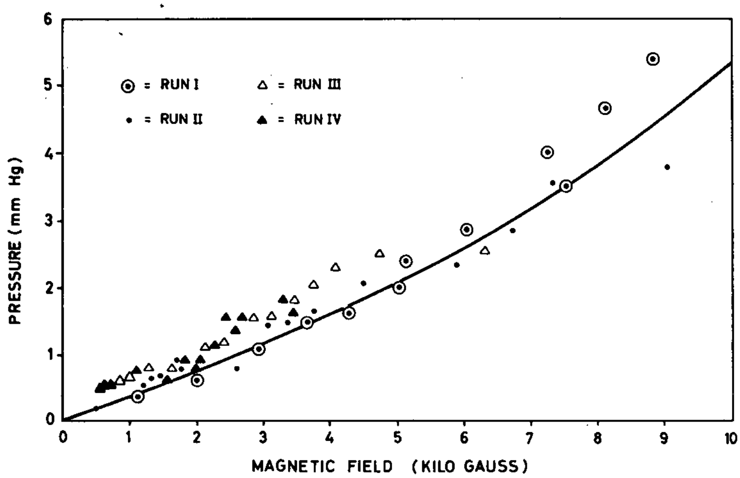


Fig. 8

Gas pressure versus magnetic field producing 98% confinement [8]. The solid curve is from Johnson and Jerde's [8] extension of quiescent plasma diffusion-conduction theory.

the resulting change in the form of the eigenfunctions did not change the critical value of the magnetic field much. However, RUGGEE [10] found experimentally that the Johnson-Jerde theory accounted for growth rates of the instability better than the Kadomtzev-Nedospasov theory did. HOLTER [11] adapted the Johnson-Jerde theory to electron-hole plasmas in semi-conductors obtaining good agreement with experiment.

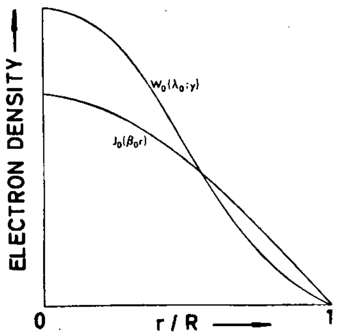


Fig. 9

Plot of the quasi-static part of the radial density distribution in the positive column of an arc discharge in slowly rising (W_0) and stationary (J_0) magnetic fields from the theory of Johnson and Jerde [12]

But the mystery still remained. Why did Johnson's experiments show a stable plasma where theory and other experiments showed instability? The major difference between Johnson's experiments and others was that he used a pulsed magnetic field to avoid overheating his magnet. It is true that a changing magnetic field produces an electric field E_θ, and that for an increasing field $\vec{E}_\theta \times \vec{B}_z / B_z^2$ is an inward radial drift but it seemed so small that it had been neglected at first. Now it was incorporated in the JOHNSON-JERDE theory [12] with the surprising result that more of the E versus B plane was unstable. The reason for this is that the inward $\vec{E}_\theta \times \vec{B}_z / B_z^2$ drift causes the steady state part of the density distribution to be more sharply peaked in the centre of the tube than the previous $J_0(2.4\, r/R)$ had been. This is shown in Fig. 9. The steeper density gradient is the driving force for the more unstable plasma. However, the greater confinement of the plasma away from the walls with the rising magnetic field caused a smaller axial electric field to be needed in the discharge. So even though the electric field required for instability was made smaller, the electric field required to maintain the discharge was made even smaller. The net result was stabilizing.* Both curves on Fig. 4 were lowered; but the E_0 curve was lowered more than the E_c curve. Thus, the magnetic field, B_c, of the intersection was moved to the right. In fact the Johnson-Jerde theory [12] shows that for fixed values of pR stabilization can be achieved to arbitrarily large magnetic fields. For instance, a discharge tube of radius 10 cm with a neutral gas pressure of 2.4×10^{-3} mm Hg pressure of helium can be stably confined by an applied magnetic field rising at the rate of 2.2 kG/ms. After 50 G has been reached the rise rate can be reduced to 0.5 kG/ms without lessening the confinement.

Because of the basic and possibly technological importance of Johnson's stabilization principle, it seemed desirable to considerably extend the range of his measurements. Such an extension has evidently been carried out by Prof. Saitsev at Moscow State University who was kind enough to describe

* This has been shown experimentally to be true also for plasmas in solids by Ancker-Johnson, Boeing Scientific Research Labs., Progress Report (Aug. 1964).

his work to Dr. Ancker-Johnson when she visited Moscow in 1962. Thus R. R. Johnson, when he learned of this from his colleague, Ancker-Johnson, was freed from the need to extend these measurements himself. Instead he devoted himself to testing another interesting consequence of the theory; that the unstable plasma should have the macroscopic property of paramagnetism.

As we all know, plasma is supposed to be diamagnetic. But the time-dependent theory of the oscillations of a positive column showed that the unstable "screw" was "right handed" or "left handed" depending on whether the axial current flow was in the direction of $\vec{B}_z$ or opposite to it. This current flowing through the helical form of the perturbed plasma would thus produce a magnetic field within the helix of the same sign as the applied $\vec{B}_z$. As Furth has pointed out, this augmented magnetic pressure on the axis is precisely what makes a perturbation of this form expand against diffusion forces which would restore the unperturbed state. R. JOHNSON has carried out a non-linear analysis [13] of the oscillating state and found among other things that this relation remains valid for finite amplitude. He conducted an experiment [14] in which he measured the diamagnetism of the stable plasma ($B < B_c$) and then noted the sudden nearly complete reversal in phase and increase in amplitude of his sensing signal as the plasma was run into the unstable region ($B > B_c$). This confirmed theoretical expectation. The size of the effect was, however, very small - the internal magnetic field jumps by only a few tenths of a milligauss in the change from diamagnetism to paramagnetism. However, since the internal magnetic field is a little larger, the externally supplied field can be reduced slightly before the critical induction is reached within the plasma. Thus small displaced hysteresis loops would be generated as shown in Fig. 10. J. DRUMMOND [15] predicted this on the basis of R. Johnson's measurements and the above argument in June 1963. Within a few weeks hysteresis loops had been found by ANCKER-JOHNSON [16] experimentally in electron-hole plasmas in a semi-conductor. But instead of being a few tenths of a milligauss in magnitude, they were over 100 G in magnitude [17], as shown in Fig. 11. In addition, they are electronically adjustable by means of the current through the semi-conductor. In some ways these ferromagnetic-like loops make desirable digital computer elements for associative-type memories. Not only can their location be adjusted electronically, but they signal their "on" condition by continuous, adjustable oscillations. The hysteresis loops are so large in the E_ω versus E_0 plane [17], as obtained from data like Fig. 12, that in principle hundreds of such elements could be selectively and rapidly ($< 1\,\mu s$) turned on and off and "read" by a single wire [18].

This example shows the extension of plasma physics into a whole new area of applications. It came about by the scaling of a small effect in a large gas chamber into a large effect in a tiny crystal. But this is only the beginning of the interaction. The stabilization schemes for plasmas such as the use of wall mirrors introduced by Ioffe can be tested on plasmas in solids much more easily than on the large fusion machines. In particular, Furth pointed out that Ioffe bars ought to stabilize the helical instability. On the other hand, Dr. Velikhov pointed out that a steady symmetrical flow of current along the conduction column would be destabilized by attraction

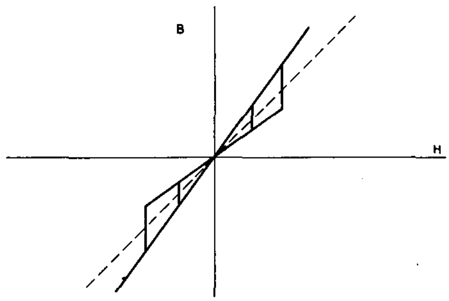


Fig. 10

Predicted [15] hysteresis of plasma.
The dotted line is free space response

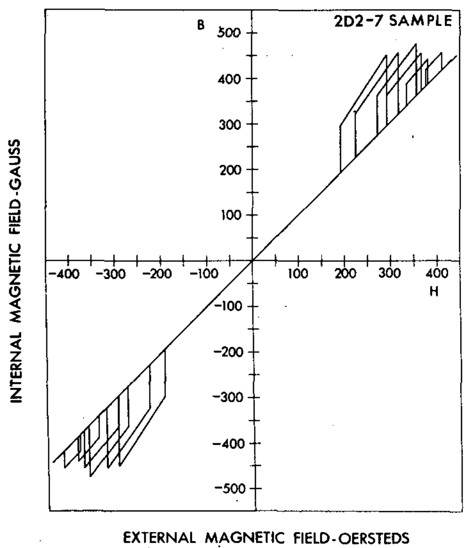


Fig. 11

Hysteresis loops inferred from experimental data taken on electron-hole plasmas in InSb at 77°K.
After Ancker-Johnson [17]

to a parallel current flow in one of the Ioffe bars. So the experiment was conducted. Ancker-Johnson used quadrupole Ioffe bars as shown in Fig. 13. As the current in the Ioffe bars increased, the instability was usually surpressed as measured both by oscillation amplitude and "DC" voltage level necessary to maintain the steady plasma current flow. Then a new form of instability showing a small amplitude incoherent oscillation and higher "DC" voltages developed at larger Ioffe currents. This is shown in Fig. 14. Many such data are plotted in Fig. 15. In addition to stabilization of the helical instability, ANCKER-JOHNSON and BERG [19] have shown that Ioffe bars significantly increase the lifetime of free electrons and holes in semi-

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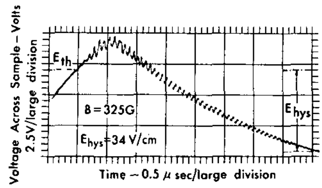

Fig. 12

Oscillogram of voltage across a small sample of p-type InSb at 77°K as a function of time showing the outbreak of oscillation at large voltage and its persistence to low voltage: a highly reproducible effect.
After Ancker-Johnson [17]

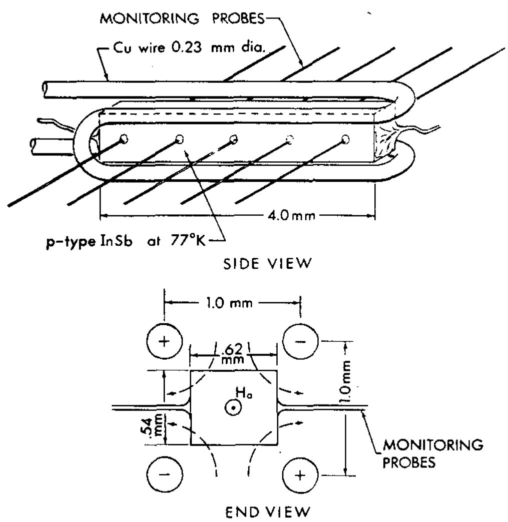

Fig. 13

Quadrupole Ioffe bars placed around p-type InSb at 77°K

conductors with zero "DC" current. All this was accomplished within a few months because of the small magnitude and extent of the magnetic field and plasma currents that were required for the tests on plasmas in solids. Extensive work is continuing.

2. HELICONS

Spies discovered a natural phenomenon during the first world war. The German scientist, Barkhausen, was tapping ground return currents of allied phones. He used an audio-amplifier with unusually long input leads and heard occasionally descending "whistles". He recognized this as a natural phenomenon, named them "whistlers" and published his findings shortly after the war [20]. Nine years later ECKERSLEY [21] noted correlation of whistlers with solar activity and that the whistlers often occurred in trains

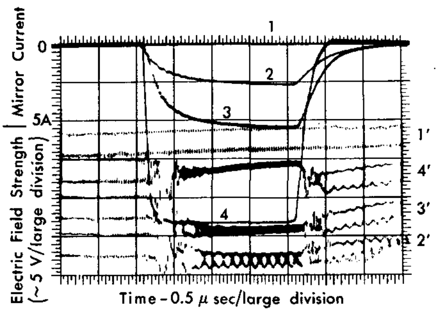

Fig. 14

Multiple exposure of oscillograms showing progressive effects of *increasingly large Ioffe bar currents (measured downwards)* labelled 1, 2, 3 and 4. These currents were pulsed on after the screw instability had fully developed as shown by the corresponding plasma voltage traces labelled 1', 2', 3', and 4'. The Ioffe bar currents all have the same "O" level (the top line of the graph) but the "O'S" of the voltage traces have been displaced to prevent overlapping of their traces. Traces 1 and 1' show zero Ioffe bar current and the undisturbed voltage oscillations. As the Ioffe bar current is increased (traces 2, 3 and 4), the oscillations are surpressed (traces 2', 3', and 4') and the "DC" voltage level first reduced (trace 2') and finally increased (trace 4'). After Ancker-Johnson [19]

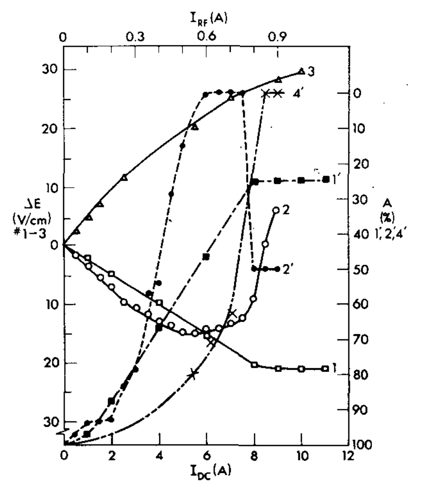

Fig. 15

Graph of the change of "DC" voltage/cm (curves 1, 2 and 3) ΔE versus "DC" Ioffe bar current I_{DC} and graph of percentage decrease of oscillation amplitude (curves 1', 2', and 4'), A, versus I_{DC} or versus "RF" Ioffe bar current (curve 4'), I_{RF}. Note how relatively very small the required "RF" current (84 Mc/s) was for removing the natural screw oscillations. After Ancker-Johnson [19] preceded by "clicks" with about 3 sec. spacing. In 1931 Tremellen [22] correlated the "clicks" with local lightening flashes. Two years later BURTON and BOARDMAN [23] measured the frequency of whistlers as a function of time. In 1935 ECKERSLEY [24] derived a dispersion law for electromagnetic waves in a plasma in a magnetic field from the APPLETON-HARTREE "magnetoionic" theory [25]. We shall use instead the standard reference text book by STRATTON [26] who gives for the index of refraction N, for left/right hand circularly polarized high-frequency radio waves in an ionized gas

$$N^2 \equiv \frac{c^2k^2}{\omega^2} = \epsilon_r - \frac{ne^2/m\epsilon_0}{\omega^2 \pm FeB_0\omega/m} \approx \frac{n_e}{\omega\epsilon_0\mu_0H_0}\left(1 + \frac{\omega}{\omega_c}\right), \tag{2.1}$$

$$\text{for } \frac{n_e}{m\epsilon_0\mu_0H_0} >> \epsilon_r \text{ and } \frac{\omega}{\omega_c} \ll 1,$$

where c is the speed of light in a vacuum, ϵ_r is the relative dielectric constant of the gas, $B_0 = \mu_0 H_0$ is the magnetic induction, m the electronic mass, and $\omega_c = eB_0/m$. Ions have been regarded as held fixed by collisions with neutrals. The validity of this dispersion law has been tested and confirmed in laboratory gas plasmas by GALLET et al. and by CONSOLI et al. [27], MAHAFFEY [28] and by DELLIS and WEAVER and most recently by LEHANE and THONEMANN [29] who obtained propagation in plasmas whose densities were more than 50 times the cut-off density.

$$n = m\epsilon_0\omega^2/e^2.$$

Retaining only the first term in the expansion of the right side of Eq. (2.1), we have

$$\omega \approx \frac{k^2H_0}{ne} \tag{2.2}$$

which results in a group velocity

$$v_g = 2\left(\frac{\omega H_0}{ne}\right)^{\frac{1}{2}}. \tag{2.3}$$

Thus the time required for a wave group to travel over a fixed distance is proportional to $\omega^{-\frac{1}{2}}$ which agrees with BURTON and BOARDMANN's data [23]. In 1953 STORY [30] made use of this fact to obtain an estimate of the average electron density experienced by the waves in following the earth's magnetic field through the outer atmosphere: $\sim$ 600 cm^{-3}.

The directly related effect in solids was also discovered by accident. In 1961 BOWERS, LEGENDY and ROSE [31] were attempting to measure the effective resistivity of sodium at 4.2°K in a magnetic field. Their experimental arrangement is shown in Fig. 16. Typical results are shown in Fig. 17. What was expected was a series of curves like the top one, but with different decay rates at different values of magnetic field strength, H_0.

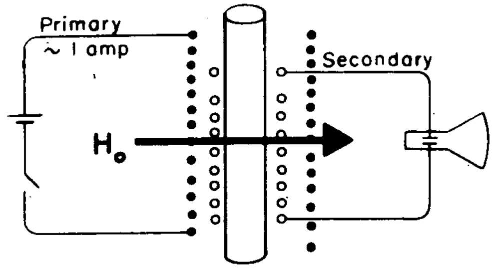

Fig. 16

Schematic diagram of the Bowers, Legundy and Rose [31] experiment on sodium at 4.20° K

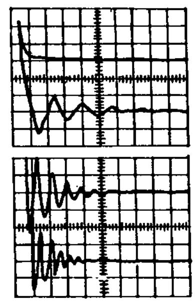

Fig. 17

Oscillograms of voltage developed across secondary coil of Fig. 16 versus time after opening primary circuit. The top trace is for $H_0 = 0$, the second for $H_0 = 3\,600$ G, the third for $H_0 = 7200$ G and the last for $H_0 = 10\,800$ G. Abscissa is 50 ms per large division.
Specimen properties: resistivity at 300° K = 7500 × resistivity at 4.1° K
$\omega_c\tau$ at 4.2° K $\approx$ 40 for $H_0 = 10\,000$ G.
After Bowers, Legundy and Rose [31]

What was found was a series of oscillations which decayed much more slowly than the expected $e^{-t/\tau}$ (with τ the mean free time of electrons against momentum transfer collisions) and had a frequency directly proportional to magnetic field strength. It was found that the proportionality constant was k^2/ne with $k \approx 2\pi/$(diameter of samples) in agreement with Eq.(2.2) for a standing "whistler" in the metal.

As it happened, KONSTANTINOV and PEREL [32] had predicted the existence of such waves in metals the preceding year and AIGRAIN [33] independently predicted for semi-conductors that the damping time should be

$$\approx \left| \frac{n_- - n_+}{n_- + n_+} \right| \frac{\omega_c}{\omega} \tau + \tau \ , \tag{2.4}$$

where $n_\pm$ is the concentration of positive/negative charge carriers. Since in solid sodium there are no positive charge carriers, the damping time becomes $(1 + \omega_c/\omega)\tau$ which agrees roughly with the Bowers, Legundy and Rose data.

Aigrain gave the name "Helicon" to such a wave because it had a characteristic mass. The plane wave solution $\exp i(\vec{k}\cdot\vec{r}-\omega t)$ of the Schrödinger equation has the same dispersion law as Eq.(2.2) if the mass of the free "particle" is taken as $\hbar ne/2H_0$. For sodium in a thousand gauss field the "mass" of a helicon is about equal to the mass of an electron.

ÅSTROM [34] first noted that if both positive and negative carriers are free to move then the first term in Eq.(2.1) will cancel with an equal magnitude but opposite sign term that comes from the positive carriers not allowed for in Eq.(2.1). Thus the second term of the expansion, which is independent of sign, will dominate. In this case the dispersion law becomes

$$\omega = kv_A; \quad v_A = H_0(\mu_0/\rho)^{\frac{1}{2}}; \quad \rho = n(m_- + m_+), \tag{2.5}$$

where v_A is the velocity first deduced by ALFVÉN [35] in 1942 to explain some features of sun spots.

In 1951 LUNDQUIST [36] published measurements of the phase velocity and damping of Alfvén type waves in liquid mercury. He obtained only fair agreement with theory probably because of the low conductivity of mercury. In 1954 LEHNERT [37] approached the problem differently: in an asbestos suit. He obtained a little better results with liquid sodium. BOSTIK and LEVINE [38] in 1952 had used an ionized gas toroid but had conditions such that the sound velocity exceeded the Alfvén velocity and somewhat obscured the results.

In fact the experimental situation seemed so hopeless that in 1957 COWLING [39] stated: "... A wide gap must persist between the best that laboratory experiments can provide and the almost perfect M.H. waves believed to be possible in the sun and stars. Extrapolation across so wide a gap is almost impossible; in cosmic work one must normally be guided by theory and hope that the theory overlooks no essential features".

But only two years later a good experimental measurement was made of the velocity of Alfvén waves in a most unexpected way. GALT, YAGER, MERRITT, CETLIN and BRADFORD [40] were measuring the absorption coefficient of micro-waves incident on the semimetal pure bismuth. At the time, the authors were unaware of it, but as BUCHSBAUM and GALT [41] pointed out two years later, this is equivalent to measuring the velocity of Alfvén waves in the bismuth. The reflection coefficient for waves incident on a medium of relatively very high refractive index N, is

$$R \approx 1 - \frac{4}{N}. \tag{2.6}$$

Since in pure bismuth there is an equal concentration of electrons and holes, Eq.(2.5) applies and Eq.(2.6) becomes

$$R \approx 1 - 4\frac{v_A}{c} = 1 - \frac{4}{c}\left(\frac{\mu_0}{\rho}\right)^{\frac{1}{2}} H_0. \tag{2.7}$$

In the experiment of Galt, Yager, Merritt, Cetlin and Brailsford, lack of reflection was measured as absorption. So the absorption coefficient should be proportional to the applied magnetic field. Their results given in Fig. 18

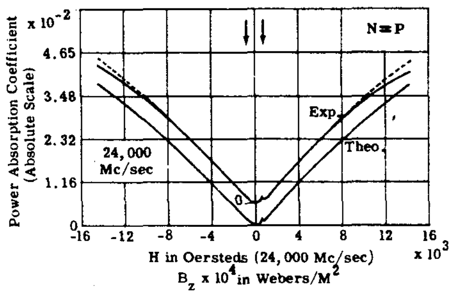

Fig. 18

Graphs of absorption coefficient of circularly polarized electromagnetic microwaves normally incident on pure bismuth versus H, the magnetic field normal to the bismuth. The linear relationship shows Alfvén waves are being excited in Bi^{41} surface after Galt et al [4]. The "O" level of the experimental curve has been elevated so that the experimental and theoretical curves can be distinguished. This necessity is remarkable in a field so noted previously for little comparison between theory and experiment [39]

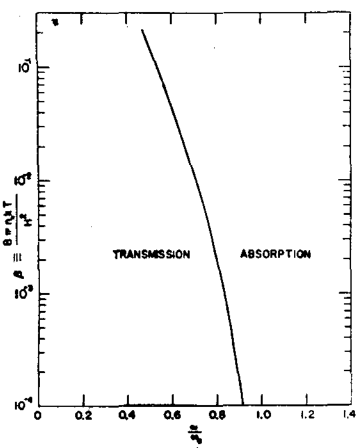

Fig. 19

Graph of $\beta = \dfrac{8\pi n_0 kT}{H^2}$ versus fractional shift in cyclotron absorption edge [42]

show this. The little "pips" just to the right of "O" on the curves represent cyclotron absorption Doppler shifted by about a factor of two. Such had been calculated by J. DRUMMOND [42] in 1958 for gaseous plasmas. One of his results is shown in Fig. 19. Two somewhat surprising things that may be seen from this curve are that the shift is many orders of magnitude larger than the diamagnetic shift in the cyclotron frequency, and is only a function of the ratio of electron pressure to magnetic pressure. MAHAFFEY [28] semi-quantitatively confirmed these results by experiments in gaseous plasmas. STERN [43] has pointed out that for most metals, the Fermi energy, $\frac{1}{2}mv_F^2$, of conduction electrons would be so great that the Doppler shifted frequency would be much greater than the helicon frequency,

$$\omega_c \approx v_F k >> \omega \tag{2.8}$$

and that this could be used to produce a point-by-point map of the curvature of a convex Fermi surface. Using Eq.(2.2) in Eq.(2.8) it can easily be seen that

$$B = C\omega^{\frac{1}{3}}, \tag{2.9}$$

where the C involves the Fermi velocity. Eq.(2.9) has been tested by TAYLOR [44] for the spherical Fermi surface of sodium with results shown in Fig. 20. The slope of this line yields a Fermi velocity only 1% different from the theoretical value.

This example shows how plasmas in solids have removed the "wide gap" between the best that laboratory experiments could otherwise provide and the theory of magnetohydrodynamic waves. Again, of course, it opens a new field of applications for the ingenuity of plasma physicists and engineers.

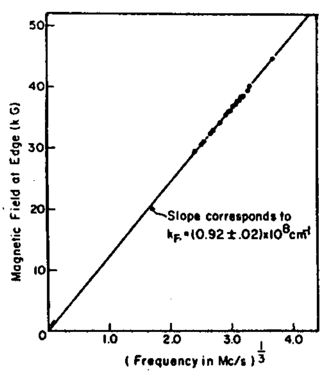

Fig. 20

Magnetic field of Doppler shifted cyclotron absorption edge versus the one third power of the frequency of helicons excited in sodium at 4°K. After Taylor [44]

3. SUMMARY

We have followed here in some detail two examples of the development of related phenomena in plasmas in gases and in solids. There are other examples of phenomena in electron-hole plasmas that have bearing on fusion research. One of these is the pinch effect. Extremely high non-equilibrium concentrations of plasma ($>10^{18}/cm^3$) have been obtained [45] with a relatively small, inexpensive apparatus. The fact that this state is so highly constricted implies that it could be used (probably in a toroidal geometry) for an initial filling of a solid state model of a shear stabilized machine or of the several new $\int dl/B$ stabilized configurations which are being proposed [46]. It is, of course, desirable to test theory against experiment in a sufficiently large machine. But would it not be desirable to select the design to be built by means of preliminary tests on small model plasmas in semi-conductors? There the small extent and small magnitude of the required magnetic fields (mass of positive carrier $\lesssim 1/2000$ of H^+ mass) contribute to making the experiments quick and inexpensive. An objection to this is the small mean free time τ of carriers in solids. However, $\omega_c\tau$ can be made $\gg 1$. As shown in section 1, the effects of instabilities and the stabilizing effects of the proper sign of magnetic field curvature can easily be detected for plasmas in solids.

Time does not permit an extended survey of the other significant related phenomena such as injection and electron-plasmon scattering. Instead, let me refer you to an exhausive survey of the current state of research on plasmas in semi-conductors by Dr. Ancker-Johnson to appear shortly [46]. At the same time, I would like to thank her for her many helpful discussions of this and related material.

The most exciting vista has opened up to plasma physics: that a single theoretical concept such as Alfvén waves can correlate phenomena in tiny solids at 4°K and in sun spots at millions of degrees.

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PLASMA THEORY AND OBSERVATIONS IN SPACE

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The subject of space physics is distinct from astrophysics and everybody knows what space physics is.

This introduction concerns the magnetosphere, a volume surrounding the earth which will gradually be defined. Its lower boundary is the ionosphere in which incoming particles and photons make collisions. The collision physics occurs in the ionosphere and also chemistry because the composition is still similar to air and the dominant positive ion is O^+. In the magnetosphere we have collision-free plasma physics and the positive ions are protons. Most collisions occur below 200 km altitude. The altitude where protons begin to dominate depends sensitively on the temperature and varies from 500 to 2000 km.

Certain American satellites have made particularly important discoveries and three are listed:

	Launched	Apogee
Explorer XII	1961	12 R_E
Mariner II	1962	Venus
Imp	1963	30 R_E

The measurements made in space are utterly different from those in the laboratory. The magnetic field is easy to measure, the electric field difficult, but the most remarkable are the measurements of the particles which constitute the plasma. This book cannot fail to demonstrate how all plasma theory flows from either the Vlasov equation, the Boltzmann equation, or the Fokker-Planck equation, each of which has for its dependent variable the velocity distribution function f, defined so that $f\,d\vec{x}\,d\vec{v}$ is the number of particles in the element of phase space $d\vec{x}\,d\vec{v}$. Now the instruments used in space essentially measure f. Some detectors accept particles from only a small cone of directions and only in a small range of energy. Others are not so simple, but f has been measured already by several different instruments and in different energy ranges.

Starting now from outside the magnetosphere all we need to know about the sun is that it continually pours out a stream of plasma. This was predicted by PARKER [1], is known as the solar wind, and was found by Mariner II [2], the observations of course being confined to the neighbourhood of the ecliptic plane. Parker predicted that the wind would be hypersonic and consequently would flow almost radially away from the sun. Mariner did not spin so the plasma probe was aimed at the sun. Measurements were made in several narrow energy ranges, giving f for protons. The wind was found to be hypersonic, so that good values were obtained for the mean speed and, by fitting a Maxwellian, a rough temperature. The Mach number ranged

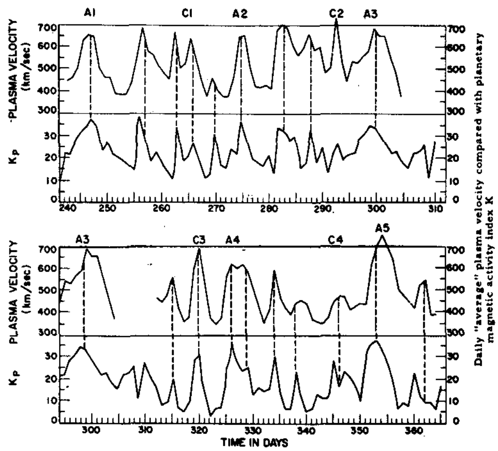

Fig. 1

Measurements with Mariner

from 5 to 10. Figure 1 shows the mean speed, the time scale being day numbers of the year (365 = Dec 31).
The density is not shown, but is typically a few protons per cubic centimetre occasionally rising by an order of magnitude. The speed is seen to vary by only a factor of 2. K_p plotted in the lower curves is a logarithmic measure of magnetic disturbance at the ground, obtained by combining the records of all observatories. The correlation is seen to be good. The nature of the phenomenon is like wind meeting an obstacle, but it is remarkable that the range of variation in K_p represents two orders of magnitude in the amplitude for only a factor of 2 in the wind.

The solar wind carries out a magnetic field, whose energy density is comparable to the thermal energy density. Parker predicted that the lines of force would be wound into spirals by the combination of the solar rotation and the wind. The equation

$$\vec{E} + \vec{v} \times \vec{B} = 0 \tag{1}$$

should be true when $\vec{v}$ is the actual plasma velocity which is approximately radial. If further the pattern of magnetic field rotates with the sun, (1) must be true when $\vec{v}$ is the velocity corresponding to rigid rotation of the sun. If (1) is simultaneously true for two different velocities, $\vec{B}$ must be parallel to their difference, and, since the speeds happen to be nearly equal, $\vec{B}$ should be at about 45° to the radial direction, with either sign possible. The Imp data [3] shows that on about half of the days observed the field lies quietly near one of these directions, apart from a component perpendicular to the ecliptic. On other days the field jitters about continually. Several times when the field was quiet its direction suddenly reversed and this was followed by quiet field in the opposite direction. Mariner also found sudden changes which were clearly travelling shocks that had originated from the sun. Sudden increases were observed on the plasma probe and sudden changes in the magnetic field.

Coming now to the magnetosphere it is helpful to recall the classic theory of Chapman and Ferraro which omitted any interplanetary magnetic field. They found that the earth's field would be confined to the interior of a "cavity", from which the solar plasma would be excluded. This model is sufficient to show that the wind finds an obstacle whose dimensions are an order of magnitude bigger than the solid earth. Because the flow is hypersonic a stand-off shock is then expected upstream from the effective boundary where there is a stagnation point. Figure 2 shows a model for the case of southward

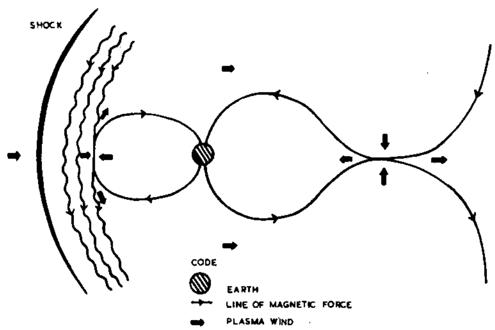

Fig. 2

Model for the case of southward interplanetary field

interplanetary field. The importance of this case will be discussed later. The boundary is found on the day-side at 8 to 13 R_E and the shock standing off about a further 5 R_E. Some properties of the shock have been obtained from the MIT*plasma probe [4] on Imp.

One energy range of electrons was observed. The flow energy of electrons in the solar wind is of course small, and electrons are observed only when they have picked up energy comparable to the protons. Imp was spinning and in the solar wind the proton record showed a spike when the detector was aimed near the sun. Isotropization by the shock would be revealed as a steady response with little spin modulation. NOERDLINGER [5] predicted that the electrons would appear rapidly behind the shock, while the randomization of protons would be relatively slow, and Imp found the sudden appearance of electrons, while the randomization of protons was gradual over a distance $\sim 1 R_E$.

Olbert (private communication) at MIT has also been able to study the energy distribution of protons after averaging some of the data. The quantity actually measured is $v^3 f$ at five energies, and the middle energy is just where $v^3 f$ has a maximum. Comparing with a Maxwellian whose maximum is adjusted to that observed, Olbert finds a significant suprathermal tail. The shock can also be detected by the magnetometer. The variance of a number of successive measurements is high behind the shock and drops

* Massachusetts Institute of Technology.

abruptly at the front. Collision-free shocks then are one basic phenomenon available for study in space and another is the current sheet at the boundary. In Fig. 2 the day-side boundary occurs at those lines of force which are shown.

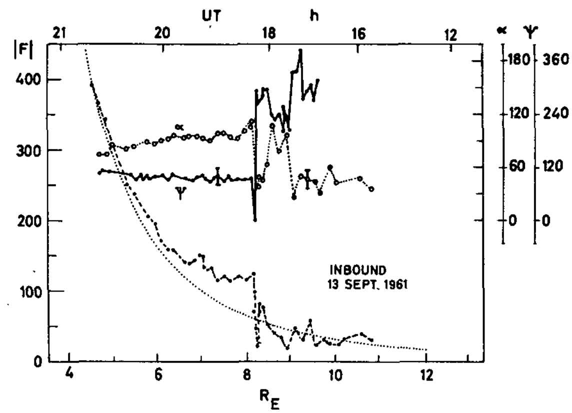

Fig. 3

Magnetic field on the day-side measured by Explorer XII

Figure 3 shows the magnetic field on the day-side measured by Explorer XII [6]. The angles are like latitude and longitude for the spacecraft. The strength of the field exceeds the dipole value just inside the boundary, agreeing with the crude theory which predicts a factor of 2. Note that the unit of field strength is the gamma = 10^{-5} G. At the boundary the magnitude drops in agreement with the known increase of plasma pressure. In Fig. 3 the change of ψ by about 180° shows that the direction nearly reverses, and this is found on more occasions than not. Figure 2 with its southward interplanetary field has such a reversal, but it is not clear why the field just outside the boundary is so often near southward. The mean interplanetary fields observed by both Mariner and Imp were significantly southward, but the southward predominance is much greater just outside the boundary.

On the day-side it seems that the magnetic field is not very important for the flow or at least the shape of the boundary. The boundary shape has been computed ignoring the interplanetary field and rough shapes for the shock also. The interior field and the shapes of the boundary and shock fit the computations reasonably well on the day-side, but not at all on the night-side. Observations on the night-side show greater resemblance to Fig. 2. This is based on a process of reconnection of lines of force at a hyperbolic null as in the boundary current sheet. This process has been described before by DUNGEY [7], and PETSCHEK [8] has recently made further progress with the theory. The essential of the reconnection is that $\vec{E} \neq 0$ at the null (where $\vec{B} = 0$). In Fig. 2 a southward line of force comes up to the boundary and then breaks, the two halves becoming attached to the appropriate polar lines from the earth. The outer part of the line continues to move with the wind while the inner part moves at $\sim 10^{-3}$ of the wind velocity over the polar cap. Thus the polar lines are stretched back in the anti-

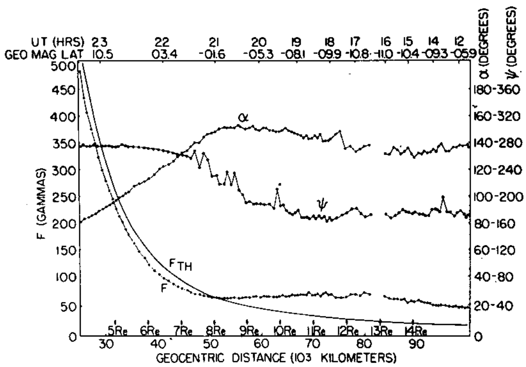

Fig. 4

Night-side observations by Explorer XIV

solar direction like a comet's tail. Figure 4 shows night-side observations by Explorer XIV [6] south of the equatorial plane.

The steady field at $> 10\,R_E$ points away from the sun as in Fig. 2. Eventually the polar lines meet up and are reconnected and dash off to catch up with the solar wind. The parts attached to the poles move towards the sun. In this model the definition of the magnetosphere is not obvious. The interior of the lines shown in Fig. 2 are included, but the north and south boundaries over the polar caps will not be discussed here.

The electric field is difficult to measure in space unless one can measure $\vec{v}$ and use equation (1). Indirect observations do however confirm the pattern of flow shown in Fig. 2. This is supposed to be a steady state, so that curl $\vec{E}$ is taken to vanish, but in any case curl $\vec{E}$ can in principle be obtained by measuring $\partial\vec{B}/\partial t$. Now on the basis of (1) $\vec{E}$ may be assumed to be perpendicular to $\vec{B}$ so that the lines of force are equipotentials. Then, if the electric field in the ionosphere were known, the electric field could be deduced on any line of force connected to the earth. Several different phenomena at high latitudes all show the same pattern. These are the motion of ionospheric irregularities deduced from radio observations, the motion of auroral features and magnetic disturbances. The latter can be represented by equivalent overhead currents, which are obtained simply by rotating the magnetic disturbance vector through a right angle. The currents found show the same pattern as the ionospheric motions, but flow in the opposite direction. Now each of these phenomena could be caused in more than one way, but, the fact that they show a common pattern is good justification for seeking a common cause. In 1961 five workers independently suggested that the common cause was an electric field given by (1) for the motions. The currents are explained by the Hall effect or by saying that the electrons move with the velocity given by (1) while the positive ions in the lower part of the ionosphere are kept almost still by collisions with neutral molecules. Hence the currents flow in the opposite direction to $\vec{v}$. Part of the flow pattern, known to geophysicists as DS, is shown in Fig. 5 which shows auroral motions.

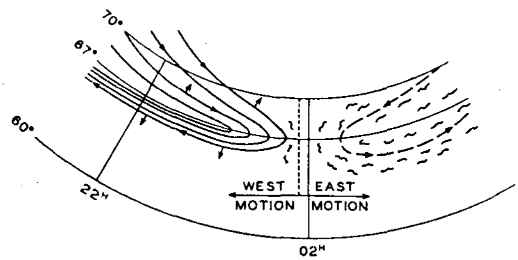

Fig. 5

Auroral motions

In the complete pattern the two eddies close and the motion is away from the sun over the pole and back round the sides. According to the interpretation the flow lines are equipotentials. The potential is positive on the morning side and negative on the evening side, the total potential difference amounting at times to tens of kilovolts. The DS disturbances are detectable down to about 60° geomagnetic latitude. The variation with time is not too clearly known. The DS pattern is commonly present and no radically different pattern has been found. Sometimes one eddy is much stronger than the other. Sometimes no significant magnetic disturbance is detected. About 10γ is needed to show a systematic pattern and up to 1000γ is observed, so that the strength of the electric field probably varies by an order of magnitude, though variations in ionospheric conductivity account for some variations in the magnetic disturbance seen. Slow variations in the strength of the disturbance are common and so are sudden enhancements known as bays. Very recently Fairfield (private communication) has correlated the strength of the magnetic disturbance at a station at very high magnetic latitude with the direction of the field outside the magnetosphere observed by Explorer XII at the same time. He found that the disturbance was strong when there was a definite southward component at Explorer XII, and that when there was a definite northward component at Explorer XII the high latitude disturbance died away. This result tends to support the model of Fig. 2.

The observations discussed so far concern mainly rather turbulent high β-plasmas. The interior of the magnetosphere, in which the radiation belts are situated, provides a quiescent plasma with $\beta \sim 0.1$. This is discussed in chapter II, but one kind of phenomenon observed from the ground may be added here. Observations are made at frequencies of ~ 1c/s (ULF) and a few Kc/s (VLF) at "conjugate points", that is opposite ends of a line of force. Sometimes bursts of emission are observed alternately at the conjugate points. The ULF could be emitted by protons at gyro-resonance, the modulation resulting from the bouncing [9]. The VLF could be the same phenomenon with electrons. A satellite could be designed to look for bounce variations in the particles, an identical pair of detectors pointing in opposite directions being required.

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EFFECTS OF ELECTROMAGNETIC PERTURBATIONS ON PARTICLES TRAPPED IN THE RADIATION BELTS

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1. INTRODUCTION

Since the radiation belts were discovered by Van Allen in 1958, observations of trapped particles have rapidly built up a large body of information. Knowledge of the neutral atmosphere as well as the ionosphere shows that for energetic particles the probable time before colliding with another particle of any kind may be extremely long. Then the only feature known to affect the motion of the particle is the electromagnetic field and, conversely, over a long time even weak electromagnetic disturbances can be important. Consequently, electromagnetic disturbances should be important in determining the form of the radiation belts, and it will be seen that certain features encourage an interpretation of this kind.

The physics of the radiation belts may be regarded as a part of plasma physics, namely the realm in which collisions are negligible. This needs qualifying in that there is a boundary layer (the ionosphere) where collisions are important, and this is analogous to laboratory plasma containment devices. The energy range of trapped particles is wide, but includes the energy range required for fusion reactors. The mean free time in the radiation belts is extreme, but the neglect of collisions yields a great simplification in theoretical work, and an understanding of collision-free plasmas is expected to be useful. Observations in space have great advantages. The quantity measured by a particle-detector sensitive to a limited range of energy and with a limited cone of acceptance is the velocity distribution function, which is fundamental in theoretical work. Local electric and magnetic measurements are also made with very little disturbance by the spacecraft. The disadvantage is that simultaneous measurements cannot be made at many different points.

Another advantage of the radiation belts to the theoretician is that the magnetic disturbances are in fact weak compared with the undisturbed field, except near the outer boundary of the magnetosphere. Then the disturbance can be treated as a perturbation and need only be calculated on the undisturbed trajectory of the particle, as will be done in most of this article. Also, the undisturbed trajectory is adequately described by the well-known adiabatic theory, which is summarized below.

2. ADIABATIC THEORY OF TRAJECTORIES

Relativistic corrections will be omitted. The particle velocity $\vec{v}$ has components $v_{\parallel}$, parallel to the magnetic field $\vec{B}$, and $\vec{v}_{\perp}$, a two-dimensional vector perpendicular to $\vec{B}$. Magnitudes of vectors will be denoted by the same symbol in normal type. The pitch angle α is defined by $\tan\alpha = v_\perp / v_\parallel$ with $0 \leq \alpha \leq \pi/2$. In a uniform magnetic field and no electric field, $v_\parallel$ and α remain constant while $\vec{v}_\perp$ rotates with angular velocity $\Omega = eB/mc$, where e and m are the charge and mass of the particle and c is the velocity of light. The Larmor or gyro frequency is $\Omega/2\pi$, the gyro radius is $v_\perp/\Omega$, and the pitch length of the helical trajectory is $2\pi v_\parallel/\Omega$.

The adiabatic theory (ALFVÉN and FALTHAMMAR [1]) uses "adiabatic invariants", which are quantities that remain approximately constant provided the field "seen" by the particle does not vary too rapidly in time. In this article the objective is to estimate changes in these invariants, and their smallness can then be verified a posteriori. Proofs of their approximate invariance are well-known and will be referenced. The first invariant, due to Alfvén and named the "magnetic moment invariant", will be taken as

$$\mu = v_\perp^2/B = v^2 \sin^2\alpha/B. \tag{1}$$

The reason for the name is that the equivalent magnetic moment due to the gyration of the particle is $mv_\perp^2/2B$, but the mass is omitted in this non-relativistic treatment. The invariance of μ determines the "bouncing" of a particle at a "magnetic mirror". This is a region in which B varies along a line of force and the particle approaches, spiralling around the line in the direction of increasing B. It is assumed that v remains constant. As B increases, (1) shows that α must increase, and bouncing occurs when α reaches $\pi/2$. Then $v_\parallel$ is zero and the trajectory reverses, spiralling back into the region of weaker B. This argument has assumed that v or the particle energy is constant. The energy of radiation belt particles is believed to vary only slowly with time, depending on the electric field. The high conductivity probably prevents any important steady electric field parallel to $\vec{B}$, and the other possible electric components will be treated later as perturbations. Returning to bouncing, the "mirror field", B_m at the "mirror point" where the particle bounces is determined by (1) as

$$B_m = v^2/\mu. \tag{2}$$

For a dipole field and spherical co-ordinates with axis parallel to the dipole, θ being co-latitude and r measured in earth radii

$$B_r = 2B_0 \cos\theta/r^3, \tag{3}$$

$$B_\theta = B_0 \sin\theta/r^3, \tag{4}$$

$$B = B_0(1 + 3\cos^2\theta)^{\frac{1}{2}}/r^3, \tag{5}$$

where B_0 is the field at the ground at the equator. The equation of a line of force is

$$r = L\sin^2\theta, \tag{6}$$

where L is the distance of the equatorial point. When expressed in earth radii, L is the standard labelling parameter for magnetic shells. From (5) and (6)

$$B = B_{eq}(1 + 3\cos^2\theta)^{\frac{1}{2}}(\sin\theta)^{-6},$$

where $B_{eq} = B_0/L^3$ is the magnetic field at the equatorial point of the shell. Then from (1), when μ and v are constant

$$\sin^2\alpha = \sin^2\alpha\ \mathrm{eq}(1 + 3\cos^2\theta)^{\frac{1}{2}}(\sin\theta)^{-6},$$

where α_{eq} is similarly the equatorial value. On a line of force B has a minimum on the equator and a trapped particle has mirror points at symmetrical positions in the northern and southern hemispheres. The particle bounces alternately at these two points or equivalently oscillates between them. Then it has a second natural period, the "bounce period", given by

$$\tau_b = 2\int_{M_S}^{M_N} \frac{ds}{v_{\parallel}} \tag{7}$$

where ds is the element of length along the line of force and M_N and M_S are the northern and southern mirror points.

The applicability of adiabatic theory requires that the average motion of the particle be nearly parallel to the magnetic field as so far assumed, but adiabatic theory includes to first order the drift motion perpendicular to $\vec{B}$, which has three parts. Assuming the electric field $\vec{E}$ is perpendicular to $\vec{B}$, a velocity $\vec{v}_B$ can be defined by

$$\vec{E} + \vec{v}_B \times \vec{B}/c = 0 \tag{8}$$

and it is well-known that the temporal variation of $\vec{B}$ is then given correctly by the description in which the lines of force move with velocity $\vec{v}_B$ (DUNGEY [2]).

Equation (8) also shows how $\vec{E}$ depends on the frame of reference, and it follows that the part of the drift velocity which depends on the frame of reference is $\vec{v}_B$, independent of the charge or mass of the particle. The consequence of an electric field then is a drift $\vec{v}_B$. The other two components of drift result from non-uniformity of the magnetic field. A gradient of field strength normal to $\vec{B}$ gives a component

$$\vec{v}_\nabla = \frac{m c v_\perp^2}{2e B^3}\vec{B} \times \vec{\nabla}\vec{B}. \tag{9}$$

Curvature of the field lines gives a component $\vec{v}_c$ determined by

$$e\vec{v}_c \times \vec{B} = m c v_{\parallel}^2 \vec{K} \tag{10}$$

where $\vec{K}$ is the curvature vector. The motion as described so far is well explained by ALFVEN and FALTHAMMAR [1]. In the dipole geomagnetic field, both $\vec{v}_\nabla$ and $\vec{v}_c$ are westward for positive particles and eastward for negative particles.

It is necessary to discuss the displacement of a particle due to drift motion integrated over a bounce period. It is convenient to introduce labelling parameters α, β for the lines of force. These could be the latitude and longitude of, say, the northern feet of the lines of force, but any α, β such that the surfaces of constant α and constant β are independent and contain the lines of force are sufficient for the present purpose. Then the drift can be represented by $d\alpha/dt$ and $d\beta/dt$. For a particle starting at P and bouncing at M_N then at M_S and returning almost to P

$$\delta\alpha = \int_P^{M_N} \frac{d\alpha}{dt}\,dt + \int_{M_N}^{M_S} \frac{d\alpha}{dt}\,dt + \int_{M_S}^{P} \frac{d\alpha}{dt}\,dt.$$

It is now assumed that the local drift velocity at any point does not change appreciably in the time τ_b, the effects of such changes being discussed at some length below. Then, since the drift is independent of the sign of $v_\parallel$,

$$\delta\alpha = 2\int_{M_N}^{M_S} \frac{d\alpha}{dt}\,dt \tag{11}$$

and similarly for $\delta\beta$. With the same assumption, then $\delta\alpha$ and $\delta\beta$ are the same for any point P on the same line of force.

For trapped particles of more than about 10 keV the drift due to non-uniformity of the geomagnetic field generally dominates over that due to electric fields. Protons then drift round the earth westward and electrons eastward. If the field is static and provided they are not too near the magnetospheric boundary they return to the same line of force on which they started, and hence there is a third period of their motion τ_d, the time for drift round the earth.

Thus the motion has three periods and it can be deduced from Hamiltonian theory that there is an adiabatic invariant associated with each, and having the form of an action. The validity of each invariant requires that the field should not change appreciably in a time equal to the associated period, so the stringency of the condition increases in the order in which the invariants are discussed. The magnetic moment invariant, already discussed, is associated with the gyroperiod and the other two are derived by NORTHROP and TELLER [3]. The second invariant, associated with τ_b, is called the "longitudinal invariant" and is

$$I = \int_{M_S}^{M_N} v_\parallel\, ds = v\int_{M_S}^{M_N} \cos\alpha\, ds, \tag{12}$$

the second form in (12) being valid if $\vec{E}$ is perpendicular to $\vec{B}$. This is important because I has to be computed, and, with this proviso, computation is necessary for a range of pitch angles, but not for a double array of pitch angles and energies, It may be noted that, if μ/v^2 is known, this determines B_m from (2), and if also I/v is known this determines the magnetic shell on which the particle must be. It should be noted that the term "magnetic shell" here means the surface generated by the lines of force on which the guiding centre of the particle moves in its drift round the earth. This is why a particle returns to the same line of force after drifting round the earth, if the field is static, even when the field is not symmetrical. The third invariant, associated with τ_d, is called the "flux invariant" and is the magnetic flux linked by the magnetic shell determined by μ/v^2 and I/v. This invariant determines how the shell moves if the field varies in a time long compared with τ_d, but will be little used in this paper.

With each period of the motion is associated a phase variable as well as an adiabatic invariant. For the bounce and drift motions the phase variables may be taken as latitude λ and longitude ϕ. For the gyration the phase variable is an angle ψ describing the direction of $\vec{V}_\perp$ and may be referred to any plane containing $\vec{B}$ such as the meridian plane, but a precise specification will not be needed. The adiabatic invariants and phase variables are summarized in Table I. These six variables provide an alternative co-ordinate system instead of the position and velocity vectors $\vec{r}$ and $\vec{v}$ for the particle, and it will be found that various mixtures of co-ordinates are useful, a total of six co-ordinates apart from time being needed.

TABLE I

ADIABATIC INVARIANTS AND PHASE VARIABLES

Name	Period	Adiabatic invariant	Phase variable
Gyration	$2\pi/\Omega$	Magnetic moment μ	ψ
Bounce	τ_b	Longitudinal I	Latitude λ
Drift	τ_d	Flux	Longitude ϕ

3. NUMERICAL VALUES

It is standard practice, following McILWAIN [4], to label magnetic shells by the parameter L, which for a dipole field is the distance of an equatorial point on the shell from the centre of the earth as in (6) measured in earth radii. Values of L in the radiation belts range up to 10. For a dipole field the flux invariant is a function of L only. The flux invariant is the flux linked by the magnetic shell and it is simplest to calculate the flux outside the shell. From

$$\int_L^\infty B_0 L^{-3}\, 2\pi R_E^2\, L\,dL$$

one finds that the flux invariant is proportional to L^{-1}. Also, L and ϕ can be used as labels for lines of force, such as those called α and β in the last section. For the actual geomagnetic field L is re-defined in terms of the longitudinal invariant, as the L of the magnetic shell in a dipole field on which a particle with the same value of I would be. Then the value of L differs slightly for different particles on the same line of force. Here, however, in discussing the effects of disturbances, nothing is lost by idealizing the geomagnetic field as a dipole field. A consequent simplification is that I/vL is then a function of only one variable, which may be taken as pitch angle or the latitude of the mirror points.

From (5)

$$\Omega = \Omega_0 r^{-3}(1+3\cos^2\theta)^{\frac{1}{2}} = \Omega_0 L^{-3}\sin^{-6}\theta\,(1+3\cos^2\theta)^{\frac{1}{2}} \tag{13}$$

where, with $B_0 = 0.32$ G, $\Omega_0 = eB_0/mc = 5.7\times10^6$ rad/s for electrons and 3.1×10^3 rad/s for protons. Thus Ω depends on the mass, but not the velocity, of the particles, and the value on a given line of force varies considerably with latitude.

The values of τ_b and τ_d have been computed by HAMLIN et al. [5], who found that good approximations are obtained from some simple formulae. Their approximations give

$$\tau_b \approx \frac{4LR_E}{v}(1.30 - 0.56\sin\alpha_{eq}) \tag{14}$$

$$= 3.3\times10^9\,(L/v)(1-0.43\sin\alpha_{eq})\,\mathrm{s}$$

where v is in cm/s, and

$$\tau_d \approx \frac{2\pi\Omega_0 R_E^2}{3v^2 L(0.35+0.15\sin\alpha_{eq})}$$

$$= \frac{3.9\times10^9}{WL(1+0.29\sin\alpha_{eq})}\,\mathrm{s} \tag{15}$$

where W is the energy of the particle in eV. It may be noted that τ_b involves the particle velocity while τ_d involves the energy. It is seen from (14) and (15) that τ_d/τ_b is roughly v/WL^2, which is about $6\times10^7\ W^{-\frac{1}{2}}\ L^{-2}$ for electrons and $1.4\times10^6\ W^{-\frac{1}{2}}\ L^{-2}$ for protons. The value of $\Omega_{eq}\tau_b/2\pi$ is the same as τ_d/τ_b within a factor of 2. The values of these ratios must be large for adiabatic theory to be valid, and this is true over most of the interesting range. For $L = 10$ and $W = 10^6$ eV, we have 600 for electrons, which is adequate, but 14 for protons may not be. However, the energies of interest tend to vary like $L^{-\nu}$ with $2 < \nu < 3$, and at $L = 10$ the interesting energies are less than 1 MeV.

4. CHANGE OF THE MAGNETIC MOMENT INVARIANT BY DISTURBANCES

Only weak disturbances will be considered and it will be assumed that the undisturbed field satisfies the condition for adiabatic theory - that the change in the field in one pitch length of the particle's trajectory is small. Then we may use a constant field $\vec{B}_0$ as a first approximation and assume that the perturbation of this field over many pitch lengths is small. To study the change of $\vec{v}$ we use

$$m\frac{d\vec{v}}{dt} = e(\vec{E} + \vec{v} \times \vec{B}/c) \tag{16}$$

and obtain

$$\frac{m}{2e}\frac{d}{dt}(\vec{v} \times \vec{B}_0)^2 = \vec{E} \cdot (\vec{B}_0 \times \vec{v} \times \vec{B}_0) + (\vec{v} \cdot \vec{B}_0)\vec{B} \cdot (\vec{v} \times \vec{B}_0)/c. \tag{17}$$

The component of $\vec{E}$ on the right-hand side of (17) is perpendicular to $\vec{B}_0$ in the plane of $\vec{v}$ and $\vec{B}_0$ and the component of $\vec{B}$ is perpendicular to $\vec{B}_0$ and $\vec{v}$. Both of these directions rotate with the gyration of the particle. Now, since the disturbance is weak, an appreciable change in $|\vec{v} \times \vec{B}_0|$ can result only if the contributions from many successive gyrations reinforce each other, so that only special components of the disturbance need be considered, namely those such that the field at the particle rotates at the gyrofrequency. It is convenient to distinguish between changes of the field with time and in space, along and across the field line.

A particle with $v_\perp = 0$ moves precisely along a field line and the changes it sees in the field are due to variations with time and along the field line. Particles with $v_\perp \neq 0$ see these changes and additional changes due to variation of the field across the field lines. Because the particle is gyrating across the field lines, the latter variation as seen by the particle is generally periodic with the gyroperiod and includes components rotating at the gyrofrequency. When the adiabatic conditions are satisfied the effect of variations across the field lines is well approximated by working to the first order in the gyroradius $v_\perp/\Omega$, and then the change in $v_\perp$ is just such as to keep μ constant.

Both terms on the right-hand side of (17) involve the vector $\vec{v} \times \vec{B}_0$, which gyrates and may be represented by a complex number which varies like $\exp i \int \Omega\, dt$, Ω being a slowly varying function of time such that $d\Omega/dt \ll \Omega^2$. Then the integral of the right-hand side of (17) with respect to time is like a Fourier integral, with $\exp i \int \Omega\, dt$ instead of only $e^{i\Omega t}$. This suggests that over a period of several gyrations at least the change in $v_\perp$ depends on a narrow band in the spectrum of the disturbance $\vec{E}$ and $\vec{B}$ near the gyrofrequency as seen by the particle. This was first pointed out by Welch and Whitaker and investigated by DRAGT [6]. PARKER [7] also discussed acceleration of electrons involving this resonance. Consequently, it is useful to make a thorough study of the effect of sinusoidal waves on a particle in a uniform magnetic field, as a guide to the more general problem.

Let the z-axis be parallel to $\vec{B}_0$, and consider waves whose components vary sinusoidally with $z - wt$, w being the phase speed parallel to $\vec{B}_0$. Provided $w < c$, a frame exists in which the wave is static, its components being independent of time. If $w > c$, there is a frame in which the wave components are independent of z, but here attention is confined to the former case $w < c$. For a static wave the electric field has a potential, and the energy of the particle including the potential energy is conserved. It is convenient to work in this frame and to calculate the change in $v_\parallel$, the change in $|v_\perp|$ then being determined by the conservation of energy.

The effect of the wave will be approximated by integrating along the unperturbed trajectory specified by

$$x = a\cos\Omega t,\quad y = a\sin\Omega t,\quad z = v_\parallel t$$

where a is the gyroradius $v_\perp/\Omega$. The components which cause changes in $v_\parallel$ are E_z, B_x and B_y, and the most general forms for these in the frame in which the wave is static are

$$E_z = E\cos(\beta + kz),\quad B_x = b_x\cos(\beta_x + kz),\quad B_y = b_y\cos(\beta_y + kz) \tag{18}$$

where E, b_x, b_y, β, β_x and β_y can vary with x and y. The rate of change of $v_\parallel$ is given from (16) by

$$(m/e)\,dv_\parallel/dt = E_z - (v_\perp/c)(B_y\sin\Omega t - B_x\cos\Omega t). \tag{19}$$

It is seen that the magnetic term involves gyration, but the term in E_z does not. If a wave has a longitudinal electric component it is effective when the particle sees zero frequency, but the change is in $v_\parallel$, not $v_\perp$.

In one gyroperiod the phase of the wave at the position of the particle changes by $2\pi kv_\parallel/\Omega$. Consider then the changes in $v_\parallel$ in successive gyroperiods. Since these differ only in the phase of the wave, they must be proportional to $\cos(\gamma + 2\pi nkv_\parallel/\Omega)$, where γ is some constant and n refers to the nth gyroperiod. On summation of a large number of successive gyroperiods, these changes in $v_\parallel$ will tend to cancel out unless $kv_\parallel/\Omega$ is nearly an integer. If $kv_\parallel/\Omega$ is nearly an integer, on the other hand, the contributions from successive gyroperiods will reinforce each other because they are in phase, and this will be called a resonance. It is of interest to calculate the change of $v_\parallel$ in one gyroperiod at a resonance and we put $kv_\parallel = N\Omega$, N being an integer for resonance.

First assume the wave components do not vary with x or y. Then substituting (18) into (19), integrating over a gyroperiod, and writing $\psi = \Omega t$ gives

$$B_0\,\delta v_\parallel = \int_0^{2\pi} [cE\cos(\beta + N\psi) - v_\perp(b_y\cos(\beta_y + N\psi)\sin\psi - b_x\cos(\beta_x + N\psi)\cos\psi)]d\psi$$

$$\begin{aligned}
&= 2\pi cE\cos\beta, \text{ if } N = 0\\
&= \pi v_\perp(b_x\cos\beta_x + b_y\sin\beta_y), \text{ if } N = 1\\
&= \pi v_\perp(b_x\cos\beta_x - b_y\sin\beta_y), \text{ if } N = -1\\
&= 0, \text{ if } |N| > 1.
\end{aligned} \tag{20}$$

It should be emphasized that $\delta v_\parallel$ is not particularly big at a resonance and that the resonance depends on reinforcement from many gyroperiods. In the simple case of no variation with x or y, (20) shows that $\delta v_\parallel$ vanishes except for particular values of N. The combinations of b_x and b_y occurring in (20) for $N = \pm 1$ are circularly polarized components, such that the particle sees the magnetic disturbance vector gyrating in the same direction as the particle itself gyrates and remaining perpendicular to the particle velocity. The cases of $N = \pm 1$ are called "gyroresonance" and the three resonances shown by (20) are likely to be the most important. When variation with x and y is included resonance can occur at any N. The occurrence of resonances with quite large values of N has been observed in the inverse phenomenon, which is the effect of the plasma on the wave. This is expressed by a dispersion equation, which is found to involve an integral over velocity space of a series of terms with denominators $\omega + kv_{\parallel} - N\Omega$, the vanishing of which is the resonance condition. If low energy particles are important, the wave is strongly affected near $\omega = N\Omega$. The Alouette satellite carried an ionosonde, i.e. a pulse transmitter and receiver, and the plasma was found to resonate at multiplies of the electron-gyrofrequency - sometimes up to the fifteenth (JOHNSTON and NUTTALL [8]).

Consider the term involving E in the integral in (20), when the variation of E and β with x and y becomes a variation with ψ through (17), and for simplicity consider variations in E and β separately. If β is independent of ψ, but E is a polynomial in x and y of order ν, $\delta v_{\parallel}$ vanishes when $N > \nu$. If E is independent of ψ but β is linear in x and y, the formulae for integral N

$$\int_0^{2\pi} \cos(N\psi - \xi\sin\psi)\, d\psi = 2\pi J_N(\xi) \tag{21}$$

and

$$\int_0^{2\pi} \sin(N\psi - \xi\sin\psi)\, d\psi = 0$$

show that $\delta v_{\parallel}$ contains a factor $J_N(Ka)$ where K is of the order of the rate of change of phase β across the lines of force. If Ka is small $J_N(Ka) \approx (Ka)^N / 2^N N!$ showing that the resonance with $N = 0$ is likely to be most important. Thus, in general, if E and β do not vary much in a gyro-radius, only a few resonances are important. The effect on $\delta v_{\parallel}$ of variations of B_x and B_y with x and y is similar. Since the principal resonance is $N = 1$ or $N = -1$, the order of the Bessel function in (21) becomes $N - 1$ or $N + 1$ (DUNGEY [9]).

In the frame of the wave, as used so far, the electric field has a potential which limits the possible change in the particle energy and at resonance the particle energy returns to its original value exactly after a gyration. Consequently $\delta v = 0$, then $\delta v_{\parallel} = -v \sin\alpha\, \delta\alpha$ and

$$\delta\alpha = -\delta v_{\parallel} / v_{\perp}. \tag{22}$$

This is the change in one gyration in the frame of the wave. In another frame moving relative to this in the direction of $\vec{B}_0$, $\delta v_{\parallel}$, and $\delta v_{\perp}$ are the same, but in general the energy changes. If w is the phase speed of the wave in the direction of $\vec{B}_0$, the energy changes by $w\,\delta v_{\parallel}$. It may be noted that $N = 0$

implies that $v_\parallel$ vanishes in the frame of the wave and hence that $\delta v_\perp = 0$ and $\delta\mu = 0$.

In a uniform $\vec{B}_0$ and a wave field given by (18), the change in $v_\parallel$ and hence z would eventually become important, and then the approximation of integrating along the unperturbed trajectory would be invalid. For very small amplitudes the change in z remains unimportant for many gyroperiods and then it is found that "particle trapping" occurs, such that $v_\parallel$ oscillates about the resonant value (DUNGEY [9]). In practice, in the magnetosphere only low energy particles are likely to resonate with a wave for long enough for this trapping effect to be important. Because Ω and $v_\parallel$ change with time in a non-uniform field, the particle just passes through resonance, and it is important to estimate the changes in $v_\parallel$ and $v_\perp$ resulting from such a passage through resonance or equivalently the number of gyrations the resonance is effective.

Since the phase velocity w may vary with z, it is best to work in an arbitrary frame. The condition for resonance is

$$kv_\parallel + \omega = N\Omega \tag{23}$$

where ω is the angular velocity of the wave. At any point let k_{res} be the value of k determined by (23), so that $(k - k_{res})$ passes through zero at a resonance at say $z = z_{res}$. Both k and k_{res} vary with z, and it will be assumed that a linear expression gives an adequate description, written as

$$k - k_{res} = k'(z - z_{res}). \tag{24}$$

When (18) is substituted into (19) the right-hand side varies sinusoidally with kz and the part belonging to each resonance N varies sinusoidally with $(\omega - N\Omega)t$, which is equivalent to $-k_{res}z$. When k and k_{res} vary, the resonant part of the right-hand side can adequately be taken to vary sinusoidally with $\int(k - k_{res})dz$. Using (24) and integrating over time gives

$$\int \cos\left[\tfrac{1}{2}k'(z - z_{res})^2\right]dz/v_\parallel,$$

the part with sine instead of cosine not contributing.

The main contribution to this integral comes from small values of $z - z_{res}$ and the limits may therefore be taken as $\pm\infty$. Its value is then $(\pi/k')^{\frac{1}{2}}/v_\parallel$ showing that the number of gyrations of effective resonance is $(\Omega/v_\parallel)(2\pi k')^{-\frac{1}{2}}$. If this number is not large compared to one, the resonance is not sharp and the perturbation of the resonant particles is not much greater than that of the other particles. If $k' \sim k/R_E$, as it may well be, this number is $(\Omega/v_\parallel)(R_E/2\pi k_{res})^{\frac{1}{2}}$. This will be used as a rough guide, though it should be checked for particular instances.

Finally, it must be recognized that a trapped particle will pass through resonance many times, either because there is a series of short-lived disturbances or because the disturbance has a fixed frequency and the resonant frequency varies as the particle moves, in which case there are two resonances in each bounce period. It will be assumed and is usually plausible that the phase factors in the values of $\delta\alpha$ for successive resonances are uncorrelated, and then successive changes must be added as in a random walk to give a probable change of $(\Sigma(\delta\alpha)^2)^{\frac{1}{2}}$. The individual changes in α are small and the effect on the velocity distribution may then be described as a diffusion in pitch angle. This may be described mathematically by adding a diffusion term to Liouville's equation (BORN [10]), as will be discussed in detail in section 8. Loss and injection are neglected entirely in this paper. Liouville's equation

$$\partial f/\partial t + v_i\, \partial f/\partial x_i + (dv_i/dt)\partial f/\partial v_i = 0$$

states that, if one follows a particle in phase space, f remains constant. Any diffusion term representing diffusion in pitch angle will have the form $F\partial/\partial\alpha(G\partial/\partial\alpha(Hf))$, where the product FGH is the diffusion coefficient D and only an estimation of the order of magnitude of D is of interest here. The distinctions between F, G and H need some care and will be considered in section 8.

Another possible form of disturbance is noise specified only by a spectrum. The particle then behaves as a resonator and $\delta\alpha$ depends on the amplitude of the noise in a certain bandwidth centred on the resonant frequency. The value of $\delta\alpha$ over a long time may be obtained by combining the values for segments of time according to a random walk, but if the change in the resonant frequency is small in relation to the spectrum of the noise the same result is obtained by treating the resonant frequency as fixed. Taking the phase as coherent over time t the bandwidth is proportional to t^{-1}, hence $b \propto t^{-\frac{1}{2}}$ and $\delta\alpha \propto t^{\frac{1}{2}}$ as before.

5. APPLICATIONS OF THE CHANGE IN MAGNETIC MOMENT INVARIANT

To apply the results of the previous section to the radiation belts, some knowledge of waves in the magnetosphere is required. RAO and BOOKER [11] give an extensive survey, but here a greatly simplified picture will be used. For frequencies below the proton gyrofrequency $\Omega_p/2\pi$ the phase speed will be taken as the Alfvén speed $V_A = (4\pi\rho)^{-\frac{1}{2}} B$. For frequencies between $\Omega_p/2\pi$ and about one-third of the electron gyrofrequency $\Omega_e/6\pi$ the simple whistler dispersion equation

$$(w/V_A)^2 = \omega/\Omega_p \tag{25}$$

will be used. Higher frequencies will not be considered, nor will the very slow waves near the proton gyrofrequency. Very slow waves resonate with many particles and may be heavily damped in consequence. From (13) with $\Omega_0 = 3.1\times10^3$ rad/s for protons, $\Omega_p/2\pi$ ranges from 1 kc/s to about 1 c/s, and the spectrum in this range in the magnetosphere is practically unknown. The value of V_A in the magnetosphere is typically between 1000 and 3000 km/s.

Investigations to date have been concerned mainly with gyroresonance. The resonance with N = 0 may be important, but, though it was convenient to include N = 0 in the previous section, it was seen there that resonance with N = 0 does not change μ, and so it will be excluded from this section. DRAGT

[6] considered gyroresonance for protons. He considered frequencies much less than Ω_p so that the resonance condition is approximately

$$\omega = \Omega V_A'/v_{\parallel} . \tag{26}$$

To be consistent $v_{\parallel} >> V_A$ and it may be noted that a proton with a speed of 10^4 km/s has an energy of 520 keV. Also, when $v_{\parallel} = 10^4$ km/s, $2\pi/k_{res} = 2\pi v_{\parallel}/\Omega$ is about $10L^3$km, and the corresponding number of gyrations of effective resonance about $25L^{-3/2}$. Thus for high energy protons the resonance is generally not very sharp. From (22) and (20) $\delta\alpha \sim 2\pi\, b/B_0$. The value of b is not expected to vary greatly with L. For b = 1 gamma, $\delta\alpha \sim 10^{-2}L^3$ degrees. DRAGT [6] suggested that scattering by hydromagnetic waves leads to the loss of trapped protons and, assuming the hydromagnetic spectrum cuts off above a few c/s, found a maximum energy of trapping which decreases rapidly with L. Further progress requires a better knowledge of the wave spectrum.

DUNGEY [10] and CORNWALL [12] have discussed the effect of whistlers on energetic electrons. The approximate resonance condition corresponding to (26) combined with (25) gives

$$\omega = (m_p/m_e)(V_A/v_{\parallel})^2\,\Omega_e. \tag{27}$$

Unless $v_{\parallel}$ is comparable to the speed of light this gives a value at least as big as Ω_e in which case (25) is invalid, and in fact less energetic electrons resonate with frequencies near $\Omega_e/2\pi$, but here we consider $v_{\parallel} \sim c$ and ω is then an order of magnitude less than Ω_e and (25) is adequate. The number of gyrations of effective resonance is about 200 $L^{-3/2}$. The largest amplitude expected is about 1 gamma and then $\delta\alpha \sim 0.3\ L^{3/2}$ degrees. Combined with whistler statistics a random walk as described at the end of section 4 gives the right order of magnitude for the lifetime of fission electrons (BROWN and GABBE [13]), and both DUNGEY [9] and CORNWALL [12] account for the rapid variation of lifetime with L in the neighbourhood of L = 2 in terms of the spectrum of whistlers. Fission electrons have energies ~1 MeV, and in the resonance condition (23) $kv_{\parallel}$ is typically a few times ω, so that $\omega/\Omega < \frac{1}{2}$. The wave frequency ω in the frame of the earth is then determined by the dispersion, which is known empirically from whistler studies. In the neighbourhood of L = 2 it is found that the frequency required for resonance is proportional to L^{-6} and so varies rapidly with L. At L = 2 the resonant frequency is between 10 and 20 kc/s, and for the higher frequencies required for resonance at L < 2 the spectrum of shistlers shows quite a rapid fall approximated by amplitude $\propto f^{-1}$. With $f \propto L^{-6}$ this then leads to a rapid increase in lifetime with increasing L, similar to the behaviour observed. The main consequence of whistlers then is the short lifetime (a few days) for electrons of ~1 MeV in the region 2 < L < 3, known as the "slot".

The same mechanism for 40 keV electrons in the outer belt (L > 3) has been studied by Kennel (private communication) and found to be important, but no details can be included here though some information on the outer belt is now added. Whistler studies by CARPENTER [14] show that the electron density suddenly drops by an order of magnitude between L = 3 and L = 4, the relevant consequence of which is that the Alfvén speed at L = 4 is about the same as at $L = 2$, so that the ratio of the resonant frequencies at fixed particle energy at $L = 4$ and $L = 2$ is 2^{-3} rather than 2^{-6}. A more important qualitative difference is that in the outer belt disturbances in the relevant frequency range are emitted by the trapped particles themselves, presumably due to instability involving the inverse of the phenomenon discussed here. These emissions form an interesting subject of their own, and have considerable variety. There are many types with a relatively narrow instantaneous spectrum which glissando up or down in frequency, but also hiss of relatively broad band, which is used by Kennel. Observations of 40 keV electrons on the Injun satellite (O'BRIEN [15]) show two phenomena which need explaining: a steady background, in which the flux of dumped particles is substantially less than the flux of particles mirroring at the altitude of the satellite, and "splashes" in which both these fluxes increase to the same value. Assuming that splashes are due to a true time variation rather than to the satellite passing through an arc, they could be caused by a rapid randomization of pitch angles, while the background could be accounted for by a modest steady hiss. Measurements of the hiss were also made by the Injun satellite and Kennel has compared the observations with the requirements, and concluded that they were compatible.

6. CHANGE OF THE LONGITUDINAL INVARIANT BY DISTURBANCES

Disturbances which change the longitudinal invariant have not yet been extensively investigated, and this section will be little more than a list of mechanisms. The problem is similar to that of the magnetic moment invariant, but there is a greater variety of geometries and the disturbances cannot be adequately approximated by plane waves. It is now assumed that μ remains constant and then, if L and W are calculated, I may be deduced, and it is most convenient to consider changes in L and W. Thus, while it is useful to classify disturbances by the adiabatic invariants which are changed or preserved, the adiabatic invariants are not the easiest variables to work with. Here it may be recalled that the $N = 0$ resonance (see section 4) does not change μ, and it does change W but not L, so that I changes and this is one mechanism for the list. This mechanism is unlikely to be coherent for more than a fraction of a bounce period, however, and it may well be that the other mechanisms, which involve resonances with a multiple of the bounce period, are more important, though of course this depends on the amplitude of different disturbances, which are not well known.

For resonances with multiples of the bounce frequency, as in the resonances of section 4, the frequency seen by the particle is not in general the same as the frequency seen in the frame of the earth, this time because of the east-west drift. Although the disturbances cannot be represented by plane waves they may be Fourier-expanded in time and longitude, varying as $\exp i(\omega t - m\phi)$ where m is an integer. They then have an east-west angular phase velocity ω/m and the resonance condition analogous to (23) is

$$\omega - 2\pi m/\tau_d = 2\pi N/\tau_b . \tag{28}$$

The bounce motion of the particles is not sinusoidal nor is the variation of the disturbance along a line of force, so that the value of N may not be particularly significant. The distinction between odd and even N is important and it may be generally expected that low values of N are most important. In this section the possible electric and magnetic polarizations will be considered. The changes in W and L are integrals over a bounce period, and it will be found that for several components they are just weighted means, but for two the weighting factor changes sign with $v_{\parallel}$, and then odd harmonics are required.

Just as waves with $N = 0$ in section 4 did not alter μ, it may be asked whether disturbances with $N = 0$ according to (28) will alter I. The proof of the longitudinal invariant (NORTHROP and TELLER [3]) assumes that at a fixed point the field does not change appreciably in a bounce period, or $\omega\tau_b \ll 1$ which together with $N = 0$ requires $2\pi m \ll \tau_d/\tau_b$. The values of τ_d/τ_b given in section 3 show that this condition may well be met when m is a small integer. A further argument suggests that this condition is not needed, however. In a frame rotating with the disturbance at angular velocity ω/m the disturbance is static, and the fact that such a frame is not an inertial frame is probably unimportant, so the transformation will be treated as Newtonian. The electric field then has a potential, partly due to the disturbance, but this is of first order and the important part is a steady $\vec{E}$ in the direction of the principal normal to the lines of force due to the transformation. The transformation must here be treated as a rotation so that the velocity of the frame $\propto$ L, and, with $B \propto L^{-3}$, the electric field $\propto L^{-2}$ and the potential $\propto L^{-1}$. For a given ω/m the change in W is then in a fixed ratio to the change in L. Now the preservation of I determines this ratio $\delta W/\delta L$ and the required value must correspond to $N = 0$ or $\omega/m = 2\pi/\tau_d$. But we now see that, for any disturbance with $N = 0$, $\delta W/\delta L$ must have this same value, and then I is preserved. This additional situation in which I is preserved depends on the existence of a frame in which the disturbance is static. Given this, I is preserved for any disturbance with $N = 0$.

A simple effect requiring odd harmonics is that due to an electric component $E_{\parallel}$ parallel to $\vec{B}$, which changes W and is effective if $E_{\parallel}$ is directed oppositely during the northward and southward passages of the particle. This has been proposed by CHAMBERLAIN [16] to explain aurorae, the disturbance being provided by an instability discovered by KRALL and ROSENBLUTH [17], and this will be further discussed after some other components have been introduced. The only other effect which resonates at odd harmonics is due to a magnetic component b_n in the direction of the principal normal to the line of force. This tilts the line of force in the meridian plane so that the mean velocity of the particle is also tilted and, if b_n reverses sign when the particle bounces, the particle zigzags, as shown in Fig.1, projected on to a meridian plane. The mean velocity is obtained assuming that the latitude does not change appreciably in many gyroperiods. The component of this mean velocity in the direction of the principal normal is $b_n v_{\parallel}/B$ and the rate of change of L is obtained by multiplying this by $|\vec{\nabla} L|$, which conservation of flux shows to be $BL^3 \sin^3\theta/B_0 R_E$. Then

$$dL/dt = b_n v_{\parallel} L^3 \sin^3\theta/B_0 R_E \tag{29}$$

and this must be averaged over a bounce period to give the result of a motion such as that shown in Fig. 1.

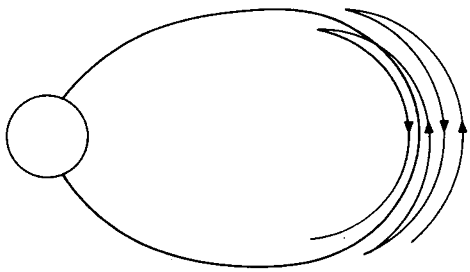

Fig. 1

Projection of the particle trajectory

The component b_n must be related to $\vec{E}$ by the Maxwell equation

$$\partial b/\partial t = -c \operatorname{curl} E \tag{30}$$

and curl $\vec{E}$ may arise from $\partial E_{\parallel}/\partial_{\phi}$ or $\vec{B} \cdot \vec{\nabla} E_{\phi}$. The effect of $E_{\parallel}$ on the particle has already been discussed but not that of E_{ϕ}. It has been implicitly assumed that $E_{\parallel}$ and b_n are even functions of latitude, but then the E_{ϕ} required in (30) will be an odd function of latitude. Then for odd harmonics the particle will see E_{ϕ} as even harmonics, and E_{ϕ} also has resonant effects. There is a change in W due to the particle's drift and a change in L given by (29) with $b_n v_{\parallel}$ replaced by cE_{ϕ}.

It has been seen that (30) relates components b_n, $E_{\parallel}$ and E_{ϕ}, E_{ϕ} being an odd function of latitude, if b_n and $E_{\parallel}$ are even. Consequently a disturbance involving one of these components must involve one of the others and usually both. When there is a disturbance involving b_n a change of frame in an east-west direction changes $E_{\parallel}$ but not (to first order) b_n or E_{ϕ}. In a frame in which the disturbance is static, $E_{\parallel}$ and E_{ϕ} must be derived from a potential and a change in W arises from the steady E_n. The change in L is determined by (29) for the effect of b_n and the corresponding equation involving E_{ϕ}. One may expect a tendency for these effects to cancel out, but no more than a tendency. Chamberlain's mechanism is of the kind discussed here, but he emphasizes $E_{\parallel}$, and Krall and Rosenbluth's stability study uses a plane stratified model and considers waves travelling strictly perpendicular to $\vec{B}$. The results are so interesting that investigation of a more realistic model should be worthwhile, and the other effects listed here would probably then need to be taken into account. Chamberlain's acceleration arises from a finite first order part of $\vec{E} \cdot \vec{B}$, a quantity which tends to be small in a plasma. It may be noted that $\vec{E} \cdot \vec{B}$ is independent of the frame of reference, and the first order part is of interest and is $E_{\parallel} B + E_n b_n$. In Krall and Rosenbluth's model E_n is due to a gradient of plasma pressure and it may be questioned whether the perturbation in plasma pressure is adequately taken into account in their treatment, there being some ambiguity when there is no spatial variation parallel to $\vec{B}$. It may be that $E_{\parallel} B + E_n b_n$ is over-estimated by Chamberlain, but even so there would still in general be a change in L and W when b_n and E_ϕ are included.

The remaining effects involve components seen as even harmonics. Previously E_ϕ was considered as an odd function of latitude, but it is noted here that if E_ϕ is an even function of latitude there is still a resonance when the particle sees E_ϕ with the same sign at both equatorial crossings. Similar considerations apply to the remaining components to be discussed. An equatorial perturbation E_ϕ is likely to be associated with a varying field strength or perturbation component $b_\parallel$. The betatron effect must then be included by a contribution to dW/dt of $\frac{1}{2}m\mu\partial b_\parallel/\partial t$. PARKER [18] discussed the betatron mechanism, acting near the mirror point. A likely concomitant is an east-west gradient of $b_\parallel$, which leads to a contribution to dL/dt due to v_∇ given by (9) with $\vec{\nabla} b_\parallel$ substituted for $\vec{\nabla} B$.

The final component b_ϕ causes drifts in two different ways. Including b_ϕ in the $\vec{B}$ preceding the vector product sign in (9) gives a v_∇ which changes L, and b_ϕ gives a component of curvature $\vec{B}\cdot\vec{\nabla}(b_\phi/B)$ which gives a v_c according to (10) in the direction of ∇L. These two effects may tend to cancel, and must do so if I is preserved, as for instance if the disturbance is static and axially symmetric such that $\partial b_\phi/\partial_\phi = 0$.

Hardly any numerical estimates of these effects have been made. The values of τ_b range from a fraction of a second to a few minutes, thus falling in the ULF range in which micropulsations are observed at the ground. Some micropulsations may themselves be due to the dumping of trapped particles, in which case the field at the ground is not simply related to that in the magnetosphere, but some may have their sources further out in which case the relation between the disturbances at the ground and in the magnetosphere depends only on their propagation through the ionosphere. There is good hope that this question will soon be clarified by better observations on the ground and by satellite observations. At present one may very crudely assume an amplitude of $\sim 1\gamma$ independent of L. Equation (29) gives a typical rate of change of L as $(b_n/B_0)(L^4/\tau_b)$ showing that it increases rapidly with L. For $b_n \sim 1$, $b_n/B_0 \sim 3\times10^{-5}$, so that the characteristic time is always many bounce periods, but disturbances of this amplitude are not negligible.

7. DISTURBANCES WHICH CHANGE ONLY THE THIRD INVARIANT

In the last section we have worked with changes in W and L and now, if I is preserved, it is necessary only to consider changes in one of these and W is most convenient. The causes of changes in W are electric components as discussed in the last section, but $E_\parallel$ can be omitted because it required the particle to see odd harmonics of the bounce frequency, which implies a change of I. There remain the betatron effect and E_ϕ. It may well be that non-linear effects are important in this case, for instance if the disturbance is a storm which moves the boundary in from L = 10 to L = 8. Then a particle at large L may suffer a large change in energy and hence a large change in the particle's L, if μ and I are preserved. Non-linear discussions of the effects of storms have been given (PARKER [19], DAVIS and CHANG [20]) but it is useful to consider the simpler linear situation analogous to sections 4 and 6. Now since the undisturbed motion is simply a drift with constant

$d\phi/dt$, the only resonance corresponds to $N = 0$ of the last section. The rate of change of W must be averaged over a bounce, but then simply integrated over time with ϕ increasing by $2\pi t/\tau_d$. Disturbances which resonate in this way, with periods as seen at the ground of minutes and fairly large values of m, may be important, but more is known of impulsive disturbances for which resonance is not important. We therefore consider large-scale disturbances with a time-scale not many times less than τ_d. The importance of such disturbances was first suggested by KELLOGG [21].

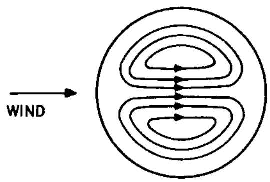

Fig. 2

DS pattern

The contributions to dW/dt from the betatron effect and E_ϕ are comparable if curl $\vec{E}$ is comparable to E_ϕ/LR_E, and this may well be the case for disturbances due to motion of the boundary, for instance. There is also evidence for disturbances in which curl $\vec{E}$ may be negligible, namely bays. Most of the disturbances on high latitude magnetograms are bays, if the term "bay" is not defined to require a sudden beginning. There is a great range of variation in their forms as they appear on magnetograms, but they nearly all conform to the DS pattern shown in Fig. 2 when the disturbance vectors at the different stations are plotted on a map (FAIRFIELD [22]). Many workers (AXFORD and HINES [23], CHAMBERLAIN [24], DUNGEY [25], FEJER [26], KERN [27]) have interpreted bays as being due to an electric field in the ionosphere, normal to the flow lines of both ionospheric movements and electric currents, which both have the DS pattern as shown in Fig. 2. This figure shows the ionospheric movements, the electric currents flow in the opposite direction, causing magnetic disturbance at the ground in the direction obtained by rotating the movement vector 90° clockwise, and the postulated electric field also has the latter direction. The pattern is generally interpreted as being due to the flow of the solar wind past the earth, and it can occur in a steady state, though the onset of a bay must involve some change, if only an intensification of the flow. At the ionospheric level curl $\vec{E}$ must be small, and this may well be true far out in the magnetosphere also, though this has not been investigated. Figure 2 also shows that the E_ϕ of the DS pattern is westward on the nightside and hence increases W there, and on the dayside E_ϕ is eastward and decreases W. Thus a bay is most effective if its duration is $\sim\tau_d/2$ which is a possible value since bays last 10 - 30 min. In this case, at large L, δW may be as much as 20 keV and certainly several keV. If $\tau_d/2$ is smaller than the duration of the bay all particles will drift more than 180° in longitude during the course of the bay and cannot then gain energy all through the bay, but δW will still be of the same order. If $\tau_d/2$ is larger the particles drift less than 180° and the magnitude of δW is correspondingly decreased. The potential of the DS field decreases rapidly as L decreases, and is undetectable for L less than about 4, except for very strong bays. This mechanism is being investigated further by the author.

The betatron effect is likely to be important for sudden impulses which occur at a rate of several a day (NISHIDA and CAHILL [28]). The value of $\delta W/W$ should be comparable to $\delta B/B$, and δB probably varies much less with distance from the earth than B does. Consequently, this effect also increases rapidly with L and should be substantial at large L. Sudden impulses grow in a few minutes, decay much more slowly, and are seen all over the earth with the same sign. It is possible that the δW of the decay cancels most of that of the growth phase, but it need not be so, for instance if the magnitude varies with longitude. This mechanism has been investigated by Mead and Nakada using Parker's non-linear formulation.

Observations of protons in the outer belt by DAVIS, HOFFMAN and WILLIAMSON [29] suggest that disturbances which change only the third invariant are important. These protons have an exponential spectrum represented by $-e^{-W/W_0}$ and the variation of W_0 with L shows a strikingly good fit to the variation for an individual particle with μ and I constant. The value of W_0 is not very different for different values of α_{eq}, but the slight differences in $\partial W_0/\partial L$ for different values of α_{eq} agree with the preservation of μ and I. Because the intensity of these protons varies very little with time it has been possible to study their distribution function f (NAKADA, DUNGEY and HESS [30]).. It is found that $\partial f/\partial L$ at fixed μ and I is always positive, which will be found on the basis of a diffusive mechanism to indicate an external source. Also $(\partial f/\partial L)_{\mu,I}$ is small for $L > 5$, which will be found to suggest that the distribution is dominated at $L > 5$ by disturbances which preserve μ and I. Provided the individual disturbances are not too strong their effect can be represented by a diffusion term as in section 4, and it will be shown in the next section that this has the form $\mathcal{J}^{-1}\partial/\partial L(\mathcal{J}D\partial f/\partial L)$, where the derivatives are taken at constant μ and I, D is the diffusion coefficient, and $\mathcal{J}$ is the Jacobian such that $\mathcal{J}\delta\mu\delta I\,\delta L$ is the volume of phase space corresponding to the range of values $\delta\mu$, δI, δL, and is L^{-2} apart from a constant factor (see Appendix). Then $\mathcal{J}D\partial f/\partial L$ represents the inward flow of particles corresponding to the diffusion and, if the source is at the outside, this must increase monotonically with L. Now $\partial f/\partial L$ is observed to be much smaller at $L = 6$ than at $L = 4$ and hence D must be much larger at $L = 6$. Estimates of D have been obtained from rough statistics of both impulses and bays. Both yield values of D which increase rapidly with L, as required, and both suggest characteristic times of diffusion at $L = 6$ of the order of one week.

Assuming this interpretation, impulses and bays have important effects on protons of 10 to 1000 keV in a time-scale of days, and they must affect electrons of the same energy in the same way, since τ_d depends on energy but not mass. Observations show that the electron distribution is much more variable in both space and time, so that disturbances which change μ and I for electrons must be relatively rapid. The energy spectrum of electrons has not been measured in the same way as for protons, but the overall picture obtained from observations at different energies shows a softening of the spectrum with increasing L, which again suggests an external source.

8. THE FORM OF THE DIFFUSION TERMS

At present the precision of this subject is little better than orders of magnitude but, anticipating improvement, it is worth considering the form of the diffusion terms. Following the argument in terms of a random walk, as in section 4 or BORN [10], the diffusion terms should have a Fokker-Planck form. Microscopic consideration to be described in this section, on the other hand, shows that the phenomena are more akin to stirring and it will be shown here that this approach determines the form of the diffusion terms, the statistical properties of the disturbances being represented by a single coefficient. Now the Fokker-Planck equation contains two coefficients representing the disturbances, one involving the mean square displacement and the other the mean displacement. Then the Fokker-Planck equation can agree with our single coefficient equation only if the two Fokker-Planck coefficients are related in a particular way. Leverett Davis (private communication) has shown that the coefficients obtained by DAVIS and CHANG [20] are so related and hence that both approaches are valid, and only one coefficient needs to be computed for any species of disturbance. The reason for this simplification will now be described.

The equation to which the diffusion terms are to be added is Liouville's equation, and the objective may therefore be regarded as finding an improvement to Liouville's equation. Now the disturbances, which have been discussed, have been quite specific, only the frequency of occurrence of disturbances of different kinds and different amplitudes being treated statistically. Consequently, in treating the effect of a single disturbance, with the fields completely specified, the trajectories of particles could be computed and Liouville's equation used to find the effect on the distribution exactly. Liouville's equation states that f remains constant following a particle trajectory in phase space, that is f is convected in phase space, and hence the effect is like stirring. Now in all the cases discussed the changes suffered by an individual particle depend critically on the relevant phase variable, with the result that after the disturbance f varies with this phase variable, whereas it has been generally assumed that the undisturbed f does not vary with any of the phase variables. The stirring occurs after the disturbance simply from the undisturbed motion and is due to the variation of the natural periods with the adiabatic invariants. For example, consider the disturbances of section 7, where only particles with given values of μ and I need be considered. The disturbance makes f longitude-dependent and, immediately after the disturbance, it is likely that the longitude-dependence will have some similarity on different shells. The drift motion, however, rotates the distribution on each shell at angular velocity $2\pi/\tau_d$, which varies with L. The result is that the contours of f are wound into a spiral. The resonances with the Larmor and bounce periods lead to stirring in a closely analogous way.

To derive the diffusion terms it will be assumed that the contours of f are wound into a tight spiral or analogous fine structure, the validity of this assumption being briefly discussed later. It is then assumed that the fine structure is smoothed out in a particular way. The effect of stirring increases the fineness of the fine structure indefinitely and there is clearly a limit to the smallness of detail which can be meaningful, since f is a statistical description of particles and is not meaningful if applied to volumes of phase space so small as to contain only a few particles. Thus the use of Liouville's theorem, which conserves entropy, has the consequence, as soon as there is any disturbance, that smoothing is required, which increases the entropy. In fact there is a physical mechanism, which probably causes smoothing before the fine structure becomes meaningless, and is represented by a true Fokker-Planck term. This is due to noise at higher frequencies than those included before, which causes a random change in the velocity of a particle. For example, noise from the sun at ~ 10 Mc/s would contribute, and, possibly, noise generated by the trapped particles themselves. This true Fokker-Planck term must be small enough to allow the long lifetime observed for some particles.

The mechanism of stirring may be illustrated by considering f to be a function of L and ϕ, and the undisturbed motion to be given by $dL/dt = 0$ and $d\phi/dt = \omega_d$, which is dependent on L. A diffusion equation will be written for f as

$$\partial f/\partial t + \omega_d \, \partial f/\partial \phi = K \nabla^2 f. \tag{31}$$

The right-hand side is inserted to do the smoothing, K being a small diffusion constant and the detailed form of the term being unimportant. The Fokker-Planck term would have the Laplacian in velocity space, but this discussion is only an illustration and the variables chosen are unimportant. The fine structure set up always involves a rapid variation in velocity space, so that the Fokker-Planck term is always effective in smoothing.

Let f be expanded as a Fourier series in ϕ

$$f = \sum_{-\infty}^{\infty} f_n e^{in\phi}.$$

Then (31) gives separate equations for each f_n

$$\partial f_n/\partial t + i n \omega_d f_n = K e^{-in\phi} \nabla^2 (e^{in\phi} f_n). \tag{32}$$

The first approximation, neglecting K altogether, is

$$f_n \approx f_n(0)\, e^{-in\,\omega_d t} \tag{33}$$

which represents the rotation of the distribution at angular velocity ω_d. The approximation (33) gives

$$\partial f_n/\partial L \approx (\partial f_n(0)/\partial L - i n t f_n(0)\, d\omega_d/dL)\, e^{-in\,\omega_d t}, \tag{34}$$

and the term linear in t represents the steady increase in $|\partial f_n/\partial L|$ due to the winding of the spiral. Now $|\partial f_n/\partial \phi|$ is constant in this approximation and it is therefore the derivatives with respect to L which represent the growth of fine structure. It is assumed that K is very small, so that $K\nabla^2 f(0)$ is negligible, and that the only part of the diffusion term to be included is that which represents the fine structure. Then the dominant term in $\nabla^2 f_n$ is

$-(nt\, d\omega_d/dL)^2 f_n$ and using this approximation in (32) gives

$$f_n \approx f_n(0) \exp[-i n \omega_d t - \tfrac{1}{3} K (n d\omega_d/dL)^2 t^3], \tag{35}$$

which shows how f_n decays. Although the diffusion involves derivatives with respect to L, the result of this smoothing is a decay of all f_n's except f_0, so that, as $t \to \infty$, f tends to the average over ϕ of its value at $t = 0$. The higher Fourier components decay first, the decay becoming very rapid when $t \sim K^{-\frac{1}{3}}(n\, d\omega_d/dL)^{-\frac{2}{3}}$. The numerical values will not be discussed here and in the following it will be assumed that f is replaced by its average over the phase variables after each disturbance. It is clear, however, that the time required for stirring must be many times the relevant natural period, and, while this is not long in the case of the gyroperiod, it is doubtful whether thorough smoothing would occur between disturbances which preserve μ and I. It should be emphasized that a lack of thorough smoothing between disturbances does not affect the main discussion of the representation of the effects of disturbances. What the rate of smoothing controls is the amount of dependence of f on the phase variables, and this can be measured, though not easily.

The form of the diffusion terms can now be deduced on the assumptions that during the disturbances Liouville's theorem is valid and that afterwards f is replaced by its average over the phase variables. It should be mentioned that in the averaging process the number of particles was conserved, though this was trivial in our example. Now in the notation introduced in section 4, the product FGH is the diffusion coefficient, but here we wish to obtain F, G and H separately. This will be done by consideration of special cases.

Suppose that a disturbance disturbs particles in a limited region and that in this region f is a constant, so that all its partial derivatives vanish. Applying Liouville's theorem to the disturbance f remains the same constant and then, since f has no variation, no fine structure can develop. Thus no change results from the disturbance and this tells us that the diffusion term must vanish when all the derivatives of f vanish. This requires that H be a constant and allows us to put H = 1. It also leads to the relation previously mentioned between the two Fokker-Planck coefficients. Clearly the effect of the diffusion terms is to reduce the derivatives of f, and if the diffusion effects alone operate for a long time f will tend to become a constant, as in a perfectly stirred state, though of course f must decrease with energy at high energies in any practical situation.

The coefficient F is determined by conservation of particles. As an illustration consider the diffusion term $F\, \partial/\partial L(G\partial f/\partial L)$. Crudely, $G\, \partial d/\partial L$ represents a rate of flow of particles across a surface of constant L, but the Jacobian must be considered. If $\mathcal{J}$ is the Jacobian as in section 7, the rate of change of the number of particles between L_1 and L_2 resulting from the diffusion is

$$\int_{L_1}^{L_2} F\, \partial/\partial L(G\partial f/\partial L)\mathcal{J}\, dL = [\mathcal{J} FG\partial f/\partial L]_{L_1}^{L_2} - \int_{L_1}^{L_2} G(\partial f/\partial L)\partial/\partial L(\mathcal{J} F) dL. \tag{36}$$

Suppose now that initially all the particles were confined to a small range of L well inside the range between L_1 and L_2. The total number of particles must not change and the first term on the right-hand side of (36) vanishes, so the other term must also vanish. For this to vanish for any f satisfying the mild restriction that the particles are confined to a small range of L, $\partial(\mathcal{J}F)/\partial L$ must vanish and so we may put $F = \mathcal{J}^{-1}$.

The form of the diffusion terms has now been shown to be like

$$\mathcal{J}^{-1}\,\partial/\partial L(\mathcal{J} D\partial f/\partial L)$$

where D is the diffusion coefficient. If the disturbances give a mean square step $(\overline{\delta L})^2$ and have a frequency ν, random walk theory (BORN [10]) shows that $D = \frac{1}{4}\nu(\overline{\delta L})^2$. The Jacobian takes a simple form when the co-ordinates are μ, I and L and the element of phase space is integrated over the phase variables, as shown in the Appendix.

ACKNOWLEDGEMENTS

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APPENDIX

The Jacobian

If co-ordinates α, β, like those introduced in section 2, but with the additional requirement that $\vec{\nabla}\alpha \times \vec{\nabla}\beta = \vec{B}$ are used, a Hamiltonian formulation can be set up, and the Jacobian for a Hamiltonian system of co-ordination is always unity. This is the underlying reason why the Jacobian is simple, but the Jacobian can also be derived in an elementary way as follows.

The volume element in ordinary space is taken for a section of a magnetic tube and is dSds, where ds is the element of length along a line of force and dS the cross-section of the tube is $B_{eq}dS_{eq}/B$ or $d\alpha d\beta/B$, where B_{eq} and dS_{eq} refer to the same tube at the equator. The volume element in velocity space is $v_\perp d\psi dv_\perp dv_\parallel$, which can be written $\frac{1}{2}Bd\psi d\mu dv_\parallel$. The volume element of phase space given by the product is then $\frac{1}{2}B_{eq}dS_{eq}d\psi d\mu dv_\parallel ds$. Now this is to be integrated between mirror points and it is necessary to check that the plausible relation

$$\int_{M_1}^{M_2} \delta v_\parallel ds = \delta I \qquad (37)$$

is true. To make this clear we should consider the precise meanings of

δI and of the element of volume in phase space required. The meaning of δI is the difference in I for two particles of the same μ and L, but slightly different $v_{\parallel}$, each particle moving according to adiabatic theory. This δI is given by (37), if $\delta v_{\parallel}$ is the difference in $v_{\parallel}$ of the two particles when they are at the same point on the line of force. The motion of the particles is determined by conservation of μ and W, and then $m v_{\parallel}\, \delta v_{\parallel} = \delta W$ is constant along the line of force (even if there is a static electric field parallel to $\vec{B}$) and

$$\delta I = \delta W/m \int_{M_1}^{M_2} ds/v_{\parallel} = \tfrac{1}{2}\tau_b\, \delta W/m.$$

Now the element of phase space required is that which lies between the trajectories of two particles with the same μ and L, but slightly different I, and so is the integral in (37) with $\delta v_{\parallel}$ varying in the same way again. Thus the volume element integrated over the bounce phase is $\frac{1}{2} B_{eq} dS_{eq}\, d\psi\, d\mu dI$.

If L is always defined as r/R_E in the equatorial plane, $dS_{eq} = R_E^2 L dL d\phi$. Then the Jacobian for ϕ, ψ, L, μ, I is $\frac{1}{2} R_E^2 L B_{eq}$. If one integrates over ψ a factor of 2π should be inserted, and similarly if one integrates over ϕ in the case of a symmetrical field. For a dipole field $\mathcal{J}$ is $\frac{1}{2} R_E^2 B_0 L^{-2}$. In the diffusion terms constant factors in $\mathcal{J}$ are not involved and so for a dipole field one may use simply L^{-2}.

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ASYMPTOTOLOGY*

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Asymptotics is the science which deals with such questions as the asymptotic evaluation of integrals, of solutions of differential equations, etc., in various limiting cases. Elements of this science may be learned from the works of VAN DER CORPUT [2], ERDÉLYI [3] and DE BRUIJN [4] and advanced aspects from the numerous references in FRIEDRICHS's article [1]. By asymptotology I mean something much broader than asymptotics, but including it; pending further elaboration, I would briefly define asymptotology as the art of dealing with applied mathematical systems in limiting cases.

The first point to note here is that asymptotology is an art, at best a quasi-science, but not a science. Indeed, this explains much of my difficulty both in expounding my material and in finding an appropriate occasion to do so. It explains, too, why I am unable to support the corpus of my dissertation with the hard bones of theorems but must be content with a cartilage of principles, into seven of which I have distilled whatever of asymptotology I have been able to formulate appropriately and sufficiently succinctly.

The aspect of the definition of asymptotology just given which is most in need of explanation is the concept of applied mathematical system. An applied mathematical system is merely the mathematical description of a physical (or occasionally biological or other) system in which the variables expressing the state of the system are complete. The importance of formulating problems in terms of complete state variables constitutes a preliminary principle, not particularly of asymptotology but of applied mathematics in general, the Principle of Classification (or, perhaps better, of Determinism). It is illustrated by the overpowering tendency, in treating classical mechanical problems, to enlarge the configuration space to a phase space, since the phase (configuration together with its rate of change), but not the configuration alone, constitutes a complete description of a classical mechanical system. Consider also the tendency, in treating probabilistic mechanical problems, to switch over from this original description, which is incomplete because, for instance, the mechanical "state" at one time does not determine the "state" at another time, to a new description in terms of a probability distribution function of the old "states", which function evolves "deterministically" in time and is therefore preferable as a state description. This Principle is obviously closely related to the notion of a well posed problem emphasized by Hadamard. Its particular relevance to asymptotology comes about because only after one has singled out ("determined") an indi-

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vidual solution (or completely "classified" the family of solutions) can one reasonably inquire into its asymptotic behaviour.

Asymptotology is important because the examination of limiting cases seems to be the only satisfactory effective method of proceeding with the analysis of complicated problems (systems) when exact mathematical methods are of no (further) avail (and is often preferable even when they are). It is of value both for obtaining qualitative information (insight) about the behaviour of a system and its solutions and for obtaining detailed quantitative (numerical) results. Thus it is hardly surprising that examples, from trivial ones to the most profound, are found everywhere throughout the fields to which analysis (in the technical sense as a branch of mathematics) is applied.

An excellent example of asymptotology is the familiar HILBERT [5] or CHAPMAN-ENSKOG [6] ("HCE" from now on) theory of a gas described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{v}} = \lambda \int d^3 v \, d\Omega \, |\vec{v} - \vec{v}'| \, \sigma [\overline{f}\overline{f}' - ff'] \tag{1}$$

in the limit of high density ($f \to \infty$) or equivalently of frequent collisions ($\lambda \to \infty$). Another example is the CHEW-GOLDBERGER-LOW [7] theory of the so-called VLASOV [8] system of equations governing an ideal collisionless plasma and its electromagnetic field in what is often called the strong magnetic field (or small gyration radius) limit but is formally best treated [9] as the limit of large particle charges. In the general theory of relativity there is the fundamental EINSTEIN-INFELD-HOFFMAN [10] derivation of the equation of motion of a "test particle" (one not influencing the space-time metric, i.e., one of negligible mass) by treating it (its world-line, rather) as an appropriate singularity in the metric and letting the strength of the singularity approach zero. Hydrodynamics is rich in asymptotology (theory of shocks as arising in the limit of small viscosity and heat conductivity, theories of strong shocks and of weak shocks, shallow water theory, and so on) and so is elasticity. Kirchoff's laws for electrical circuits can be properly derived from Maxwell's equations only by going to the limit of infinitely thin conductors (wires). Simple examples also abound and are encountered daily by the practising applied mathematician and theoretical physicist. Naturally it is not practical to discuss deep examples in detail here, so I shall have to confine myself to brief remarks about them, relying for illustration mainly on simple and often trivial instances.

It should now be apparent, I hope, that whatever features such important, wide-spread, and diverse examples may have in common, and whatever lessons for future application may be gleaned from studying them, are well worth formulating and eventually standardizing. Even the many (most? far from all, as I know from my acquaintance) applied mathematicians (etc.) who have become familiar by experience with asymptotological principles — at least in the sense of knowing how to apply them in practice — must inevitably benefit from the introduction of a standard terminology and of the clarity of expression it permits. Implicit knowledge, no matter how widely distributed, deserves explicit formulation, but I am aware of no efforts in this direction which attempt to go anything like so far as I am doing here, though there are some related suggestions in Friedrichs's article.

The final possible obscurity in our previous tentative definition of asymptotology is what it means to deal with a system. To clarify this we might alternatively define asymptotology as the art of describing the behaviour of a specified solution (or family of solutions) of a system in a limiting case. And the answer quite generally has the form of a new system (well posed problem) for the solution to satisfy, although this is sometimes obscured because the new system is so easily solved that one is led directly to the solution without noticing the intermediate step.

To illustrate first by a trivial example: suppose it is desired to follow the (algebraically) largest root x of the simple polynomial equation

$$3\epsilon^2 x^3 + x^2 - \epsilon x - 4 = 0 \tag{2}$$

in the limit $\epsilon \to 0$. There is one root of order ϵ^{-2} obtained by treating the first two terms as dominant, $x \approx -\frac{1}{3}\epsilon^{-2}$, for which indeed the other two terms are relatively negligible (even though one of them is absolutely large, of the order ϵ^{-1}), but which is negative. The other two roots are finite, obtained by neglecting the terms with ϵ factors, $x \approx \pm 2$, the one sought having the plus sign. If we desire it to higher order, incidentally, we may put (2) for this root in the "recursion" form

$$x = 2\left(1 - \tfrac{3}{4}\epsilon^2 x^3 + \tfrac{1}{4}\epsilon x\right)^{\frac{1}{2}}, \tag{3}$$

expand out the right side in powers of ϵ, and generate better and better approximations for x by continually substituting the previously best approximation into the right side. But this is irrelevant to the present point, which is that (the problem of the algebraically largest root of) the original cubic Eq. (2) has been replaced by (the problem of the algebraically largest root of) the quadratic equation $x^2 - 4 \approx 0$, or more exactly $x^2 - (4 - 3\epsilon^2 x^3 + \epsilon x) = 0$, the quantity in parentheses being treated as known.

In the HCE treatment of system (1) in the limit $\lambda \to \infty$, the original integro-differential equation in the seven independent variables $t, \vec{x}, \vec{v}$ gets replaced by the set of coupled partial differential (hydrodynamic) equations

$$\begin{aligned} &\frac{\partial \rho}{\partial t} \approx -\frac{\partial}{\partial \vec{x}} \cdot (\rho \vec{u}), \\ &\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \frac{\partial}{\partial \vec{x}} \vec{u} \approx -\frac{1}{\rho}\frac{\partial p}{\partial \vec{x}}, \\ &\left(\frac{\partial}{\partial t} + \vec{u} \cdot \frac{\partial}{\partial \vec{x}}\right)(\rho^{-5/3} p) \approx 0 \end{aligned} \tag{4}$$

in the four independent variables $t, \vec{x}$; here, ρ, $\vec{u}$, p are of course the usual velocity-space moments of f.

These examples clearly illustrate the first asymptotological principle, which is in fact largely the raison d'être of asymptotology. This Principle of Simplification states that an asymptotological (limiting) analysis tends to simplify the system considered. This can occur in at least three general ways.

The basic way systems simplify is merely by the neglect of terms (or, in higher order analyses, at least treatment of small terms as if known, as in the case of the cubic equation earlier). Thus the polynomial equations $x^5 - \epsilon x + 1 = 0$ and $x^6 + ax^4 + \epsilon x^3 + 1 = 0$, without getting lower in degree as the cubic did, nevertheless become simple enough in the limit $\epsilon \to 0$ to be explicitly solvable algebraically. Differential equations in irregular domains approximating regular ones may in the limit become solvable by separation of variables. In other cases the coefficients may become so simple in the limit as to permit solution by Fourier or other transform. These are typical instances of perturbation theory; there are of course also many instances where the simplification which occurs does not appreciably facilitate the further analysis of the system.

A derivative way in which systems simplify, sometimes striking in effect, is the decomposition of the system into two or more independent systems among which the solutions are divided, so that the particular solution of interest satisfies a system with fewer solutions and hence usually in some sense of lower order. Thus the cubic polynomial equation considered earlier split up into a quadratic equation and what is effectively a linear equation. That is, the root of order ϵ^{-2} was obtained by neglecting the two last terms and writing $3\epsilon^2 x^3 + x^2 \approx 0$, and although this is cubic it has two trivial unacceptable roots $x \approx 0$ (corresponding to the solutions of the quadratic for finite roots) and is therefore equivalent to the linear equation obtained by dividing through by x^2.

The third (also derivative) way systems simplify, often spectacularly, is through the splitting off of autonomous subsystems. By an autonomous subsystem of a system is meant a part of the system (part of the conditions together with part of the unknowns) which is complete in itself, i.e., forms an applied mathematical system in its own right, so that it can (in principle, at least) be solved before the rest of the system is considered. The qualifier "autonomous" is by no means superfluous. Thus the system $f(x, y) = 0$, $g(x) = 0$ for the two variables x, y has the autonomous subsystem $g(x) = 0$. It has also the non-autonomous subsystem $f(x, y) = 0$ for y, non-autonomous because it is not definite (well-posed) until x has been determined, which requires the other part of the system.

Systems with autonomous subsystems occur much more often than one may at first realize, since there is an instinctive tendency to concentrate attention on the subsystem and forget that it is part of a larger problem. A particularly contemporaneous illustration of this is provided by the gravitationally determined motion of the sun, a planet, and an artificial satellite; the subsystem of the sun and planet alone is autonomous, since their motions are unaffected by the satellite and are naturally considered to be given and definite when its motion is under consideration. But there is a very common special kind of system having autonomous subsystems which do not get overlooked just because there are too many of them for any one to be singled out naturally. Such are the initial value problems, which, if well posed for $t_0 < t \leq t_1$ with initial conditions at t_0, are also well posed for $t_0 < t \leq t_2$ for any t_2 between t_0 and t_1, so that the autonomous subsystems constitute a continuous one-parameter family.

For an illustration of the third way of simplifying, note that in HCE theory the five moments ρ, $\vec{u}$, p satisfy (in the limit, of course) the autonomous subsystem (4), which is vastly simpler than (1) in having only four independent variables instead of seven. Similarly the "general" (for finite ϵ) pair of simultaneous equations $f(x, y) = 0$, $g(x) + \epsilon h(x, y) = 0$ reduces for $\epsilon \to 0$ to the system with an autonomous subsystem considered earlier. The sun-planet subsystem split off only by virtue of the implied limit of (relatively) small satellite mass, as is apparent from the less extreme case of the earth and its natural (rather than artificial) satellite.

The second and third ways both involve a reduction in the number of solutions from which the desired one must be singled out. This is a characteristically asymptotic simplification and, as Friedrichs [1] has affirmed, it justifies the limiting process even though complications arise in other respects. For instance, a linear second order differential equation may reduce to one of first order but non-linear. The "number" of solutions must be counted in whatever way is appropriate to the instance: as an integer (e.g., for the polynomial equation); as the dimensionality or number of parameters of a family of solutions (as for an ordinary differential equation); as the dimensionality of a parameter space, or number of independent variables of a function characterizing a solution (as with HCE, where seven reduces to four); etc.

In carrying out asymptotic approximations to higher order terms we are aided by the (second) Principle of Recursion, which advises us to treat the non-dominant terms as if they were known (even though they involve the unknown solution). The simplified system then determines the unknown in terms of itself, but in an insensitive way suitable (in principle at least) for iterative generation of an asymptotic representation of the solution. This has already been illustrated for one of the finite roots of our cubic equation example. For the numerically large root of (2) we may obtain the recursion formula $x = -(x^2 - \epsilon x - 4)/(3\epsilon^2 x^2)$. However, this is far from unique; by grouping the terms differently we obtain $x = -(x^2 - 4)/(3\epsilon^2 x^2 - \epsilon)$, which is equally suitable, since x has still been solved for from the dominant terms. It would be folly to solve for x from a small term such as ϵx; iteration on $x = (3\epsilon^2 x^3 + x^2 - 4)/\epsilon$ merely produces wilder and wilder ϵ behaviour. If one solves from the dominant terms inappropriately, namely in a way which does not give the solution explicitly outright when the small terms are neglected, then one has a scheme which may or may not converge, but which, even if it does, converges at a "finite" rate, not improving the asymptotic order of the solution in each iteration. This is illustrated by putting (2) in the convergent but asymptotically inappropriate recursion form $x = -[-(x^2 - \epsilon x - 4)/(3\epsilon^2 x)]^{1/2}$, which is quite usable, however, for numerical computation.

This trivial example is so trivial that the emphasis on recursion formulas seems forced. It is true that here and in many, many other cases one can simply write down an obvious power series in ϵ and determine the terms order by order. This approach fails, however, whenever a more general representation is required, as is by no means rare. For instance I recently encountered a case where the obvious series needed to be supplemented by a single logarithmic term (which was neither the dominant nor even the next-to-dominant term); the recursion relation generates all the right terms without prejudice as to their form. Generation of terms by recursion is often very clumsy for practical purposes, apart from leading to terms of unexpected form. However, it has a great theoretical advantage when properties of (all terms of) the series are to be derived, since the recursion relation is highly adapted naturally to the use of mathematical induction. (See the final reference for an example.)

The limiting cases we keep referring to are conventionally, in asymptotics, formulated so as to be cases where a parameter (often denoted by λ) approaches infinity. Since I intend asymptotology to embrace also situations where the limit system itself (not merely arbitrarily near ones) is meaningful (perturbation problems), it is preferable now instead to use a small parameter, conventionally denoted by $\epsilon\,(=1/\lambda$ for conversion). In fact, it may not be known in advance whether the limit case is meaningful, and, whether or not it is meaningful physically, mathematically it may or may not be so depending on the description employed. This brings us to our third asymptotological principle, the Principle of Interpretation: it is a major task of asymptotological analysis to find variables in which the given problem becomes a perturbation problem (has a meaningful limit situation). This may involve nothing more than recognizing that the original variables are such, as is the case for two roots of the cubic; for the third root, however, the formal limit of (2) is meaningless, but if transformation to the new variable $y = \epsilon^2 x$ is effected first, the equation obtained for y may be solved by perturbation analysis.

The characteristic feature of asymptotic analyses proper, as opposed to perturbation analyses, is the appearance (in both senses) of overdeterminism. Thus the cubic Eq. (2) with three roots apparently reduces in the limit to a quadratic with only two; the well behaved (for $\epsilon \neq 0$) pair of simultaneous linear equations $x+y=1$, $x+(1+\epsilon)y=0$ formally reduces to a mutually contradictory pair for $\epsilon = 0$; in the initial value problem $\epsilon(d/dt)z+z=0$ $(t>0)$, $z(0)=1$, for the continuous function $z(t)$, we seemingly have $z(t)=0$ in the limit, contradicting the initial condition; and the same thing happens in many less trivial cases (such as the theories of shocks, of boundary layers, and of fast oscillations), as described in detail by Friedrichs [1]. In this connection we have the (fourth) Principle of Wild Behaviour, which tells us that apparent overdeterminism arises because (at least some of) the solutions behave wildly in the limit – wildly, that is, compared to our preconceptions, as embodied in the mathematical form of the expressions employed for representing the solutions. Thus in neglecting the cubic (in addition to the linear) term of (3) we have obviously made the implicit assumption that x is not too large (say bounded), which is correct for only two of the roots, while the third behaves "wildly" in becoming infinite (like ϵ^{-2}); the solution of the simultaneous equations is similarly wild (like ϵ^{-1}); the solution of the initial value problem, $z=\exp(-t/\epsilon)$, is wild in having a derivative which, though converging to zero for every fixed positive t, does so non-uniformly and actually becomes infinite for t approaching zero sufficiently rapidly; and similar wildnesses occur in the deeper examples mentioned.

When overdeterminism occurs, if the solution we want is among those still permitted by the formal limit system, well and good: the loss of other solutions is our gain in simplicity (in the second way). If the solution we want is among those lost, then according to the Principle of Wild Behaviour we should allow for more general asymptotic behaviour of the solution. It is one of the most troublesome difficulties of asymptotological practice to find an appropriate asymptotic form. It is impossible to prescribe a priori all asymptotic representations that may ever prove useful, but among more general representations to try are two worth specific mention as frequently successful. The first is to supplement the originally expected series with new terms, such as smaller (more negative) powers, as in the case of the cubic equation, or logarithmic ones. The second, effective in many of the deeper problems, including those just referred to (see also a detailed example from my own experience [11], and illustrated by the initial value problem just exhibited (which may in fact be viewed as an elementary boundary layer problem), is to write the unknown as the exponential of a new unknown represented by a series, the dominant term of which must become infinite (at least somewhere) in the limit if anything is to be gained by so doing.

If there can be overdeterminism there can also be underdeterminism, which means that the original well posed problem reduces formally in the limit to a problem with more than one solution. For instance, let A be a known j-by-j matrix, let b and x be j-by-1 matrices (vectors), respectively known and unknown, and consider the matrix equation $Ax = b$. Suppose that A and b depend on ϵ and that the determinant of A is zero if and only if $\epsilon = 0$. Then the formal lowest order system $A^{(0)}\ x^{(0)} = b^{(0)}$ is certainly not well posed. Since $A^{(0)}$ is a singular matrix, there exists a 1-by-j matrix $n\ (\neq 0)$ such that $nA^{(0)} = 0$; for simplicity assume that n is unique (up to a constant factor). If $nb^{(0)} \neq 0$ the limit system obviously has no solution (overdeterminism, as in the previous example of simultaneous linear equations), so assume $nb^{(0)}=0$. Then $x^{(0)}$ is not completely determined by the limit system, and we have an example of underdeterminism.

Another excellent and rather typical example of underdeterminism is again the HCE problem. Letting $\lambda \to \infty$ in (1) (after dividing through by λ) leads to the information that $f^{(0)}$ is invariant under collisions, i.e. locally Maxwellian in some (local Galilean) co-ordinate system, which is very far from determining $f^{(0)}$, since there are five parameters (ρ, $\vec{u}$, p) needed to specify such a distribution and we are left unprovided with information on how the parameters at different points of space-time are related. (The CHEW-GOLDBERGER-LOW [7] theory is another such example [9].

In such straits we are rescued by the (fifth) Principle of Annihilation, which instructs us to find a complete set of annihilators of the terms which persist in the limit, apply them to the original system, and then go to the limit after multiplying by an appropriate function of ϵ so that the now dominant terms persist in the limit. By an annihilator of a mathematical entity is meant an operator which results in zero when applied to the entity. (Of course there are complicated cases in which this produces only some of the missing information, and the same procedure must be re-applied, perhaps repeatedly.)

In the matrix example, the terms $A^{(0)}x^{(0)}$ and $b^{(0)}$ which persist in the limit are annihilated by multiplication on the left by n. Applying this annihilator to the original equation, dividing by ϵ, and taking the limit gives what may be written

$$\lim_{\epsilon=0}\{\epsilon^{-1}n[A-A^{(0)}]\}\ x^{(0)}=\lim_{\epsilon=0}\{\epsilon^{-1}n[b-b^{(0)}]\}, \tag{5}$$

or $nA^{(1)}x^{(0)}=nb^{(1)}$ if A and b are expandable in integral powers of ϵ. In the normal case this provides just the one extra condition needed to determine $x^{(0)}$, which by the condition $A^{(0)}x^{(0)}=b^{(0)}$ was determined only up to a solution p of $A^{(0)}p=0$. In the abnormal case that (5) is not an independent condition, there is a linear combination of $A^{(0)}x^{(0)}=b^{(0)}$ and (5) which gives $0=0$. The formation of this linear combination is then our new annihilator, the application of which to $Ax=b$ and $\epsilon^{-1}n[A-A^{(0)}]x=\epsilon^{-1}n[b-b^{(0)}]$ leads to a new extra condition which will normally be independent and provide the missing piece of information.

In the HCE problem there are five scalars (mass, three components of momentum, and energy) which are preserved by collisions, so that taking the corresponding moments of (1) annihilates the right side. These are therefore annihilators of the dominant terms, which is why they are applied to (1) to obtain the five hydrodynamic equations relating the values of ρ, $\vec{u}$, p (and therefore f which is expressed in terms of them) at different points of space-time.

It is through the application of the Principle of Annihilation that the Principle of Simplification is maintained. The loss of solutions in a limit simplifies a system, while the gain of solutions, or loss of information*, would "complicate" it if we were not able to recover sufficient additional conditions to make up for the information lost.

The basic way systems simplify is by the neglect of terms, as stated earlier. But it commonly happens that the relative asymptotic magnitude of two terms to be compared depends upon some knowledge not yet available or on some assumption or decision not yet made. According to the (sixth) Principle of Maximal Balance (or of Maximal Complication**), for maximal flexibility and generality we should keep both terms, i.e., we should allow for the possibility or assume that they are comparable. In the case of incomplete knowledge this is mere prudence; any term in an equation definitely smaller in order of magnitude than another term may be considered negligible, but no term should be neglected without a good reason. In the case of a pending assumption or decision, the desire to balance two such competing terms helps to determine the choice.

The most widely applicable and hence most informative ordering is that which simplifies the least, maintaining a maximal set of comparable terms. Quite often there is more than one possible maximal set of terms, with no

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* Use of this terminology is justified even from the technical viewpoint of information theory, suggesting the possibility of assigning a measure to the decrease in the number of solutions occurring in a limit.

** I now feel that "Minimal Simplification" is more appropriate here.

set including all terms of any other. (Sets of terms form a lattice ordered by inclusion.) Each maximal set corresponds to different asymptotic behaviour. The solutions may split up according to which behaviour they have (second way of simplifying), as with the cubic, or each solution may exhibit a variety of different behaviours, in different regions, as with a boundary layer phenomenon.

For instance in the case of the cubic equation, how could we know that two solutions are finite and one of order ϵ^{-2}? Put another way, why did we not assume the first and third terms to be the dominant ones, or the second and third, or so on? In this particular case there is an easy answer: if we had, we would have obtained a "solution" for which the neglected terms were not in fact negligible compared to the supposed dominant terms, i.e., the "solution" found would not have been self-consistent. But suppose there were several more terms, would we have had to try every pair? (Or suppose there were two independent small parameters δ and ϵ instead of only one.) Clearly, no matter which terms are dominant x will behave predominantly as some power of ϵ. We therefore assume the general representation $x = a\epsilon^q$ and wonder what value of q to take. One might in fact choose arbitrarily any value for q but will then generally find that for finite a only one term of (2) dominates, which is nonsensical, so that $a = \infty$ (if it was the constant term), which is not legitimate, or else $a = 0$ (if it was one of the others), which, if more legitimate, is certainly no more useful. A value of q will only be "proper" if we end up with a representation which is "maximally complicated" in that it really consists of one term $a\epsilon^q$ instead of "no terms" such as 0 or ∞. If we put $x \approx a\epsilon^q$ into (2) the successive terms vary as ϵ to the respective powers $3q+2$, $2q$, $q+1$, 0, and it is easy to see that only $q = 0$ or $q = -2$ makes two (or more) powers equal minima.

On the side it might be of interest to mention a graphical method of finding the proper values of q which apparently goes back to Newton. It is hardly needed in the present simple illustration but can be a great time-saver in more involved examples (also those of higher dimensionality). We plot each term of (2) as a point on a graph, the abscissa being the exponent of x and the ordinate that of ϵ (see four heavy points in Fig. 1); the coefficient is ignored so long as it is not zero. The specification of a definite relationship between x and ϵ (i.e. of a definite value of q) leads to the identification of the asymptotic behaviour of all terms (present or not) corresponding to points which are on a common line with a definite slope. Thus, for $x \sim \epsilon$ all points on the same down-slanting (from left to right) 45° line correspond to a common asymptotic behaviour, while for $x \sim \epsilon^{-1}$ the same holds for up-slanting 45° lines (see light dashed lines). Since the smaller the power of ϵ the larger the term, we seek lines passing through (at least) two graphed points and having no graphed points below them. We may think of finding the lower convex support lines of the set of graphed points, perhaps kinesthetically by imagining pushing a line up from below until it first hits a graphed point and then rotating it around that point until it next hits a second graphed point. It is immediately apparent from Fig. 1 that there are just two such lines and that they correspond to $q = 0$ and $q = -2$ (see heavy dashed lines). It is also clear that the point (1, 1), like all points in a semi-infinite vertical strip (see horizontally shaded area), are "shielded" by the points

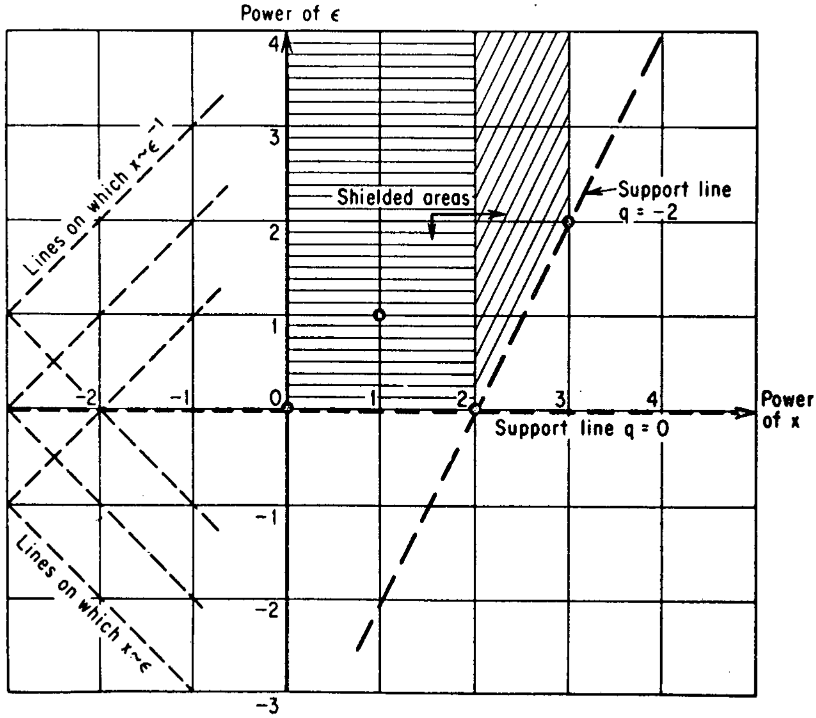

Fig. 1

Graphical methods of finding the proper values q

(0, 0) and (2, 0) and can never be on a support line; it is indeed obvious that ϵx is negligible with respect to either x^2 or 4 no matter how x varies with ϵ. Similarly there is a semi-infinite vertical strip shielded by the points (2, 0) and (3, 2) (see diagonally-shaded area). In more complicated cases we can thus exclude terms wholesale from competition.

To return to our proper business, illustration of the Principle of Maximal Complication, consider the problem of finding the lowest frequency of vibration and the corresponding form of vibration of a uniform membrane stretched between two close wires lying in a plane, one of which we take straight for simplicity. The equation for the standing vibration of a membrane is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \nu^2 u = 0, \tag{6}$$

where u is the displacement normal to the (x, y) plane, which is the rest plane of the membrane (the plane containing the wires), and ν is the frequency of vibration of the mode. Let the equations of the wires in the (x, y) plane be $y = 0$ and $y = \epsilon Y(x)$, where ϵ of course is the small parameter of closeness. We may suppose $Y(x_1) = Y(x_2) = 0$ so as to have to consider only the finite region $x_1 < x < x_2$, $0 < y < \epsilon Y(x)$. Imposing the condition $u = 0$ on the boundary of this region and (6) inside the region, we have an eigenvalue problem for the lowest eigenvalue ν and its corresponding eigenfunction u. This is one common type of asymptotic problem, asymptotic rather than "perturbational" in that there is no limit problem because the region of interest disappears in the limit. The remedy for this is well known [1]; we re-scale the variables appropriately, in this case introducing $\eta \equiv \epsilon^{-1}y$ so that the region in the (x, η) plane becomes $x_1 < x < x_2$, $0 < \eta < Y(x)$, and (6) becomes

$$\frac{\partial^2 u}{\partial x^2} + \epsilon^{-2}\frac{\partial^2 u}{\partial \eta^2} + \nu^2 u = 0 . \tag{7}$$

Taking the asymptotic behaviour of each term at its face value (but remembering that ν is not yet determined), we deem the first term negligible compared to the second, and (by the Principle) assume $\nu^2 \sim \epsilon^{-2}$ to balance the second and third terms. Introducing $\omega \equiv \epsilon\nu$ we write (7) as

$$\frac{\partial^2 u}{\partial \eta^2} + \omega^2 u = -\epsilon^2\frac{\partial^2 u}{\partial x^2}. \tag{8}$$

To lowest order we neglect the right side of (8), whereupon x degenerates from an independent variable to a mere parameter. The really proper treatment at this point, by the Principle of Recursion, would be to treat the right side of (8) as known, solve for u on the left in the form of an integral representation (involving the simple, well known, explicit Green's function), and try to obtain u iteratively. Instead we shall do something similar but simpler, more or less parallelling the lowest order version of the proper treatment. For each x we have, to lowest order, a simple eigenvalue problem with the lowest eigenstate $u = A\sin(\pi\eta/Y)$ and eigenvalue $\omega = \pi/Y$. But ω so defined depends on x, which is impermissible, so we take $A(x)$ to be a Dirac delta function, the location of whose singularity we take to be at the maximum of $Y(x)$ in order to have the smallest ω; for simplicity we assume the maximum of Y to be unique and to occur at $x = 0$. In a sense we have now solved the problem originally posed, but since our answer is singular it is not entirely satisfactory (see the next and final Principle to be formulated). Indeed, since our "solution" is singular in its x dependence, we ought to worry whether our earlier neglect of $\epsilon^2(\partial^2 u/\partial x^2)$ was justified, and we might well be curious anyway about the true detailed x dependence which we have cavalierly expressed as a delta function. Since the significant behaviour occurs near $x = 0$ we introduce $\xi = \delta^{-1}x$, where δ is a small parameter to be determined (related to ϵ). We also write $\omega = \omega_0 + \hat{\omega}$, where $\omega_0 = \pi/Y(0)$ and $\hat{\omega}$ is small. Since $\partial^2 u/\partial\eta^2 \sim -\pi^2 Y(x)^{-2}u$, from (8) we obtain

$$\left[\frac{\pi^2}{Y(\delta\xi)^2} - \omega^2\right] A \approx \frac{\epsilon^2}{\delta^2}\frac{d^2 A}{d\xi^2} . \tag{9}$$

Let $Y(\delta\xi) = Y(0) + \frac{1}{2}Y''(0)\delta^2\xi^2 + \dots$ with $Y''(0) < 0$, whereupon this becomes

$$\left[-\frac{\pi^2 Y''(0)}{Y(0)^3}\delta^2\xi^2 - 2\omega_0\hat{\omega}\right] A \approx \frac{\epsilon^2}{\delta^2}\frac{d^2 A}{d\xi^2}. \tag{10}$$

According to the Principle of Maximal Complication we choose the as yet undetermined asymptotic behaviours so as to keep all the terms in the equation and are thus led to take $\delta = \epsilon^{1/2}$ and $\tilde{\omega} = \epsilon^{-1}\hat{\omega}$, obtaining

$$\frac{d^2A}{d\xi^2} + \frac{\pi}{Y(0)}\left[\frac{\pi Y''(0)}{Y(0)^2}\xi^2 + 2\tilde{\omega}\right]A \approx 0. \tag{11}$$

On the ξ distance scale A must vanish at "infinity", and we have a well known eigenvalue problem arising in the quantum theory of the harmonic oscillator. The lowest eigenfunction is the Gaussian $A = \exp\left\{-\frac{\pi}{2}Y(0)^{-3/2}[-Y''(0)]^{1/2}\xi^2\right\}$ with real eigenvalue $\tilde{\omega} = \frac{1}{2}[-Y''(0)/Y(0)]^{1/2}$.

Incidentally, if we should be interested in the behaviour of u for $|x|$ not very small, where u decreases rapidly, a different procedure must be used. The right side of (8) cannot be neglected there, since $\omega \approx \pi/Y(0)$ does not even approximate the local eigenvalue $\pi/Y(x)$ for which the left side can vanish with $u \neq 0$. The device mentioned earlier of representing the unknown as an exponential works here; with $u = \exp v$, (8) becomes

$$\frac{\partial^2 v}{\partial\eta^2} + \left(\frac{\partial v}{\partial\eta}\right)^2 + \omega^2 = -\epsilon^2\left[\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]. \tag{12}$$

We may assume that v is expandable as a series in ϵ, $v = \epsilon^{-1}[v^{(0)} + \epsilon v^{(1)} + \dots]$, where the leading term has been taken large of order ϵ^{-1} to permit the right side of (12) to contribute. We must have $\partial v^{(0)}/\partial\eta = 0$ or the left side will dominate again, so $v^{(0)}$ is a function of x only, and to dominant terms (12) becomes

$$\frac{\partial^2 v^{(1)}}{\partial\eta^2} + \left(\frac{\partial v^{(1)}}{\partial\eta}\right)^2 + \omega_0^2 = -\left(\frac{\partial v^{(0)}}{\partial x}\right)^2.$$

Viewed as an equation for $v^{(1)}$ this can be linearized and "homogenized" by reversing the exponentiation procedure, namely by introducing $w = \exp v^{(1)}$, whence

$$\frac{\partial^2 w}{\partial\eta^2} + \left[\omega_0^2 + \left(\frac{\partial v^{(0)}}{\partial x}\right)^2\right]w = 0.$$

Together with the boundary conditions on w (that it vanish at $\eta = 0$, $Y(x)$) this is an eigenvalue problem which determines the variation of $v^{(0)}$,

$$\omega_0^2 + \left(\frac{\partial v^{(0)}}{\partial x}\right)^2 \equiv [\pi/Y(x)]^2,$$

as well as the η dependence of w (sinusoidal). All that the device has amounted to in this case, of course, is factoring out (from u) a fast varying function of x, but the use of the exponential representation has led to that procedure in a natural and systematic way.

We complete our list with the simple (seventh) Principle of Mathematical Nonsense: if, in the course of an asymptotological analysis, a mathematically nonsensical expression appears, this indicates that the asymptotology has not been done correctly or at least not carried out fully (although even incomplete it may be satisfactory for one's purposes). One may come upon expressions such as 0/0, divergent sums or integrals, singular functions, etc., and whether they are to be considered nonsensical sometimes depends on the use they are to be put to. In the membrane vibration problem just discussed the first instance of mathematical nonsense was the disappearance in the limit of the region over which the partial differential equation was to be solved, the second was perhaps the dependence of ω on x, and the third was the response to this, the use of a singular (delta) function.

Frequent in asymptotological analyses is the occurrence of phenomena on different scales of distance or time. The HCE problem is a well-known case, since if if is not prescribed Maxwellian at the initial instant, there is a relatively short period of time (the order of a collision time) during which f becomes Maxwellian, while the five moments remain approximately constant, and a relatively long period (of order λ times as long) during which the five moments (hydrodynamic variables) vary but f maintains its Maxwellian form. For an extremely simple example of the same type, consider the familiar electric circuit equation $V = RI + L\dot{I}$, where the voltage V(t) is an imposed function of time, the current I(t) is to be found, the resistance R and the inductance L are positive constants, and we choose to examine the limit $L \to 0$. Treating $L\dot{I}$ as if it were known, we immediately obtain a recursion formula for I,

$$\begin{aligned} I &= \frac{1}{R}(V - L\dot{I}) \\ &= \frac{1}{R}\left[V - \frac{L}{R}\dot{V} + \left(\frac{L}{R}\right)^2\ddot{V} - \left(\frac{L}{R}\right)^3\dddot{V} + \dots\right], \end{aligned} \tag{13}$$

which is fine except for not in general satisfying the arbitrary initial condition on I natural for the original first order differential equation. For short times (of order L) $\dot{I}$ is large and V approximately constant, so that the difference of I from its quasi-equilibrium value V/R decays like $\exp(-Rt/L)$; after this transient has died out (13) holds. Incidentally, the expression in brackets in (13) is just like the Taylor expansion in powers of L of V evaluated at the argument $t - L/R$ except for a factor of $(n-1)!$ in the denominator of the n-th term, which shows that the asymptotic series (13) for I cannot be expected to converge even if V is analytic (which does not stop it from being very useful).

In phenomena with behaviour on two different time scales there is a widely pertinent distinction to be observed between finite conservative systems on the one hand and infinite or dissipative systems on the other. For instance, the well-known problem of the harmonic oscillator with slowly varying coefficient of restitution [12], $\ddot{x}+k(\epsilon t)x = 0$, is an example of the first kind; on the short (finite) time scale k is approximately constant and the oscillator simply oscillates steadily, while on the long ($\sim \epsilon^{-1}$) time scale the frequency and amplitude of the oscillation vary in response to the variation in k. Contrast with this the behaviour of the dissipative electric circuit, where only initially the current I varies on the short time scale, swooping toward its quasi-steady value. The HCE example shows that a conservative system can act the same way so long as it is infinite; in this case the decay comes about by a process of "phase mixing", and is possible because the Poincaré recurrence time is infinite.

The asymptotic separation of time scales is the basis for an exciting recent approach in statistical mechanics [13]. Typically one obtains equations for the one-particle and the two-particle distribution functions f_1 and f_2 for a gas of appropriate characteristics, and finds that f_1 can vary only slowly, but that f_2 can vary quickly so as to phase-mix towards a quasi-steady distribution as t gets large on the short time scale while remaining small on the long time scale. The limiting distribution f_2 is a functional of f_1, which when substituted into the equation for f_1 leads to an autonomous "kinetic equation" for f_1. The irreversibility (timewise) of this kinetic equation comes about in a natural way, in that the limiting f_2 depends on which direction t is taken to the limit (on the short time scale), whether to plus or to minus infinity. It is a major triumph of this approach that the "Stosszahlansatz" can for the first time be actually derived (under moderate smoothness assumptions).

To return to the finite case, I am glad to take the opportunity of advertising a recent paper [14] in which I have elaborately worked out the asymptotic theory of finite systems of ordinary differential equations depending on a small parameter ϵ which to lowest order have all solutions periodic. Applied to Hamiltonian systems the theory leads to the existence of adiabatic invariants which are constant (integrals) to all orders in ϵ.

We are all familiar with those rather unsatisfactory research papers in which the author makes a series of largely arbitrary ad hoc approximations throughout, often dubious without (sometimes even with) the author's intuitive grasp of the situation. These "ad-hoaxes" have their place and utility, but how much more desirable and convincing is a properly worked out and elegant asymptotological treatment, with any arbitrary assumptions (like remarkable coincidences in a well constructed mystery story) made openly and above board right at the beginning where anyone can assess their merits for himself, and with the later development unfolding naturally and inexorably once a definite problem and the limit in which it is to be considered have been settled upon.

The art of asymptotology lies partly in choosing fruitful limiting cases to examine - fruitful first in that the system is significantly simplified and second in that the results are qualitatively enlightening or quantitatively descriptive. It is also an art to construct an appropriate generic description for the asymptotic behaviour of the solution desired. The scientific element in asymptotology resides in the non-arbitrariness of the asymptotic behaviour and of its description, once the limiting case has been decided upon.

One of Molière's characters observes that for more than forty years he has been talking prose without knowing it. It is doubtful that he benefited from the discovery, but I hope that you will be more fortunate and not disappointed in having by now discovered that asymptotology is what you have been practising all along.

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III

PLASMA CONFINEMENT

TOROIDAL MAGNETIC FIELD CONFIGURATIONS AND FINITE RESISTIVITY

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The following paper discusses the confinement of low-β plasma in closed vacuum-magnetic-field configurations, with particular reference to the finite-resistivity hydromagnetic stability problem. The configurational solutions and stability results given in Sections II and III are part of a collaborative effort with M. N. Rosenbluth.

I. INTRODUCTION
SHEAR-STABILIZATION AND MAXIMUM-$\int dl/B$-STABILIZATION

A simple toroidal magnetic field does not afford an equilibrium solution for an isolated (toroidal) body of plasma. One solution of the toroidal equilibrium problem was originally found by SPITZER [1], in terms of the rotational-transform technique, which insures infinite-conductivity equilibrium by causing (almost) every flux surface to be covered ergodically by a single field line. Rotational transform provides the basis for the confinement of low-β plasmas in the vacuum field of the stellarator. The variation of the rotational transform from one flux-surface to the next can be used to provide stability against low-β, infinite-conductivity hydromagnetic interchanges [1]; this technique is known as shear-stabilization.

Experimentally, little is known about the effectiveness of low-β shear-stabilization in vacuum fields, since phenomena due to the plasma-heating method have always predominated in past experiments. Theoretically [2], a small but finite resistivity (in the sense of large but finite magnetic Reynold number $S = \tau_R/\tau_H$) permits β-driven interchange modes to grow in spite of shear, but at a growth rate that is reduced from the shear-free rate $\omega \propto G^{\frac{1}{2}} \tau_H^{-1}$, to the lower rate $\omega \propto \Sigma^{-\frac{2}{3}} G^{\frac{2}{3}} \tau_H^{-\frac{2}{3}} \tau_R^{-\frac{1}{3}} k^{\frac{2}{3}}$; where $G = \left| a^2 \nabla\beta / R_c \right|$ measures the destabilizing force due to field-line curvature R_c^{-1}, $\tau_H = \frac{a}{B} (4\pi\rho)^{\frac{1}{2}}$ is the Alfvén-wave transit time over a characteristic linear dimension a, $\tau_R = 4\pi a^2/\eta$ is the resistive-diffusion time at resistivity η, k is the wave number of the interchange, and Σ is a measure of the shear. (We can define Σ^{-1} loosely as the distance in the ψ direction required for the magnetic field vector to turn through an angle $\pi/2$.) The preceding results, derived in the hydromagnetic approximation, cast a gloomy light on the possibility of long-time stable confinement; and the gloom appears to be lifted only partly by consideration of "finite Larmor-radius effects" [3].

Moreover, one fears that finite classical resistivity may be only one of a number of phenomena that can disrupt the perfect communication along the confined plasma body on which shear-stabilization depends. As long as the plasma resides in a gradient or at a maximum of B, the destabilizing force G, due to the plasma diamagnetism, is present, and one can never be certain that the associated "energy reservoir" can be prevented from driving actual dynamic modes.

In open-ended geometry, the "Minimum-B" approach championed by IOFFE [4] and others [5] resolves this uncertainty by positively removing the energy reservoir that supports the interchange mode. In closed geometry, topology prevents the placement of the plasma in a region of everywhere outwardly increasing magnetic field strength [6]. The most that can be done is to insure that B increases outwardly in some average sense, such as

$$\vec{\nabla} P \cdot \vec{\nabla} U > 0, \tag{1}$$

where the integral $U = \int dl/B$ is taken along a field line, and P is the plasma pressure, which we will consider to be isotropic.

The criterion (1) is a familiarly sufficient condition for infinite-conductivity low-β stability [7], and at the edge of the plasma becomes also a necessary condition. A simple physical meaning can be given to (1), as well as to the associated equilibrium condition [8].

In equilibrium, we have $\vec{\nabla} P = \vec{j} \times \vec{B}$, so that P is constant along field lines. In Fig. 1, a sector of a toroidal configuration is shown. We want

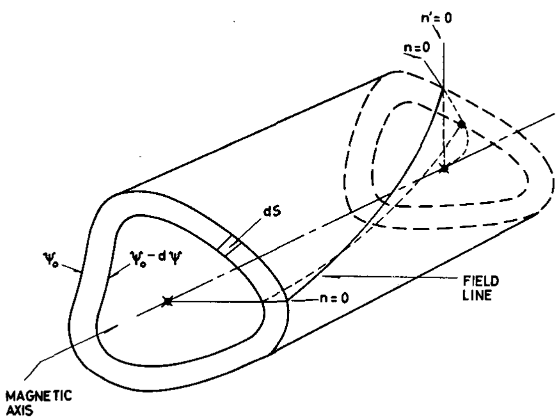

Fig. 1

Section of flux-tube illustrating generalized co-ordinates

to find the condition that the magnetic flux surfaces ψ_0 and $\psi_0 - d\psi$ can be constant-P surfaces. If a single field line covers the surface ψ_0 on successive passes around the major circumference of the torus, thanks to rotational transform, then automatically an equilibrium exists [1]. When there is no transform, we require that the pressure increment dP from surface ψ_0 to surface $\psi_0 - d\psi$ should be constant and equal to

$$dP = B\, dJ_{\perp} \tag{2}$$

where $dJ_\perp$ is the transverse current density per unit length along field. The surface current density $dJ_\perp$ has no divergence, hence the total current crossing a field line

$$dI = \int dl\, dJ_\perp \tag{3}$$

is constant for all field lines in ψ_0. From (2) and (3) we then have that $\int dl/B$ must be constant on ψ_0. The equilibrium condition is thus that constant-P surfaces must be constant-U surfaces [8]. The stability analysis follows the same approach. If the plasma surface lies on ψ_0, then the outward displacement of a tube of plasma at initial pressure dP is accompanied by the appearance of a surface current

$$dI = \int_{l_1} dl\, dJ_\perp = \int_{l_2} dl\, dJ_\perp, \tag{4}$$

where l_2 is a field line on the outward side and l_1 on the inward side of the tube, and where

$$dP^* = B\, dJ_\perp \tag{5}$$

holds on both l_1 and l_2, with dP^* representing the local pressure in the perturbed state. From (4) and (5) we have that the stability condition $dP_2^* > dP_1^*$ corresponds to

$$\int_{l_1} \frac{dl}{B} > \int_{l_2} \frac{dl}{B}$$

in accordance with (1). An alternative physical interpretation of $\int dl/B$ is as the flux-tube-volume per unit flux [7]; and stability properties are then derived in terms of plasma expansion during interchange. For finite-resistivity interchanges in sheared fields, the interpretation given here is more illuminating.

As we shall see later, the $\int dl/B$ stability criterion is applicable only when the magnetic scalar potential χ is well-defined (as it may not be when there are internal floating conductors), in which case $\chi = \int dl B$ taken around a field line is the same for all lines, and we may interpret the condition

$$\vec{\nabla} \int dl/B < 0$$

as

$$\vec{\nabla} \int d\chi / B^2 = \langle \vec{\nabla} B^{-2} \rangle < 0. \tag{6}$$

Configurations that are $\int dl/B$-stable evidently depend on communication along field lines to counterbalance regions of favourable and unfavourable local contributions to $\langle \vec{\nabla} B^{-2} \rangle$. Thus one can say that finite resistivity impairs both shear and $\int dl/B$-stabilization; but stability analyses such as that given below, or in [19] show that in the limit of high conductivity ($S \to \infty$) the $\int dl/B$-technique tends to become more effective, with a maximum instability growth rate $\omega \propto G\,\tau_R^{-1} k^{-2} L^2$, where L is the distance between "good" and "bad" regions. If the correlation length of such modes in the direction of $\vec{\nabla}P$ is of order k^{-1}, then the resultant diffusion rate will exceed ordinary resistive diffusion only by some geometric factor $\sim (L/a)^2$. As reference [19] points out, however, this correlation length may remain large for small k^{-1}, and the prospects of effective $\int dl/B$-stabilization then decline. The resolution of this uncertainty must be sought in non-linear analysis and in experiment.

TABLE I

HIGH-S INSTABILITIES

Shear-stabilized	$\int d\ell/B$-stabilized	Approx. growth rate
No	No	$G^{1/2}\,\tau_H^{-1}$
Yes	No	$G^{2/3}\,\tau_H^{-2/3}\,\tau_R^{-1/3}\,\Sigma^{-2/3}\,k^{2/3}$
No	Yes	$G\,\tau_R^{-1}\,k^2\,L^2$
Yes	Yes	Smallest of above

II. SOME $\int dl/B$-STABLE CONFIGURATIONS

Closed configurations satisfying the equilibrium condition on P and U (in the absence of rotational transform) are relatively easy to find [9]. The stability condition (1) can also be met readily in systems employing "floating" rigid conductors inside the plasma volume [10, 11]. During the past year, a considerable number of solutions to (1) have been found without resorting to floating conductors. In the latter category, which is probably of the main practical interest, there are two basic types of solution, corresponding to the use or non-use of rotational transform as a help in obtaining favourable $\vec{\nabla}U$ (as distinct from its usual function of providing equilibrium and shear).

A. Periodic multipole solutions

We will treat a thin, straight flux tube about the z-axis, keeping only lowest-order terms in r, and assuming that infinite periodic solutions with finite positive stability will go over smoothly into toroidal solutions of large aspect ratio. We use the scalar potential

$$\chi = \int f dz - \frac{1}{4} f' r^2 + \sum_{\ell=1}^{\infty} \frac{1}{\ell} g_\ell r^\ell \cos \ell\theta \tag{7}$$

where $\vec{B} = \vec{\nabla}\chi$. We consider f and g to be arbitrary periodic functions of z. Various combinations of g_l's can be used to obtain solutions where U is maximal on axis or at least has a closed constant-U surface on which the outward gradient of U is negative.

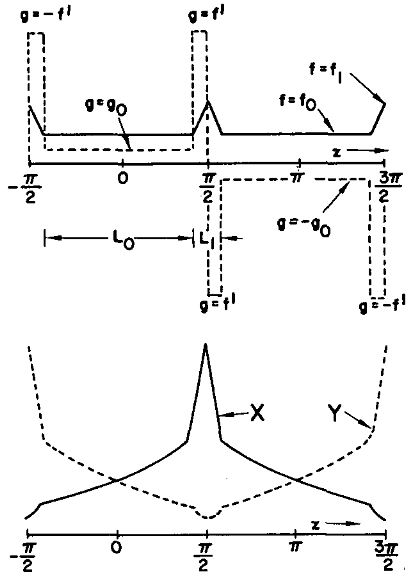

Fig. 2

Idealized solution for periodic quadrupole configuration. Here g refers to g_2

We begin with the simple quadrupole case [12] $g_\ell \equiv 0$ for $\ell \neq 2$, $f(z) = f(z+\pi) = f(\pi - z)$, $g_2(z) = -g_2(z+\pi) = -g_2(\pi - z)$, which is illustrated in Figs. 2 and 3. In Cartesian co-ordinates, the equations of the field-lines are

$$X = X_0\left(\frac{f_0}{f}\right)^{\frac{1}{2}} \exp\left(\int_0^z dz_1 \frac{g_2}{f}\right)$$
$$Y = Y_0\left(\frac{f_0}{f}\right)^{\frac{1}{2}} \exp\left(-\int_0^z dz_1 \frac{g_2}{f}\right). \tag{8}$$

Using $dl/B = dz/B_z$, we find

$$U = \int_0^{2\pi} \frac{dz}{B_z} = \int_0^{2\pi} \frac{dz}{f}\left[1 + R_0^2 \frac{f_0}{f^2}\left(\frac{f''}{4} - \frac{g_2'}{2}\right) \exp\left(2\int_0^z dz_1 \frac{g_2}{f}\right)\right], \tag{9}$$

where $R_0^2 = X_0^2 + Y_0^2$. The equilibrium surfaces of constant P are thus seen to be flux tubes having circular cross-section in the $z = 0, \pi, 2\pi \ldots$ planes. The stability condition (1) can be met for a confined plasma if the coefficient of R_0^2 is negative, i.e., if

$$0 > \int_0^{\pi} \frac{dz}{f^4}\left(\frac{1}{2}f' - g_2\right)\left(\frac{3}{2}f' - g_2\right) \exp\left(2\int_0^z dz_1 \frac{g_2}{f}\right). \tag{10}$$

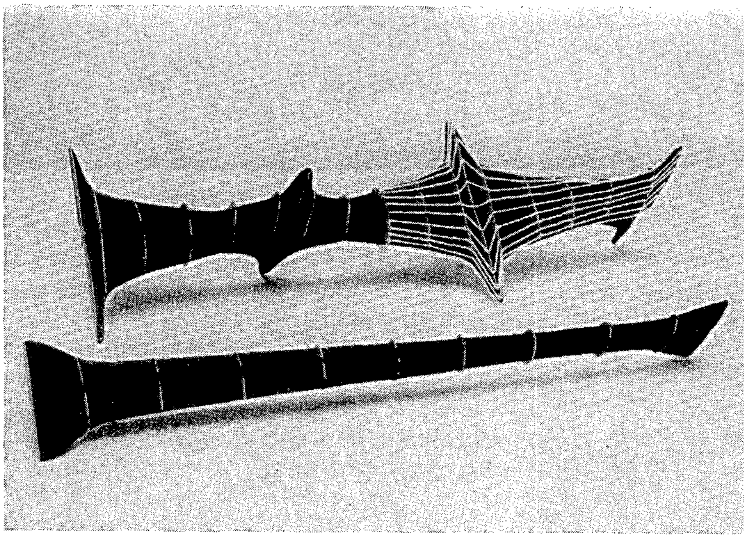

Fig. 3

Models corresponding to solution of Fig. 2. Top model is on axially foreshortened scale

Stabilizing contributions arise only where $g_2 \approx f'$, but the periodicity conditions (designed to give X-Y symmetry in U) imply that for every such region there is another where $g_2 \approx -f'$, tending to give a much larger destabilizing contribution. The only solution is to weight the favourable contributions heavily by means of the exponential factor, which implies a certain minimum modulation $Q = X_{max}/X_{min} = Y_{max}/Y_{min}$. The lowest possible Q-value, namely $2+\sqrt{3}$, is obtained for $L_1 \to 0$ and $g_2 = \pm\alpha f'$, α being selected to minimize the integral in (10). For more "practical" solutions like that in Figs. 1 and 2, Q must be taken of order 20. Typical finite-R calculations, carried out numerically, show that the negative outward slope of U is maintained only out to $R_0 = 0.1$, and that the local peak in U at $R_0 = 0$ is less than 1%.

In the preceding example, the f and g_2 fields have played two distinct roles: to control the shape of the basic flux surface (e.g., set the magnitude of Q); and to determine the integral in (10). The first role is affected primarily by the behaviour of f and g_2 in the intervals L_0. The second role is affected primarily by the magnitudes of f' and g_2 in the intervals L_1. A radical improvement can be introduced, following the suggestion of JOHNSON [13] by setting $f' = g_2 = 0$ in L_1, and using g_1 and g_3 instead. One can make g_1 follow the previous symmetry pattern of f', while g_3 follows that of g_2. Calculating U, one obtains an integral analogous to that in (10); and in the limit of small L_1 the contribution comes exclusively from L_1. The stability condition then is

$$0 > \left\{ X^2\left(\frac{\pi}{2}\right) - X^2\left(-\frac{\pi}{2}\right) \right\} \int_{(\pi-L_1)/2}^{(\pi+L_1)/2} dz\, g_1 g_3 . \tag{11}$$

Taking $g_1g_3 < 0$ at $z = \pi/2$, we now find that $Q > 1$ is sufficient to provide a maximum of U on the z-axis. Finite-R calculations now also indicate much stronger maxima and wider regions of favourable $\vec{\nabla}U$.

Another highly significant improvement on the original periodic-multipole concept was made by LENARD [14]. In place of the previous system of alternating "flux-shaping" regions L_0 with "stability-contributing" regions L_1, he suggested continuous sequence of identical regions L_1 - like that about $z = \pi/2$. In such a region, the field line $\theta = 0$, $r = X$ is given a "good" contribution to $\vec{\nabla}U$, and the line $\theta = \pi/2$, $r = Y$ is given a "bad" contribution. To obtain uniform stability properties, Lenard added a rotational transform, thus causing every line to pass through both X and Y-type regions. The resultant configurations have been more elegant in shape, and have also shown more favourable stability properties, because of the injection of a new effect - the rotational transform is typically weaker in the regions of favourable contribution to $\vec{\nabla}U$, so that the field lines tend to linger there, and the "good" contributions become more heavily weighted than the "bad".

B. Stagnation-point solutions

The use of rotational transform to improve $\int dl/B$-stability can be exploited in its purest form by causing separatrices to appear, on which $\vec{\nabla}U$ is wholly favourable. If such a separatrix bounds a set of nested closed-flux surfaces, then, just within the separatrix, $\vec{\nabla}U$ tends to be strongly favourable; while, just beyond, the field lines go to the wall. This technique can be applied to Lenard's type of multipole torus, yielding very "fat" confinement regions. An even more powerful embodiment of the technique can be found by going to a new type of configuration.

We will consider here a scalar potential

$$\chi = z - 2R_cI_1(kr)\ \sin(\theta + kz) + Ak\theta \tag{12}$$

corresponding to a uniform B_z-field on which is superimposed a helical $\ell = 1$ field and the field of a current $kA/2$ on a rigid conductor at $r = 0$. We will look for stable helical flux-tubes winding about the central conductor, but leaving it accessible for current-input and mechanical support, so that it need not "float".

For simplicity, the case $kR_c \ll 1$ will be treated here. The field-line equations integrate to give

$$\cos\psi = \frac{R^2 - R_1^2 + 2R_1R_c + 2A\log\dfrac{R}{R_1}}{2R_cR} \tag{13}$$

for a field line described by $\psi = \theta + kz$, $R = r$. The individual field lines are characterized by the parameter R_1, the smallest R for which $\psi = 0$ on a given line.

When $A = 0$, Eq.(13) describes circles in the $r-\theta$ plane, of radius R_c, and with their centres displaced from $r = 0$ when $R_1 \neq R_c$ (see Fig.4). In the $r-\psi$ co-ordinate system, the field line also describes circles,

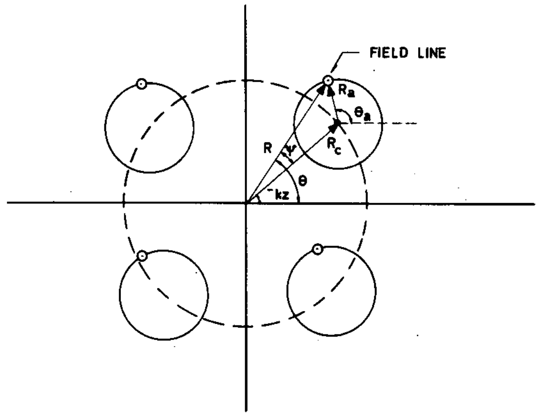

Fig. 4

Behaviour of field line R, ψ for $A = 0$.

centred on $r = R_c$, $\psi = 0$, and of radius $R_a = R_c - R_1$. To order kR_c, the circular motion in the r–ψ–system produces a line integral of $1/B_z$ that is independent of R_1, so that $\vec{\nabla} U \equiv 0$. To higher order in kR_c, one obtains cylindrical constant-U surfaces with an unfavourable outward gradient.

For $A > 0$, one obtains the field pattern sketched qualitatively in Fig. 5. Through the stagnation point at R_s passes the helical separatrix $r = R_s, \psi = 0$, which bounds a set of nested, closed flux-surfaces surrounding the helical magnetic axis $r = R_v$, $\psi = 0$. From (13) we have that

$$\gamma = -R_c \frac{d \cos \psi}{dR}\bigg|_{R_1} = \frac{R_1(R_c - R_1) - A}{R_1^2}, \tag{14}$$

and a sketch of γ as a function of R_1 is given in Fig. 5. The first null of γ defines R_s, and the second defines R_v. We have

$$R_s = \frac{R_c - (R_c^2 - 4A)^{\frac{1}{2}}}{2} \tag{15}$$

$$R_v = R_c - R_s. \tag{16}$$

Thus the restriction on the central current is $A < R_c^2/4$. For small A, giving small R_s, one can readily see from (13) that the flux-surface just within the stagnation point intercepts the $\psi = 0$ line again at $R_{s2} = 2R_c - R_s$. A three-dimensional sketch of the configuration is given in Fig. 6.

We turn now to the evaluation of U. In a torus with rotational transform, a typical field line will pass an infinite number of times around the major circumference before closing on itself, while in the present linear periodic approximation to a torus of large-aspect ratio the line simply passes from $-\infty$ to $+\infty$. To obtain a finite integral, we must take $\int dz/B_z$ over an interval in z such that along the given field line the quantity $1/B_z$ goes through a complete period. In the preceding section this was readily

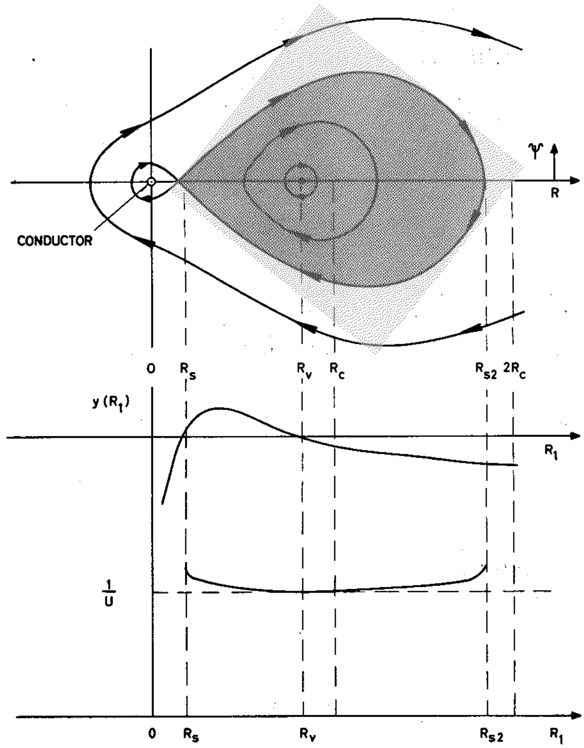

Fig. 5

Sketch of typical flux surfaces in ψ-R co-ordinates for $A > 0$.

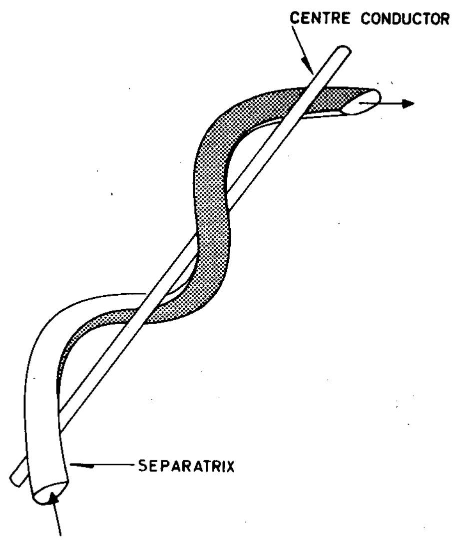

Fig. 6

Three-dimensional view of confinement region corresponding to Fig. 5.

done. In the presence of rotational transform, all the field lines on a given flux surface evidently still pass through a period of $1/B_z$ during the same z-interval; but different flux surfaces generally require different z-intervals. For example, near the separatrix the intervals tend to infinity. One therefore uses the normalized form of U,

$$U = \frac{\oint dz}{\oint_1 dz} \oint_1 \frac{dz}{B_z}, \tag{17}$$

where $\oint_1$ means the integral taken over a period in $1/B_z$ as seen along the field line, while $\oint dz$ refers to the period as seen along the z-co-ordinate (i.e., $2\pi/k$). This definition is equivalent to that given by LENARD [14]:

$$U = \lim_{N \to \infty} \frac{\int dl/B}{N},$$

where the field line is followed N times around the torus. (As will be shown below, we can also identify U, as defined in (17), with $dV/d\psi$ [15], where $V(\psi)$ is the volume enclosed by the flux surface ψ.)

For present purposes, we can write (17) as

$$U = \frac{2\pi}{k} \frac{\int_{R_1}^{R_2} dR\,(B_z dR/dz)^{-1}}{\int_{R_1}^{R_2} dR\,(dR/dz)^{-1}},$$

$$U = \frac{2\pi}{k} \left[1 + R_c k^2 \frac{\int_{R_1}^{R_2} dR\, R \tan\psi}{\int_{R_1}^{R_2} dR/\sin\psi} \right], \tag{18}$$

which can be evaluated with the aid of (13).

The calculation of U on R_v and R_s is trivial, since $\psi \equiv 0$, and one obtains

$$\frac{U_v}{U_s} = \frac{B_z(R_s)}{B_z(R_v)} = \frac{1 - R_c R_s k^2}{1 - R_c R_v k^2}, \tag{19}$$

which is maximal for small R_s, giving

$$\frac{U_v}{U_s} = 1 + (kR_c)^2. \tag{20}$$

The finite-kR_c analysis can be carried out analytically and gives a maximum of 1.23 for U_v/U_s. Inclusion of an $\ell = 2$ field can raise this ratio to 1.33.

The typical variation of U between R_s and R_v is sketched in Fig. 5. Near R_s it is helpful to write

$$U = \frac{2\pi}{k} \left[1 + R_c R_1 k^2 + \frac{\int_{R_1}^{R_2} dR(R\cos\psi - R_1)/\sin\psi}{\int_{R_1}^{R_2} dR/\sin\psi} \right] \tag{21}$$

The integral in the denominator diverges logarithmically as $R_1 \to R_s$, giving infinite positive (favourable) slope $\partial U/\partial R_1$. Near R_v one obtains

$$U = \frac{2\pi}{k}\left[1 + R_c R_v k^2 - \frac{2k^2A^2}{R_c R_v - 2A}\left(\frac{R_v - R_1}{R_v}\right)^2\right] \tag{22}$$

so that there is a maximum of U at R_v as long as $A < R_c^2/4$. In the finite-kR_c analysis, however, this maximum tends to disappear when A is small.

A second type of stable solution can be obtained for $A < 0$. The typical field-line pattern in the ψ-R system is sketched in Fig. 7. The helical flux

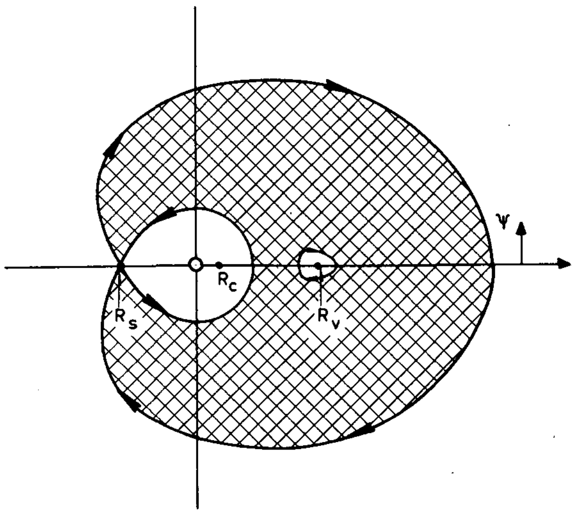

Fig. 7
Sketch of typical flux surfaces in ψ-R co-ordinates for $A < 0$.

tube in which the plasma is to be confined now tends to engulf the centre conductor; but access can be maintained through the $\psi = -\pi$ plane by stopping the plasma somewhat short of the separatrix through the stagnation point R_s. We now have

$$R_s = \frac{(R_c^2 - 4A)^{\frac{1}{2}} - R_c}{2} \tag{23}$$

$$R_v = R_c + R_s \tag{24}$$

$$\frac{U_v}{U_s} = 1 + R_c(R_c + 2R_s)k^2. \tag{25}$$

We note that small R_s (i.e., small A) is no longer the best case, and that the "well" in U^{-1} is now greater than for $A > 0$. The finite-kR_c analysis gives a maximum U_v/U_s of 1.41, for $k^2A \sim -1$; and this can be raised to 1.5 by inclusion of an $\ell = 2$ field. Near R_v we see from (22) that $A < 0$ is even better than $A > 0$. Moreover, the finite-kR_c analysis shows that the occurrence of optimum U_v/U_s for substantial values of A now permits the simultaneous maintenance of a maximum of U at R_v.

C. *Floating-ring solutions*

A number of solutions of this type are familiar by now. Two or more toroidal floating rings, carrying currents in the same direction, produce deep wells in U; a set of poloidal rings or a floating helix have a similar effect [16]. Likewise, the stellarator divertor coil produces a region of favourable $\vec{\nabla}U$, but only in the diverted part of the flux.

YOSHIKAWA [17] has pointed out that the levitron [18] can also be given favourable $\vec{\nabla}U$, essentially by the rotational transform technique: the poloidal field is weakened on the small-major-radius side of the floating ring, thus causing the (mainly toroidal) field-lines to spend most of their time where the toroidal field strength increases away from the ring. In this way, a valid solution, in the "maximum-$\langle B^{-2}\rangle$" sense, can clearly be obtained, in the limit of the small poloidal field. One can also derive a paradox: U can be written

$$\int \frac{dl}{B} = \int \frac{\rho\, d\phi}{B_\phi},$$

where ρ is the major radius and ϕ the major toroidal angle. Thus, the poloidal field strength does not directly affect U at all (it affects only the transform of the field-lines) whereas we know well that its falling off, away from the floating ring, produces an adverse gradient of $\langle B^2\rangle$. The paradox is resolved by noting that the helical field of the levitron does not allow a single-valued potential χ, so that the $\int dl/B$ criterion cannot be applied in its usual form.

III. $\int dl/B$ - STABILITY IN THE PRESENCE OF SHEAR AND FINITE RESISTIVITY

Stability according to the $\int dl/B$-criterion is loosely related to minimum-B stability [2] (cf. Equation 6), but is much less reliable, since communication between "good" and "bad" regions is required. Even for perfect conductivity, this consideration sets an upper limit on β. In what follows we will assume that β is very small, and make a general analysis of finite-resistivity effects, allowing for the presence of shear. The intention is to estimate the low-β stability properties of the configuration discussed in the preceding section.

We begin with the basic equation

$$\vec{j}_1 = \frac{\vec{B}\times\vec{\nabla}P_1}{B^2} + \sigma\frac{E_{\parallel 1}}{B}\vec{B} + \frac{\vec{B}\times\rho\,\partial\vec{v}_1/\partial t}{B^2}, \tag{26}$$

where the quantities with subscript 1 refer to a perturbation

$$\vec{E}_1 = \vec{\nabla}\phi_1 \tag{27}$$

$$\vec{v}_1 = \frac{\vec{E}_1 \times \vec{B}}{B^2} \tag{28}$$

$$\frac{\partial P_1}{\partial t} = -\vec{v}_1 \cdot \vec{\nabla} P_0 . \tag{29}$$

It is readily verified by a slight variation of the present procedure that there are no overstable modes. Let the time-dependence of the perturbation be $e^{\omega t}$, with real ω. We multiply (26) by $\vec{\nabla}\phi$ and integrate over the volume of the configuration. The left-hand side vanishes for $\vec{\nabla} \cdot \vec{j}_1 = 0$, and we obtain

$$0 = \delta W = \int d\tau \left[\left(\vec{\nabla}\phi_1 \times \vec{B} \cdot \vec{\nabla} \frac{1}{B^2} \right) \left(\frac{\vec{\nabla}\phi_1 \times \vec{B}}{B^2} \cdot \vec{\nabla} P_0 \right) + \omega\sigma \frac{(\vec{B} \cdot \vec{\nabla}\phi_1)^2}{B^2} + \omega^2 \rho \frac{(\vec{\nabla}\phi_1)^2}{B^2} \right]. \tag{30}$$

The first term on the right represents work against the plasma pressure; the second represents the work of decoupling magnetic field and fluid; the third represents the rate of increase of kinetic energy. Equation (30) sets an upper limit to ω; and since δW is stationary when ϕ_1 conforms with (26-29), the maximum ω found for any trial function ϕ_1 corresponds to a real dynamic mode.

We now introduce co-ordinates in accordance with Fig. 1. Let ψ be the flux between the magnetic axis and a given magnetic surface, which it labels. Let the magnetic potential χ be the co-ordinate along field-lines. We will restrict ourselves here to configurations where χ is single-valued. Let the third co-ordinate n be defined by

$$dn = \frac{BdS}{d\psi} = \frac{BdS}{\int BdS}, \tag{31}$$

so that n measures the fraction of the flux $d\psi$ passing between the line $n = 0$ and n. We will choose $\vec{B} \cdot \vec{\nabla} n = 0$, so that n is a flux-preserving co-ordinate. For the volume element, we have

$$d\tau = dS\; dl = \frac{d\psi\; dn\; d\chi}{B^2} . \tag{32}$$

When shear is present, the line $n = 0$ becomes distorted as one moves along χ (see Fig. 1). It is then convenient to introduce a new co-ordinate

$$n' = n - n_0(\chi, \psi),$$

where the function n_0 is such that on some chosen surface ψ_0 we have $n_0 = 0$, while on the other ψ-surfaces the points $n' = 0$ lie (for each χ-value) on the shortest line between the magnetic axis and the $n = 0$ line on the ψ_0 -surface. Since $dn' = dn$, the Jacobian in (32) is unchanged.

We now choose a perturbation of the form

$$\phi_1 = e^{\omega t} \cos 2\pi m n' \, e^{-(\psi-\psi_0)^2/2(\delta\psi)^2} \; g(\chi, n'). \tag{33}$$

(In a toroidal configuration, continuity round the major circumference then requires, strictly speaking, that we are dealing with a flux-surface ψ_0 on which a field-line passes once round the minor circumference and returns on itself after N turns round the major circumference, and that m is an integral factor of N.) Note that (33) does not describe an interchange of flux-tubes (which would be topologically impossible in a sheared toroidal field) but an interchange, about a field-line in ψ_0, of fluid tubes having the same pitch about the magnetic axis.

In (33) we take $\delta\psi$ to be small, so that the perturbation is sharply localized in ψ. The amplitude g is taken to be a relatively slowly varying function of χ and n'. When communication along the field is weak, the fastest growing mode will have g such as to concentrate the perturbation in the "bad" regions. We will neglect $(\partial g/\partial n')/g$ relative to $2\pi m$. We now have

$$\vec{\nabla}\phi_1 \times \vec{B} \cdot \vec{\nabla} \frac{1}{B^2} = B^2 \left(\frac{\partial \phi_1}{\partial n'} \frac{\partial}{\partial \psi} \frac{1}{B^2} - \frac{\partial \phi_1}{\partial \psi} \frac{\partial}{\partial n'} \frac{1}{B^2} \right). \tag{34}$$

The second term on the right is relatively unimportant, since it tends to cancel in both the n'-integration and the ψ-integration (and indeed vanishes identically for the usual interchange mode, where $g \equiv 1$).

The first term on the right in (30), which corresponds to work against the plasma pressure, can now be written

$$\delta W_p = (2\pi m)^2 \int d\psi \, dn' \, d\chi \; e^{-(\psi-\psi_0)^2/(\delta\psi)^2} \; \frac{\partial P_0}{\partial \psi} \, (\sin 2\pi m n')^2 \, g^2 \, \frac{\partial}{\partial \psi} \frac{1}{B^2}. \tag{35}$$

Because we are here assuming a single-valued magnetic potential χ (which is not always possible when there are floating conductors) we can rewrite (35) in the form

$$\delta W_P = (2\pi m)^2 \int d\psi \; e^{-(\psi-\psi_0)^2/(\delta\psi)^2} \; \frac{\partial P_0}{\partial \psi} \frac{\partial}{\partial \psi} \int dn' \, d\chi (\sin 2\pi m n')^2 \, \frac{g^2}{B^2}, \tag{36}$$

where the χ and n' integrations have been interchanged with $\partial/\partial\psi$, since the limits on the integrals do not depend on χ.

Now the zero-order quantity $1/B^2$ may have some periodicity in χ and n'; and if g varies, its periodicity will be related to this. The period of $\sin 2\pi m n'$ is determined by the quantized first-order parameter m, and will generally be incommensurate with the period of χ and g. Thus we may set $(\sin 2\pi m n')^2 \approx \frac{1}{2}$, and obtain

$$\delta W_P = 2\pi^2 m^2 \int d\psi \; e^{-(\psi-\psi_0)^2/(\delta\psi)^2} \frac{\partial P_0}{\partial\psi} \frac{\partial}{\partial\psi} \int dn' d\chi \frac{g^2}{B^2}. \tag{37}$$

For the special case $g \equiv 1$, a sufficient condition for $\delta W_P > 0$ is

$$\frac{\partial P_0}{\partial\psi} \frac{\partial}{\partial\psi} \int dn \, d\chi \frac{1}{B^2} > 0, \tag{38}$$

where we have replaced n' by n, which leaves the Jacobian unchanged. We can identify

$$\int dn \, d\chi \frac{1}{B^2} = \frac{\int dS \, dl}{\int B \, dS} = \frac{dV}{d\psi}, \tag{39}$$

where $V(\psi)$ is the volume inside flux-surface ψ. Since δW_P is the only component of δW that can be negative for $\omega > 0$, a sufficient condition for stability is then (noting $P_0 = P_0(\psi)$)

$$\frac{dP_0}{d\psi} \frac{d^2V}{d\psi^2} > 0. \tag{40}$$

We can relate (40) to (1) and (17) in the following manner. In terms of the cylindrical co-ordinates, we have for (31)

$$dn = \frac{B_z(\partial r^2/\partial\psi)d\theta}{\int B_z(\partial r^2/\partial\psi)d\theta} \tag{41}$$

in the $\chi = 0$ plane, which we take to be the $z = 0$ plane. Then (39) gives us

$$\frac{dV}{d\psi} = \frac{\int_0^{2\pi} d\theta \, [B_z \partial r^2/\partial\psi]_{z=0} \int_0^{z_T} dz/B_z}{\int_0^{2\pi} d\theta \, [B_z \partial r^2/\partial\psi]_{z=0}} \tag{42}$$

where $1/B_z$ is to be evaluated along field lines through the points θ, r, in the intersection of the $z = 0$ plane with the flux surface ψ. At the point z_T the configuration would be joined back on to itself if it were deformed into a torus. The length in z required to make $1/B_z$ go through a complete period as seen along a field line is $\oint_l dz$. An integral over this length in z will, by definition, yield the same value of $\int dz/B_z$, no matter what the starting point on the ψ-intersection with the $z = 0$ plane. Hence (42) reduces to

$$\frac{dV}{d\psi} = \frac{z_T}{\oint_l dz} \oint_l \frac{dz}{B_z},$$

which differs from (17) only by a fixed number relating z_T to the length $\oint dz$ required for a period of B_z as one moves along the z-co-ordinate (i.e., $kz_T/2\pi$).

We must now consider interchanges for which $g \not\equiv 1$; in other words, we must study the problem of communication between "good" and "bad" regions. Here we will restrict ourselves to a rough calculation. Using (33) with the assumption of a slow variation in g, letting $k_\perp$ represent the wave number in the n' direction, and with a small δ representing the "radial extent" of the perturbation, we can approximate (30) as

$$\delta W \propto \int dl \left[k_\perp^2 g^2 \vec{\nabla} P \vec{\nabla} \frac{1}{B^2} + \omega\sigma \left\{ \left(\frac{dg}{dl}\right)^2 + k_\perp^2 \delta^2 \Sigma^2 g^2 \right\} + \frac{\omega^2 \rho}{B^2} \left(k_\perp^2 + \frac{1}{\delta^2}\right) g^2 \right], \tag{43}$$

where the integral is taken in the ψ_0 surface, with arbitrary $g = g(l)$, and where Σ is the measure of shear. By $\vec{\nabla} P_0$ and $\vec{\nabla} 1/B^2$ we mean the gradients normal to ψ_0. The former is constant on ψ_0, and for the latter we will adopt a simple behaviour:

$$B^2 \vec{\nabla} \frac{1}{B^2} = \frac{1}{R_c}\left(1 + \gamma \cos \frac{2\pi l}{L}\right), \tag{44}$$

where R_c represents an effective radius of curvature, which is proportional to $d^2V/d\psi^2$ in the sense defined by (39). In what follows we will treat both the predominantly favourable case $\vec{\nabla} P_0 / R_c > 0$ and the predominantly unfavourable case $\vec{\nabla} P_0 / R_c < 0$; and we will generally be interested in $\gamma^2 > 1$, so that both "good" and "bad" regions of curvature are present simultaneously.

The choice of g that will maximize δW (i.e., will maximize ω) is governed by

$$\frac{d^2 g}{dx^2} - \frac{L^2}{\pi^2} \left[k_\perp^2 \delta^2 \Sigma^2 + \frac{1}{\omega\sigma B^2} \left\{ k_\perp^2 \vec{\nabla} P_0 \, \frac{1 + \gamma \cos 2x}{R_c} + \omega^2 \rho \left(k_\perp^2 + \frac{1}{\delta^2}\right) \right\} \right] g = 0, \tag{45}$$

where $x = \pi l / L$. Defining

$$b = -\frac{L^2}{\pi^2} \left[k_\perp^2 \delta^2 \Sigma^2 + \frac{1}{\omega\sigma B^2} \left\{ k_\perp^2 \vec{\nabla} P_0 \, \frac{1 - \gamma}{R_c} + \omega^2 \rho \left(k_\perp^2 + \frac{1}{\delta^2}\right) \right\} \right]. \tag{46}$$

$$h^2 = \frac{L^2}{\pi^2} \frac{2\gamma k_\perp^2 \vec{\nabla} P_0}{\omega\sigma B^2 R_c}, \tag{47}$$

we reduce (45) to the standard form of the Mathieu equation

$$\frac{d^2 g}{dx^2} + (b - h^2 \cos^2 x)\, g = 0. \tag{48}$$

The sign of γ in (44) depends only on the choice of $l = 0$, and should be chosen so that $h^2 > 0$. We will consider ω and $k_\perp$ to be positive.

We now solve (48) with the condition that g be regular, thus obtaining an eigenvalue condition relating b to h. This condition becomes simple in two limits:

$$2b = h^2 \text{ for } b << 1, \tag{49}$$

which corresponds to a non-localized g, and

$$b = h \text{ for } b \gg 1, \tag{50}$$

which corresponds to g localized near $x = 0, \pi, 2\pi \ldots\ldots$ There is no solution for $b < 0$. The eigenvalue conditions (49) and (50) then give us the maximum possible ω in their respective limits.

There are two δ-dependent "stabilizing terms" in (46), the shear term in Σ^2 and the inertial term in ρ. We maximize ω by the choice

$$\delta \doteq \left(\frac{\omega\rho}{\sigma B^2 k_\perp^2 \Sigma^2}\right)^{\frac{1}{4}}, \tag{51}$$

which becomes small in the high-conductivity limit. Taking, therefore, $k_\perp \delta << 1$, we reduce (46) to

$$b = -\frac{L^2}{\pi^2}\left[\frac{2k_\perp\Sigma}{B}\left(\frac{\omega\rho}{\sigma}\right)^{\frac{1}{2}} + \frac{k_\perp^2 \vec{\nabla} P_0}{\omega\sigma B^2}\ \frac{(1-\gamma)}{R_c}\right] \tag{52}$$

We have then, for $b << 1$,

$$\omega^{\frac{3}{2}} = -\frac{\vec{\nabla} P_0 / R_c}{|\vec{\nabla} P_0 / R_c|}\,\omega_0^{\frac{3}{2}}, \tag{53}$$

where

$$\omega_0 = \left(\frac{k_\perp |\vec{\nabla} P_0|}{2|R_c|\Sigma B \rho^{\frac{1}{2}} \sigma^{\frac{1}{2}}}\right)^{\frac{2}{3}} \tag{54}$$

which is the usual finite-resistivity growth rate [2].

For $b >> 1$, we have

$$\omega^{\frac{3}{2}} = -(1-\gamma)\,\frac{\vec{\nabla} P_0 / R_c}{|\vec{\nabla} P_0 / R_c|}\,\omega_0^{\frac{3}{2}} - \frac{L_0}{L}\,\omega^{\frac{1}{2}}\,\omega_0\,, \tag{55}$$

where

$$L_0 = \pi\left(\frac{\sigma B^2}{\Sigma k_\perp^2}\right)^{\frac{1}{3}}\left(\frac{2\gamma^3 R_c}{\rho \vec{\nabla} P_0}\right)^{\frac{1}{6}}. \tag{56}$$

A geometrical interpretation of L_0 can be given in terms of (51):

$$L_0 = \frac{\pi}{k_\perp \delta \Sigma} \, |\gamma|^{\frac{1}{2}} \left(\frac{\omega}{\omega_0}\right)^{\frac{1}{2}}, \tag{56a}$$

so that L_0 is roughly the distance along the field required to make two field lines on flux surfaces with separation δ diverge by a distance π/k. We note also that

$$h = 2\,|\gamma| \, \frac{L}{L_0} \left(\frac{\omega_0}{\omega}\right)^{\frac{1}{2}}. \tag{57}$$

We now distinguish between two basic types of configuration:

Type I, $\int dl/B$-stable, in the sense that the condition (40) is met. Then $\vec{\nabla} P_0 / R_c > 0$, and we select $\gamma > 0$, so as to make h^2 real. There is no small-b solution, as we have already noted in the b = 0 limit, which is $g \equiv 1$. There are two cases of large-b (localized-g) solutions:

(1) If $L \ll L_0$, then (55) yields

$$\omega \approx \left(\frac{L}{\pi}\right)^2 \frac{(\gamma - 1)^2}{2\gamma} \, \frac{k_\perp^2 \vec{\nabla} P_0}{\sigma B^2 R_c}, \tag{58}$$

which is much smaller than ω_0. The large-b (large-h) approximation is only roughly valid, as seen directly from (47).

(2) If $L >> L_0$, then

$$\omega \approx (\gamma - 1)^{\frac{2}{3}} \omega_0 . \tag{59}$$

From (57) we see that the large-b approximation holds well. In the present case, the advantage of meeting condition (40) is evidently lost.

Type II, $\int dl/B$-unstable, so that $\vec{\nabla} P_0 / R_c < 0$, and $\gamma < 0$.

(1) If $L \ll L_0$, then the highest growth-rate is achieved for $b \ll 1$, and we obtain from (53)

$$\omega = \omega_0 . \tag{60}$$

(2) If $L >> L_0$, we must go to the large-b limit of (55), and obtain

$$\omega = (1 + |\gamma|)^{\frac{2}{3}} \omega_0 . \tag{61}$$

In both cases (57) verifies the validity of the chosen b-limit (h-limit).

These results serve to document the remarks made in Section I on the relative utility of $\int dl/B$ and shear-stabilization. The condition $L < L_0$, for which $\int dl/B$-stabilization becomes advantageous, can be interpreted most conveniently in terms of (56a). We may regard $L\gamma^{\frac{1}{2}}$ and Σ^{-1} to be roughly comparable, so that the critical condition is $k_\perp \delta < 1$ (the range in which we have carried out the present analysis). From (51) and (54) we see that

$k_\perp \delta > 1$ is just the range where ω_0 equals the rate of ordinary resistive diffusion over the dimension $k_\perp^{-1}$; a range that is of little practical importance, unless as Reference [19] points out, the correlation length in the $\vec{\nabla} P$ direction turns out to be much greater than $k_\perp^{-1}$.

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EXPERIMENTS IN TOROIDAL PLASMA CONFINEMENT

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I. INTRODUCTION

From the "classical" point of view, closed magnetic confinement schemes have an advantage in that a particle must undergo many collisions before it can escape across magnetic field (unlike the case of open-ended systems, where a single collision can permit escape). The "modern" point of view on the confinement problem tends to concentrate on the danger of co-operative rather than purely collisional loss-mechanisms; however, the conclusion with respect to the advantage of closed confinement remains the same: a level of turbulence that leads to prohibitive end-losses in an open system may be relatively unimportant in a closed one.

As has been mentioned in previous papers presented at this Seminar [1, 2], the simplest closed system, a pure toroidal vacuum field, does not provide a plasma equilibrium, and must therefore be modified by the addition of rotational transform or by some deformation so as to satisfy the constant-P-constant-$\int dl/B$ condition. Historically, the major experiments have followed the first approach, because of its convenience. Three experimental routes, in particular, have been favoured; namely, (1) the pinch, where a strong plasma current provides a principal part of the confinement field; (2) the Tokomak, where the plasma current provides a field that is weak compared with the toroidal vacuum field, but retains an essential role in generating the rotational transform; and (3) the stellarator, where the vacuum field itself can provide an equilibrium.

During the past ten years these experiments have been among the principal sources of plasma-physics results in the areas of stability, diffusion theory, wave propagation, and spectroscopy, among others. Plasmas of typical density 10^{13}-10^{15} cm^{-3}, and typical temperature 10 to 100 e.V., have been produced and confined for times as great as 100 - 1000 μs. On the whole, however, these experiments cannot be considered to extend a promise of direct success as high-temperature plasma containers; and what is perhaps more important, they have contributed little definite information to the question of the feasibility of highly stable plasma confinement in closed systems.

The object of the present lecture is to state and illustrate the nature of the stability problem, and to indicate a probable direction of experimental progress.

II. THE STABILITY PROBLEM

The instabilities of closed configurations can be classified conveniently according to the "energy reservoirs" that drive them. (Here we follow the concept, though not the exact categories, of Dr. Rosenbluth's paper in these Proceedings [3].)

1. The mere fact of localization of the plasma may permit universal [3] instabilities – as in every other confinement situation.

2. Since in closed geometry the plasma cannot be placed in a perfect minimum-B position [2, 4] there is a tendency to flute instabilities. These can be counteracted by shear or $\int dl/B$-stabilization, either of which will suffice when the electrical conductivity is perfect, and neither of which may suffice in the presence of finite resistivity [2] or other effects disrupting communication along the magnetic field.

3. Anisotropic heating may create a non-thermal velocity distribution, which will support microinstabilities [3, 5].

4. A directed current along a magnetic field tends to support more serious microinstabilities [5].

5. When a substantial part of the confining magnetic field is based on plasma current, the magnetic energy content may be lowered by gross hydromagnetic modes.

The actual non-linear results of instability are generally extremely difficult to diagnose and to trace back to known linear modes. A powerful and reliable approach to diagnosis, however, is available in those experiments where the possible energy reservoirs for instability can be controlled by the experimenter. In open-ended configurations, we have the line of experimentation initiated by IOFFE [6], where the experimenter can turn on and off the energy reservoir associated in the magnetic field gradients, simply by switching between mirror and hybrid (minimum-B) geometry. At the same time the extraneous energy reservoirs associated with categories 3-5, as given above, are kept sufficiently small so that they do not obscure the configurational effects that are intended for study.

In closed configurations, no such experiment capable of clear diagnosis has been performed as yet. Inherently, toroidal pinches have every kind of energy reservoir, and exhibit a complex pattern of instabilities in practice, for which only a very partial explanation has been found in ten years of study. By largely empirical methods, certain specific regimes have been discovered that are substantially stable; but these results have not as yet furnished a useful basis for theory, or the means of extrapolating to more interesting high-temperature plasmas. Tokomaks have the great advantage of minimizing the magnetic energy reservoir based on plasma current, but they have not been able to turn off the directed current to a sufficient extent to exclude the energy reservoir for microinstabilities, and they cannot vary their basic configuration. Stellarators could in principle provide a clear-cut experiment of the Ioffe type by setting up a thermal plasma and then studying stability with shear on and off. In the past, however, the co-operative effects of the directed Ohmic-heating current have dominated the situation; unless the Ohmic heating was turned off, in which case the rapid drop of the plasma temperature has led to conductivities insufficient for the maintenance of a credible plasma equilibrium. At present the Model-C stellarator is being heated by ion-cyclotron waves instead of by directed current. Again the heating method has dominated the pattern of co-operative effects, at least in the initial experiments, but now there is a much better prospect of significant stability studies during the power-off phase. Toroidal experiments with directed current conducted at high density could ideally conform with the concept of making the electron streaming velocity negligible, at least compared with electron and ion thermal velocities. This approach to the elimination of Class-4 instabilities, however, calls for extremely high confining fields, and its total effectiveness remains both theoretically and experimentally undemonstrated.

Ioffe's experiment has been regarded as giving open-ended configurations a significant lead over closed ones, because of the topological impossibility of minimum-B in the latter case. It is far from clear, however, that shear-stabilization is an inadequate alternative to minimum-B; still less, that $\int dl/B$-stabilization is inadequate. The real lag of the toroidal configurations consists in the absence thus far of clear-cut experiments that test the theories of shear and $\int dl/B$-stabilization.

One might add parenthetically that there is some indirect evidence for $\int dl/B$-stability, coming again from open-ended experiments. As Dr. IOFFE points out [6], mirror machines with substantial neutral-gas backgrounds and hot-electron plasmas are generally very stable, presumably because of "line-tying", that is, electrical conduction along field lines to solid surfaces. The stability problem here is essentially the same as in an $\int dl/B$-stable configuration with very strong favourable properties near one point (e.g. the stagnation point in the helical example given in [2]) which corresponds to the point at which the lines are tied.

III. HARD-CORE CONFIGURATIONS

To illustrate the above remarks with concrete material, a brief account will be given of toroidal-confinement research conducted at the Lawrence Radiation Laboratory (LRL) in collaboration with S.A. Colgate, D.H. Birdsall, and C.W. Hartman.

By 1956 the deficiencies of the simple dynamic pinch had been noted and conditions for gross infinite-conductivity stability had been derived [7]. The "stabilized" pinch consists of an ordinary pinch with entrapped longitudinal B_z magnetic field, as in Fig. 1a. For stability, the radial compression ratio must be kept small (always less than 5:1), and the B_z-field must be neatly entrapped within the pinch column. Plasma heating is Ohmic, and can be thought of as the gradual extraction of the energy of the magnetic-field departure ΔB from vacuum field, on a time scale determined by the effective resistivity [8]. The hypothetical scheme for a hot-plasma container thus called for the initial creation of a sharply defined "stabilized" pinch, which was then to diffuse toward an unstable field-distribution, heating the plasma in the process to a characteristic temperature $T_c = \langle B\Delta B\rangle / 4\pi n$, where n is the plasma density. As the electron temperature rises, the diffusion process becomes correspondingly slow, permitting long confinement times.

Toroidal "stabilized"-pinch experiments [9] initiated at LRL and elsewhere in 1956/58, exhibited a gross stability behaviour fairly consistent with theory. Small-scale turbulence was evident, however, from magnetic and electric probe measurements; and while the operating times of the experi-

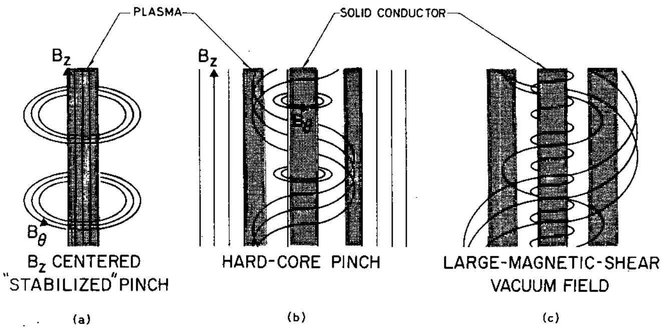

Fig. 1

(a) Field configuration of "stabilized" pinch, showing B_z-field entrapped in plasma column
(b) Hard-core pinch with tubular plasma column compressed between regions of B_z and B_θ-field
(c) Hard-core configuration confining low-β plasma tube in strongly sheared vacuum field

ments ranged up to several milliseconds, the time for leakage of the plasma-energy content was typically much shorter. At moderate values of T_c, the input power was being lost by impurity radiations. For T_c-values in the kilovolt range, the actual plasma temperature remained moderate, and anomalous particle transport across magnetic field removed the surplus energy with great effectiveness.

The cause of the difficulty remained uncertain, particularly since it was known that the infinite-conductivity condition against small-scale hydromagnetic instabilities [8] was fairly restrictive, and possibly was not being met with sufficient precision in the experiments. In order to test this point, we turned to the "hard-core pinch" (Fig. 1(b)), a configuration with a wide margin of infinite-conductivity stability, and an easily verified sufficient condition for stability:

$$\frac{d}{dr}(rB_\theta) < 0. \tag{1}$$

Linear pinch configurations were studied first [10], for reasons of convenience. In a typical operation an initial B_z-field was set up; next a plasma current was drawn between the electrodes and returned on the inside conductor (instead of on an outside return conductor, as in the usual case). An outward pinch or "unpinch" resulted, and by virtue of the pressure-balance equation

$$\frac{B_\theta}{r}\frac{d}{dr}(rB_\theta) + B_z\frac{d}{dr}B_z + 4\pi\frac{dP}{dr} = 0 \tag{2}$$

one could be sure during the rise-time of the B_θ-field that condition (1) was being satisfied.

Small-scale instability was still observed under some conditions, and we were thus able to conclude that the observed difficulty must lie outside the infinite-conductivity hydromagnetic theory. This conclusion was confirmed by the experiments of BICKERTON and co-workers [11], who suggested that finite plasma resistivity must be responsible for the discrepancy, and proposed a mode somewhat similar to the positive-column instability of KADOMTSEV and NEDOSPASOV [12]. Our own group performed some suggestive analogue experiments with sodium pinches and arrived at the theory of the "tearing mode" [13], a tearing of the tubular pinch-current layer into a helical filamentary pinch, involving a finite-resistivity mechanism similar to the "neutral-point instability" of DUNGEY [14]. The same mode was discovered independently by REBUT and co-workers under the name of "neighbouring equilibrium" [15], and has been shown by them to be conclusively the cause of the "anomalous" turbulence in hard-core pinches. Specifically, the development of the helical mode causes condition (1) to become inapplicable, and secondary infinite-conductivity instabilities then lead to the observed turbulent regime. The same course of events [16] has been predicted and observed for the so-called reverse-field theta pinches.

The complete theory of finite-resistivity hydromagnetic stability [17] contains a current-driven "rippling" mode analogous to the positive-column instability, and a plasma-pressure-driven interchange mode [2], as well as the tearing mode. The tearing mode is easily avoided [15, 17] in hard-core configurations that do not depart strongly from the vacuum magnetic field. (Specifically, the current layer cannot tear into half-wave lengths shorter than the current-layer thickness δ. There results the stability condition [17] $B_\theta/B_z \gtrsim r/2\delta$, or, with Rebut's model for the zero-order equilibrium, the condition $I_{\text{hard-core}}/I_{\text{plasma}} \gtrsim r/2\delta$. This condition has been verified experimentally by REBUT and co-workers [15]. One asks next whether the other finite-resistivity hydromagnetic modes, or the current-driven non-hydromagnetic modes, will still lead to enhanced plasma loss.

Our linear hard-core pinch experiment was unable to answer this question neatly, and we turned to the toroidal hard-core configuration, or "levitron" (Figs. 2 and 3). Here, the typical operation is to support a central ring core (a 300-lb copper conductor) on steel rods, which are withdrawn and replaced by magnetic-pneumatic-driven pistons, leaving the ring freely floating for 40 ms, during which time the plasma experiment is conducted. An initial B_θ-field is set up, and a rising B_z-field (toroidal field) then induces a transient plasma current (Fig. 4). The resultant discharge appears highly stable at first sight, but sensitive magnetic probes detect a "magnetic flutter" at a few gauss, in the 100-kilocycle range. The onset of this flutter coincides with a rapid cross-field diffusion, as detected by Langmuir probes at the wall, and is followed by a sharp rise of impurity light. The effective plasma conductivity, as measured by the rate of compression of the B_θ-field ($V_{\text{tor, vac}}$ - V_{tor} in Fig. 4) also undergoes a sharp adverse charge at the onset time of the flutter. Values of T_c as high as 4 keV are reached, while the electron temperature reaches the 100-eV mark at best. Thus the plasma-energy content is lost some 40 times during the discharge – mostly by anomalous particle transport across magnetic field; i.e. by anomalously large effective $\eta_\perp$. The $\eta_\parallel$ resistivity component, as estimated from the B_θ-field entrapment, is also somewhat low, corresponding to 20-30 eV electron temperature in a hydrogenous plasma.

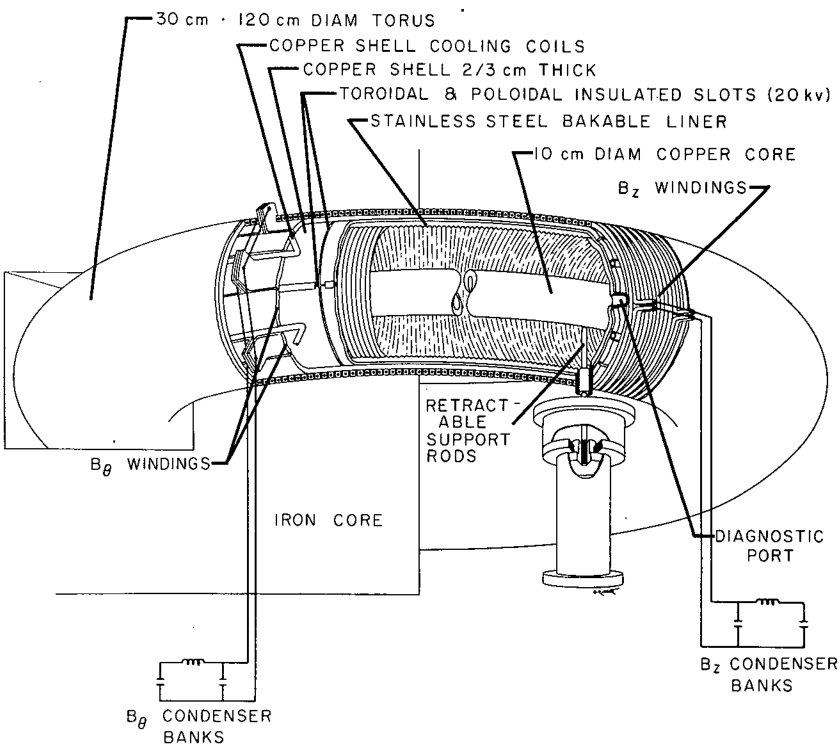

Fig. 2

Schematic of levitron

Detailed probe measurements indicate that the magnetic perturbation has an auto-correlation distance of about 5 cm across magnetic field, and a 10 times longer distance along magnetic field. At high plasma currents, the frequency of the perturbation goes up and its auto-correlation distance goes down, maintaining a "phase velocity" of about 10^6 cm/s. The radial perturbation magnetic field is generally well correlated across the thickness of the tubular current layer, which is about 5 cm. This result is rather interesting if one interprets the instability as an interchange: only a finite-resistivity interchange mode can have positive correlation of the B_r-field across the $\bar{k} \cdot \bar{B} = 0$ surface.

As usual, the diagnosis of the observed small-scale turbulence in terms of specific linear modes is difficult and uncertain. Aside from finite-resistivity hydromagnetic modes, one could appeal to Kadomtsev's direct-current-driven form of the universal instability, and also to various infinite-medium-type microinstabilities. Fortunately we have been able at least to establish that the directed ($j_{\parallel}$) current in the discharge is the cause of the observed instability.

Figure 5 shows a type of operation where B_θ and B_z rise slowly and simultaneously, so as to minimize the induced directed current. A weak Ohmic-heated discharge takes place, which ceases near the maximum field

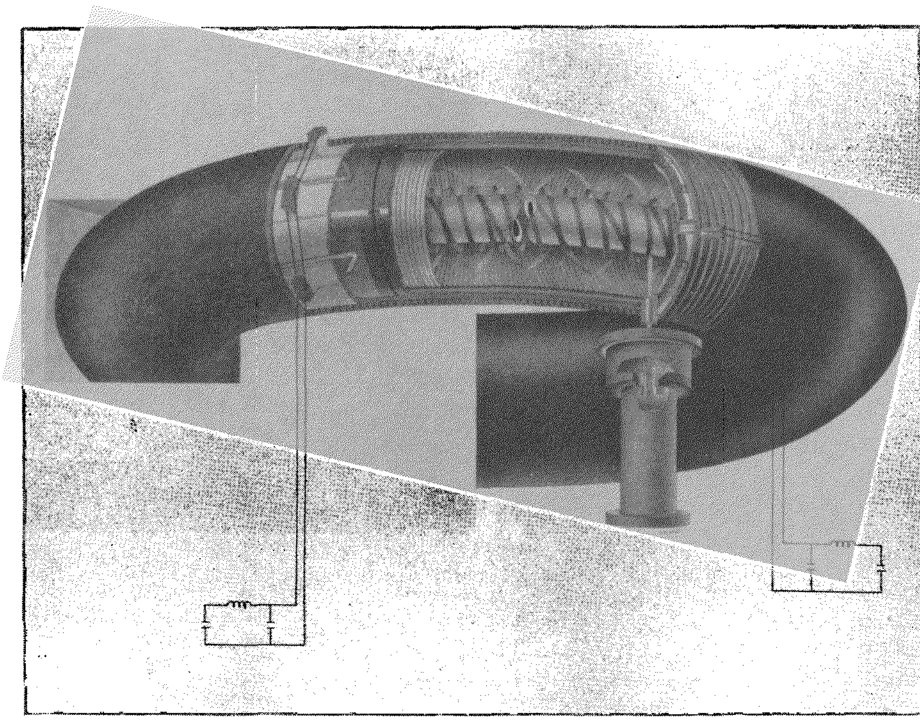

Fig. 3

Field-configuration of levitron

(near the null in induced electric field). At this time, an RF 5-megacycle heating current is induced, reheating the plasma, but failing to rekindle the earlier instability – which resumes only during the decline of the magnetic field, when a directed current is again induced. In the case of Fig. 5 the RF was applied somewhat crudely by two conductors contacting the ring core at opposite ends. The same effect has been obtained with loops encircling the minor circumference.

Another point on which good evidence has been obtained concerns the manner of the plasma-energy leakage. Linear-pinch experiments [18] with an end-injected electron beam and a fluorescent-screen detector have shown that in turbulent "stabilized" and hard-core pinches the electrons leak readily from one flux-surface to another, by following the "tangled" magnetic field lines that are evidenced by the B_r-flutter. That this process occurs readily in the levitron is confirmed by the rise of a positive plasma potential (ϕ_3 in Fig. 4) after the onset of the flutter, presumably owing to electron leakage. If one attempts to alter the plasma potential by means of a biased filament immersed in the plasma, hundreds of amperes can be drawn between plasma and liner, indicating a low-impedance path for the electrons. When this experiment is performed in conjunction with the RF-experiment of Fig. 5, the anomalously low impedance is seen to be correlated with the B_r-flutter.

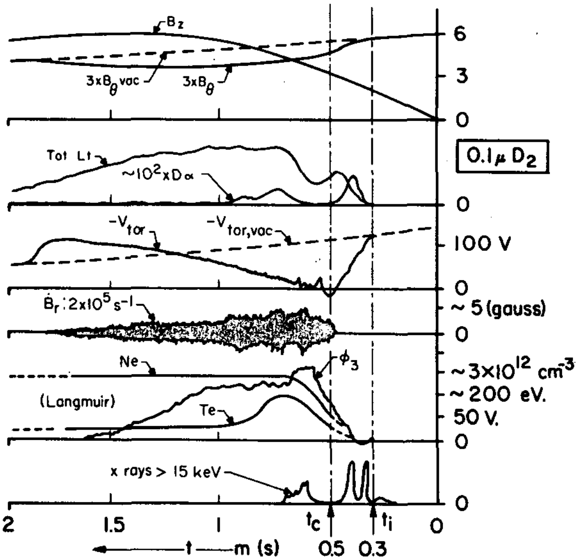

Fig. 4

Typical Ohmic-heating operation in levitron, showing onset of induced current at 300 μs and onset of instability at 500 μs. B_θ and B_z fields are measured at tube wall.

The disappearance of the B_r-flutter during RF-heating is thus of considerable diagnostic value, but the achievement of altogether stable conditions remains in doubt. Preliminary Langmuir-probe studies indicate that with RF (5-megacycle) heating the plasma diffusion rate is in some cases even more severe than with Ohmic heating. At the present time, a 10 000-megacycle heating source is being installed, which may provide results that can be more readily diagnosed.

IV. CONCLUSION

The principal problem in the experimental study of stable toroidal confinement has been the difficulty of reliable diagnosis, in the sense of the inability to interpret co-operative phenomena reliably and to prove points about the basic feasibility of the theoretical configurational stabilization techniques. The effects of the plasma-heating method have invariably obscured the points of more basic interest, as well as remaining obscure on their own account. Much progress has been made in understanding and eliminating the gross hydromagnetic instabilities that occur when confinement is based to a considerable extent on the self-field of plasma currents, but the non-vacuum-field containment configurations so arrived at are of secondary interest, since they appear neither sufficient nor necessary to obtain a hot plasma with a high degree of stability.

If toroidal confinement experiments are to attain a level of clarity and significance similar to that which the open-ended experiments of Ioffe and

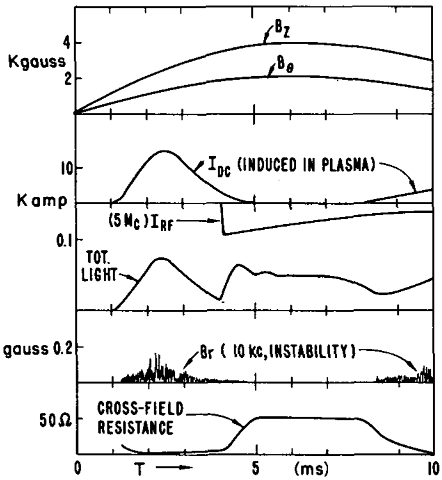

Fig. 5

Weak Ohmic heating, supplemented by RF-heating, at 0.01 μ D_2

others have achieved, it seems probable that similar inherent advantages will have to be built into future toroidal apparatus: namely, we will require a non-disruptive method of plasma generation, and we will have to allow flexibility for configurational changes that test the stability theory in a clear-cut way.

Two promising methods for highly quiescent generation of plasma appear at the moment to be electron-cyclotron heating and injection of a low-energy neutral atom beam. There would be some advantage in using these methods in conjunction. As for configurational flexibility, the stellarator and levitron are well suited to test the question of shear versus no-shear. Whether the addition of $\int dl/B$-stabilization is valuable, can be tested in a stellarator modified by application of the periodic-multipole concept, or in a helical-equilibrium configuration, as described in [2].

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MIRROR TRAPS

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I. GENERAL INFORMATION ON MIRROR TRAPS

The various magnetic configurations suggested as magnetic traps for plasma confinement can be divided into two classes of configurations. To one of them belong systems where plasma fills a certain toroidal region bounded by a toroidal magnetic surface. The best known systems of this type are the stellarator and levitron. Another class covers open-end-systems, that is to say, systems in which plasma fills a section of space which is limited along the lines of force of the magnetic field. In the first place, it pertains to simple adiabatic traps with magnetic mirrors and picket fence traps with opposite fields, as well as more complex modifications of these systems. We shall consider below solely magnetic mirror configurations.

The concept of such traps was formulated independently by Budgker (USSR) and York and Post (USA). The basic physical principle is that charged particles placed in a longitudinal axially-symmetric magnetic field, bounded at both ends by sections with a stronger field, can be reflected from these sections of stronger field when moving along the lines of force. Stronger field regions have therefore been called magnetic mirrors. The reflection from the mirrors is connected with the adiabatic invariance of the charged particles' magnetic moment $\mu = W_{\perp}/H$.

From constancy of μ it follows that only those particles can be reflected for which the angle (α) between the velocity direction and the magnetic field line of force is determined by the relation

$$\sin \alpha_0 > \left(\frac{H_0}{H_m}\right)^{\frac{1}{2}},$$

where H_0 is the field strength between mirrors and H_m is the field in the mirror. The value H_m/H_0 is called mirror ratio R, so that the reflection condition is:

$$\sin \alpha_0 > (R)^{-\frac{1}{2}}.$$

Thus, if particles with isotropic velocity distribution were placed, at a given moment of time, in the region between the mirrors then, after an interval of time equal to the time of flight between mirrors, no particles, the phase points of which are enclosed in a $2\alpha_0$-angle cone in the velocity space, would remain in the trap. Such a cone in the velocity space is called a loss cone.

However, those particles, the phase points of which fall outside this cone, will not always move in the trap since there are two obvious sources of losses. One of them depends on the accuracy to which the magnetic moment $\bar{\mu}$ is conserved during multiple reflection of the particles from the mirrors. It is known that conditions of adiabatic movement are fulfilled better the lower the changes in the magnetic field encountered by a particle (in space or in time) per one Larmor revolution, that is to say, the stronger the fulfilment of inequalities:

$$\left|\frac{H}{\nabla H}\right| \gg \rho_L ; \quad \left|\frac{1}{H}\frac{\partial H}{\partial t}\right| \gg \omega_L .$$

If the non-homogeneity of the field in the mirrors is not sufficiently small then, after a number of reflections, the magnetic moment may change to such a degree that the conditions for reflection will not be satisfied any longer and the particle will escape from the trap.

The problem of the accuracy of the adiabatic invariant has been investigated theoretically and experimentally. The clearest results were obtained in Gibson's et al. experiments which showed that, with the adiabatic parameter

$$\epsilon = \rho\left|\frac{\nabla H}{H}\right| < 0.05,$$

the particles may have 10^{10} oscillations between the mirrors. This means that one can ensure an extremely long confinement of the particles (measured for example in minutes), i.e. one can practically fully eliminate losses due to non-adiabatic conditions.

Another more important loss mechanism is connected with particle scattering during intercollisions. Due to scattering, particles may fall into the prohibited cone and escape from the trap. Since we are dealing with strongly ionized plasma, collisions between charged particles play an essential role. In this case, due to the Coulomb interaction, scattering is defined not by close but by remote collisions with small angle deviations.

The problem of particle losses due to Coulomb collisions was studied by a number of authors (Budgker, Rosenbluth, Judd et al.) and the main results are as follows. The ion flux flowing from a unit plasma volume into the loss cone, in the case of hydrogen plasma, is equal to

$$I = K\frac{n^2 e^4 Lk}{\sqrt{m_i}\; T_i^{3/2}} \frac{1}{\ln R}, \tag{1}$$

where K is numerical coefficient of the order of a unit depending on the shape of the magnetic field and the initial distribution of ions over the angles during injection and Lk is the Coulomb logarithm.

In this formula it is assumed that the mirror ratio R is sufficiently high. According to (1) the mean ion scattering time with incidence in the prohibited cone is defined by the following expression:

$$\tau_{ii} = \frac{1}{K} \frac{\sqrt{m_i}\, T_i^{3/2}}{n e^4 L k} \ln R. \tag{2}$$

With an accuracy up to $1/k$, τ_{ii} differs from the known expression for the angular relaxation time merely by the factor ln R. Hence it appears that the magnitude of the mirror ratio R has a very weak effect on the loss velocity.

The unavoidable character of the Coulomb losses is an essential drawback of mirror traps and the only means of reducing these losses is to increase the ion temperature. We shall give a numerical example to illustrate the case. At a deuterium plasma density of $n = 10^{14}$ cm^{-3} and ion temperature $T_i = 30$ keV, the mean ion life time before scattering and incidence in the loss cone (τ_{ii}) is $\simeq 0.1$ s.

Electron losses are defined by the same relations (1 and 2) by substituting $\sqrt{m_i}\, T_i^{3/2}$ for $\sqrt{m_e}\, T_e^{3/2}$. As in normal scattering

$$\tau_{ee} \simeq \tau_{ei} = \left(\frac{m_e}{m_i}\right)^{1/2} \left(\frac{T_e}{T_i}\right)^{3/2} \tau_{ii}\,. \tag{3}$$

From equation (3) one may see that the rate of losses conditioned by the Coulomb scattering, generally speaking, is not the same for ions and electrons. At $T_e/T_i > (m_i/m_e)^{1/3}$ the ion flux prevails over the electron flux. With a different inequality sign, the electron flux, on the contrary, exceeds the ion flux.

On the other hand, in the steady state, both fluxes must be equal. That is why an equilibrium is reached due to the formation of an ambipolar electric field in the plasma, which prevents the escape of particles with a smaller τ. At the same time such a field accelerates particles of the opposite sign in the direction of the mirrors and thus confinement is affected.

In this connection, let us consider in more detail the effect of the electric field on the confining properties of the magnetic mirror. Let us assume that $T_e/T_i < (m_i/m_e)^{1/3}$ so that positive ions are accelerated in the direction of the mirrors. The change of the longitudinal ion velocity component $v_\parallel$ under the effect of the electric field E is defined by:

$$m_i \frac{dv_{\parallel i}}{dt} = e E_\parallel - \mu \frac{\partial H}{\partial l}, \tag{4}$$

where $E_\parallel$ is the projection E on the direction of the magnetic line of force. After integration of (4) along the line of force from the minimum field point up to the maximum field point in the mirror H_m we obtain

$$W_{\parallel i} - W_{0i} \cos^2\alpha_0 = e(\varphi_0 - \varphi_m) - (R - 1)\, W_{0i} \sin^2\alpha_0, \tag{5}$$

where $(\varphi_0 - \varphi_m)$ is the difference of the ambipolar field potentials along the magnetic mirror. Introducing into (5) that $W_{\parallel i} = 0$, we find that in the presence of an electric field the reflection condition is:

$$\sin \alpha_0 > \frac{1}{\sqrt{R}}\left(1 + \frac{e\,(\varphi_0 - \varphi_m)}{W_{0i}}\right)^{\frac{1}{2}}. \tag{6}$$

The potential difference regulating the electron flux from the trap is determined by the electron temperature

$$e\,(\varphi_0 - \varphi_m) = \gamma\, T_e\,. \tag{6'}$$

(γ is a numerical coefficient of the order of unity). Substituting T_i for W_{0i} one can rewrite (6) in the form:

$$\sin \alpha_0 > \frac{1}{\sqrt{R}}\,(1 + \gamma\, T_e/T_i)^{\frac{1}{2}}. \tag{6''}$$

From the comparison of (6) or (6") with (1), it follows that the longitudinal electric field increases the angle of the loss cone. This new cone can be compared with the effective mirror ratio R_{eff} which is connected with the real mirror ratio R by:

$$R_{eff} = \frac{R}{1 + \gamma \frac{T_e}{T_i}}\,. \tag{7}$$

As may easily be seen, the ambipolar field is the greatest danger for the confinement of plasma in the case when T_e approaches T_i and R exceeds only slightly the value of unity.

This gives in brief the essential features of the confining properties of mirror traps if they are considered from the point of view of the kinematics of the individual charged particles and pair collisions.

II. PLASMA INSTABILITIES IN MIRROR TRAPS

The main problem, however, for plasma confinement in mirror traps is plasma stability, or rather instability, and their influence on particle confinement. A survey of various instabilities occurring in plasma was given in a number of other papers. It only remains therefore to recall the main results concerning mirror effects.

Various plasma instabilities in a magnetic field are generally differentiated by their physical natures into three types: hydromagnetic, kinetic and drift instabilities. In a magnetic field of the usual mirror trap the hydromagnetic interchange (flute) instability is revealed most clearly. Its elementary mechanism is connected with an opposite drift of ions and electrons in a radially decreasing magnetic field. Due to this instability any plasma can move radially across the magnetic field.

Flute instability starts developing at very low plasma densities. According to Kadomtsev, the instability criterion has the form:

$$\lambda_{D\,i} < (a\,R)^{\frac{1}{2}},$$

where $\lambda_{D\,i}$ is the Debye ion radius, i.e. $\lambda_{D\,i} = (T_i/4\pi e^2 n)^{\frac{1}{2}}$, a is the transverse plasma scale, and R is the mean curvature radius of the magnetic lines.

On account of this instability, plasma should be ejected to the walls, with a velocity of the order of the ion thermal velocity, in the form of separate "tongues" extending along the field lines. The flute instability is the crudest form of plasma instabilities in mirror traps.

Subsequently ROSENBLUTH et al. modified the initial flute instability theory, taking into account the effect of the ion finite Larmor radius. In this case, they found that at sufficiently high plasma density ($H^2/4\pi n\, m_i c^2 \ll 1$) the higher modes are stabilized on account of the difference of the $\vec{E} \times \vec{H}$ drifts for ions and electrons. The stability criterion has the form

$$\left(\frac{\rho_i}{a}\right)^2 > \frac{M}{(M-1)^2}\,\frac{a}{R}, \tag{8}$$

where M is the perturbation harmonic number.

Mikhailovski generalized these results to the case of lower densities. He showed that the stabilizing effect vanishes at that plasma density when the ion Debye radius becomes comparable to the Larmor ion radius.

Among the instabilities of the kinetic type, two instabilities conditioned by the anisotropic particle distribution in the velocity space are directly connected with mirror traps. The first of them is the so-called mirror or diamagnetic instability. The very term explains up to a certain point the nature of its origin. If there is an initial density perturbation δn then, due to plasma diamagnetism, the magnetic field where perturbed is either weakened (at $\delta n > 0$), or strengthened (at $\delta n < 0$). In other words, any such perturbation gives rise to local "valleys" in the field or to local "hills" – i.e. "mirrors". With anisotropic particle distribution the initial perturbation of the magnetic field will increase in time due to capture (or, on the contrary, due to the ejection) of new particles which leads to the development of instability. The criterion of mirror instability is given by the inequality:

$$\frac{T_{\parallel}}{T_{\perp}} < \beta',$$

where $T_{\parallel}$ and $T_{\perp}$ are the particle energies parallel and perpendicular to the field and

$$\beta = \frac{n\,(T_i + T_e)}{H^2/8\pi}.$$

Another kinetic instability – the Harris instability – (or the ion-cyclotron instability) has a resonance nature. It arises from the resonance coupling between the ion Larmor rotation and the longitudinal electron plasma oscillations. The minimum plasma density at which unstable electron oscillations may start building up is defined by the obvious condition:

$$\omega_{pe} > \omega_{Hi} ,$$

where ω_{pe} is the electron Langmuir frequency and ω_{Hi} is the ion-cyclotron frequency. Actually it corresponds to densities of approximately $10^7\,cm^{-3}$. Moreover the building up of instability requires a sufficiently high velocity anisotropy which, in its turn, depends on the ratio T_e/T_i. When T_e/T_i increases, the instability covers less anisotropic distributions up to

$$\frac{T_{\perp i}}{T_{\parallel i}} \simeq 2 .$$

Recently Mikhailovski developed a theory of a high-frequency drift instability of non-homogeneous plasma which, under certain conditions, can become a potential threat for mirror traps. The physical meaning of this instability is the generation of unstable drift waves in the frequency range near the ion-cyclotron frequency ω_{Hi}, $2\omega_{Hi}$, ... The criterion of this instability has the form:

$$\left(\frac{\rho_i}{a}\right)^2 > 4\,n\left[\left(\frac{C_A}{c}\right)^2 + \frac{m_e}{m_i}\right], \tag{9}$$

where C_A is the Alfvén velocity, c is the velocity of light, n the number of the cyclotron harmonic. It should be noted that the condition of stabilizing the flute instability with the finite Larmor radius, Eq. (8), may in certain cases be incompatible with criterion (9).

At the end of this brief survey of main instabilities which may occur in mirror traps, it should be pointed out that theory in many cases merely indicates the existence of certain instabilities, but does not make any definite quantitative conclusions on their actual effect on plasma confinement.

At the same time one cannot but express satisfaction with the fact that the rapid stage of theory development, with a stream of new discoveries of instabilities, is drawing, I think, to its close. It is to be hoped that theory in this connection will make no new discoveries and that it will be possible now to elucidate experimentally the degree of danger presented by each of the predicted instabilities.

Extensive experimental data have been collected to date with regard to plasma stability in mirror traps. Research work in this field is carried out in a number of laboratories using various experimental devices varying in their geometrical dimensions, strength of magnetic field, techniques of plasma formation and plasma parameters (see Table I).

The great variety of experimental conditions gave rise in the beginning to nearly the same variety of contradictory results concerning plasma stability in a magnetic field of a mirror configuration. According to theory,

TABLE I

MAIN CHARACTERISTICS OF EXPERIMENTAL DEVICES WITH ORDINARY MIRRORS

Device	Method of injection and heating	H_0 (kG)	R	Chamber diam. (cm)	Distance between the mirrors (cm)	T_e (keV)	T_i (keV)	Injection current (mA)	n	τ (s)	Observed *instabilities*
OGRA I (USSR	Dissociation on neutral gases	5	1.4	140	1200	-	80	150	$\sim 10^{8}$	$\sim 10^{-3}$	Flute, ion-cyclotron
DCX-II (USA)	Dissociation on lithium arc	12	3.3	100	300	-	300	40	$\sim 10^{10}$	0.03to 0.3	Ion-cyclotron
ALICE (USA)	Lorentz ionization of neutral atoms	30	1.4	?	30	-	20	40	$\sim 10^{8}$	0.6	Flute M=1 ion-cyclotron
PHOENIX (UK)	Lorentz ionization of neutral atoms	40	?	?	30	-	30	8	3×10^{8}	?	Flute M=1 ion-cyclotron
Table Top (USA)	Single-stage adiabatic compression	17	1.5	20	35	10-25 ?	? 3		10^{7} -10"		Flute M=1
Toy-Top (USA)	Multi-stage	17	1.3	23	35	0.100	5		$\sim 10^{13}$	$\sim 10^{-4}$	Flute M=1
Ion magnetron (USSR)	Radial electric field	8	1.5	50	140	0.020	1.5		$\sim 10^{9}$	$\sim 10^{-4}$	Flute high modes
Alexeev-Nidigh (USA)	Electron beam instability	1.5	3	?	?	130	?		10"	0.07	Stable

TABLE I (cont'd)

Device	Method of injection and heating	H_0 (kG)	R	Chamber diam. (cm)	Distance between the mirrors (cm)	T_e (keV)	T_i (keV)	Injection current (mA)	n	τ (s)	Observed instabilities
Ard-Baker, Dandal (USA)	Electron-cyclotron resonance	3.8	?	?	?	120	?		5 × 10"	0.1	Stable
Zavoysky et al. (USSR)	"Turbulent" heating + adiabatic compression	1+9	1.5	5	100	3 to 30	0.3+3		$10" + 10^{12}$	$> 10^{-3}$	Stable
Nestezikin et al. (USSR)	Shock-heating	1-2	1.5	20	150	5	10		$\sim 5 \times 10^{12}$	$\sim 50 \times 10^{-6}$	?

flute instability should develop in such a type of field, which induces a rapid plasma ejection to the walls across the magnetic field. However, although this instability appeared distinctly in some of the experiments, it was not revealed at all in a number of others.

Without proceeding into a detailed analysis of the peculiar features of each of these devices, we shall only mention that the essential difference between "stable" and "non-stable" devices is the difference in vacuum conditions. All "non-stable" devices are those in which the highest possible vacuum during plasma injection is maintained, so that during operation the neutral gas pressure as a rule does not exceed 10^{-7} torr. In "stable" installations the initial vacuum (before injection) is usually 10^{-6} to 10^{-5} torr and, what is much more important, during injection pressure falls sharply without any means of control.

In connection with this, a number of authors (Post, Rosenbluth, Velikhov) suggested the hypothesis that stable confinement regimes are reached in those cases when around the "hot" plasma confined between mirrors a sufficiently dense cold plasma is present due to the ionization of the neutral gas. The cold plasma, being in contact with the conducting walls (in the region beyond the mirrors outside the trap), ensures good conductivity along the magnetic field lines and, therefore, prevents the formation of polarized electric fields in the "hot" plasma. The recent series of investigations carried out in Livermore with adiabatic plasma heating devices have proved convincingly the validity of this hypothesis. These experiments have directly proved the influence of vacuum conditions on plasma stability with regard to the lower modes of flute perturbations.

To the family of "non-stable" experiments were added recently installations with plasma accumulation by injection of fast neutral beams. In these experiments with a simple geometry a flute type hydromagnetic instability has also been observed. This instability limits the plasma density accumulation.

Thus, although specific features of the phenomena accompanying flute instability are not described in the initial hydromagnetic theory, a qualitative agreement between theory and experiment does exist.

As to the other kinds of instabilities mentioned earlier, the ion-cyclotron instability (Harris) has been experimentally discovered in a number of instalations. Its investigation has not yet developed to the extent which would enable us to draw a definite conclusion on the degree of its negative effect on the confining properties of the trap.

III. METHODS OF INJECTING HIGH TEMPERATURE PLASMA INTO TRAPS

Many different methods of injecting plasma into mirror traps have been proposed and are being developed at present.

Since these traps have open ends there is scope for various inventions and suggestions, in contrast to closed traps where the possibilities of plasma injection are much more limited.

We shall give a brief survey of the existing methods, sub-dividing them into three groups: (a) methods of external injection of fast particles; (b) adiabatic magnetic compression and heating of the already formed cold plasma; (c) methods of cold plasma heating based on the use of external high frequency electromagnetic fields and beam instabilities.

(a) The main problem in the generation of high temperature plasma through fast particle injection into the trap is the development of efficient methods of capturing the injected particles. Since the particle is introduced into the magnetic field from outside, it can remain in the space between the mirrors on the sole condition that during its movement a sharp (non-adiabatic) change would occur of those parameters which characterize its trajectory. This means that either the particle properties (velocity, mass or charge) or the magnetic field where the movement takes place will change.

Those methods of capture which are based on abrupt change of the injected particles' mass or charge have found the most extensive application. In some cases beams of accelerated hydrogen molecular ions are used. Molecular ions are dissociated and atomic ions are formed through collision, either with residual gas particles or with the cold plasma specially injected for this purpose. In other cases, beams of fast neutral atoms are used which convert partly into ions in the magnetic field, due to the so-called Lorentz ionization, and also due to collisions with particles located in the trap.

Independently of the injection method applied, the energy of the injected particles should be equal to some tens or hundreds of keV if the cross-sections of the dissociation and ionization processes are not to be too small.

Let us consider as an example, the accumulation of plasma in the case of molecular ion injection. (A large part of the results obtained can also be applied to the injection of neutral atom beams.)

The accumulation process starts with the ion dissociation on the residual gas or on the specially generated plasma column. With the increase of the density of the captured atomic ions, the dissociation starts developing also on the accumulated particles of the plasma.

Therefore the density of the atomic ions would grow almost exponentially if there were no losses of the ions. Actually the main source of losses during the initial stage is the charge exchange on neutral gas, which limits density growth.

On the other hand, at a sufficiently high density of charged particles, the neutral gas density can decrease due to ionization by ions and electrons of the accumulated plasma. This last process, called "burn-up", plays a determining role in the possibility of obtaining high plasma densities.

In order to elucidate the relative value of each of these competing processes, let us consider the simplest equation of the accumulation process during the dissociation on the residual gas.

Let I be the injected current of molecular ions, L be the overall length of the path of the molecular ions in the trap until they come back to the injector, Ω be the plasma volume, n_i, n_0 be the ion and neutral gas densities, σ_d be molecular ion dissociation cross-section, σ_c the atomic ion charge-exchange cross-section and v_i the ion velocity; then the initial stage of accumulation will be described by the following equations:

$$\frac{dn_i}{dt} = \frac{IL}{\Omega} n_0 \sigma_d - n_i n_0 \sigma_c v_i . \tag{10}$$

If the injected current I is so low that n_0 remains practically unchanged during the accumulation process, then the obtained equilibrium concentration n_i is equal to:

$$n_i = \frac{IL}{\Omega v_i} \frac{\sigma_d}{\sigma_c} . \tag{10'}$$

With the increase of I, n_i increases; when n_i becomes, to within an order of magnitude, comparable to n_0, then the molecular ion dissociation in the gas is supplemented by the dissociation on atomic ions.

That is why the equilibrium concentration of atomic ions will exceed the value defined in Eq.(10'). In other words, with the increase of the injected current I, n_i will grow more rapidly than I, and at a certain critical value of the current I_c, a sharp change will occurr in the regime. Equilibrium can no longer be maintained and n_i will increase in time until other processes start intervening, limiting the value of n_i, for example the Coulomb scattering in the loss cone. As a result plasma concentration will increase very strongly and the neutral gas in the plasma region will be fully ionized.

The critical current I_c depends in a complex way on a number of technical and physical factors characterizing the vacuum system of the experimental device. For a rough evaluation of this current one may use the following conditions:

$$n_i \sigma_d L \simeq 1 .$$

The meaning of this condition is as follows. Due to dissociation of the molecular ion beam on the atomic ions accumulated in the volume, the intensity of the beam coming back to the injector begins to decrease sharply. This leads to a decrease of n_0 and to a corresponding decrease of charge exchange losses.

Introducing n_i from Eq.(10) in Eq.(11) we obtain the following expression for the I_c value.

$$I_c \simeq \frac{\sigma_c}{\sigma_d^2} \frac{\Omega v_i}{L^2} .$$

The plasma accumulation process during molecular ion dissociation was analysed in detail in a series of studies by Symon, Golovin et al. They showed that, depending on experimental conditions, the I_c value can change within a wide range. For the OGRA device $I_c \simeq 1$ A at $E_i \simeq 200$ keV, $L = 10^5$ cm and $\Omega = 10^7$ cm^3.

(b) Another frequently used method of filling the trap with high temperature plasma consists in injecting not single particles but plasma clusters from outside. The plasma is captured and subsequently heated by a magnetic field slowly (adiabatically), increasing with time. The heating of plasma by this method is based on the acceleration of charged particles in electric fields, induced by the time varying magnetic field.

There are two ways of heating plasma adiabatically, by radial compression and by longitudinal compression.

Plasma is heated by radial compression. Due to the conservation of the adiabatic invariant connected with the transverse motion of particles $\mu = W_\perp / H$ = constant, we have:

$$W_\perp(t) = \frac{H(t)}{H(0)} W_\perp(0) = \alpha W_\perp(0).$$

The change of the plasma radius is inversely proportional to $\sqrt{\alpha}$, so that plasma density is $n(t) = \alpha n(0)$.

Plasma is heated by converging mirrors (longitudinal compression). In this case the transverse energy remains unchanged and the longitudinal energy increases according to the conservation of the other adiabatic invariant: $\oint p_\parallel \, dl$, connected with the motion of particles along the magnetic lines.

The kinetic energy of the longitudinal motion increases in inverse proportion to the square of the distance between the mirrors.

$$W_\parallel(t) = \left[\frac{L(0)}{L(t)}\right]^2 W_H (0) = k^2 W \ (0).$$

Particle density increases in proportion to the first power of this distance:

$$n(t) = k n(0) .$$

At simultaneous longitudinal and radial compression

$$n(t) = \alpha k n(0) .$$

(c) Apart from the methods of impeding fast charged and neutral particles and the method of adiabatic heating of plasma, the method of filling traps with high temperature plasma has been recently developing on the basis of various types of instabilities. In particular, the so-called turbulent heating method is being successfully developed in the Soviet Union although, as mentioned by Sagdeev, the accuracy of the term "turbulent heating" is not yet clear. These experiments are described in detail in Sagdeev's paper (these Proceedings). Successful experiments are being carried out in Oak Ridge (USA) for obtaining high temperature electron plasma using energetic electron beams. Besides these methods, plasma heating methods are being developed with ion-cyclotron waves where electromagnetic fields are used with a frequency close to the ion-cyclotron frequency. In other cases super-high-frequency electromagnetic fields are being used with frequencies close to the electron-cyclotron frequency. In this last case too, one obtains high electron temperature plasma.

This does not seem to exhaust the list of possible methods for obtaining high temperature plasma in traps, since there is no limit to the ingenuity of the human mind. In particular, experiments are being started at present in a number of laboratories for obtaining hot plasma in traps by heating the initially solid hydrogen with an intense laser beam, as was proposed by Dawson and Engelhardt.

IV. THE HYBRID MAGNETIC FIELD EXPERIMENTS

Before coming to the results of the hybrid magnetic field experiments, I should like to talk about the history of these experiments. When we started investigating plasma behaviour in an ordinary mirror trap in 1958, we soon faced the difficulty of poor confinement. Plasma lifetime was of the order of a few hundred microseconds for a plasma density of about $10^9\ cm^{-3}$, $T_e \simeq 20$ eV, $T_i = 1.5$ keV. This fact could not be explained either by charge exchange losses, nor by Coulomb scattering. A whole series of measurements proved that these losses are conditioned by plasma effects. The greatest part of the plasma loss is due to plasma transfer across the magnetic field to the side-walls and not due to particle escape through the mirrors.

Subsequently, it was found that plasma reaches the walls in the form of separate long "tongues" or plasma tubes which extend along the whole length of the trap. This pointed clearly to the fact that there was an instability phenomenon externally similar to flute instability. We had, however, to provide an answer to the following question: if we truly have a case of flute instability, why is plasma lifetime so long? Indeed, according to the early Rosenbluth – Longmire theory, plasma lifetime would be defined in order of magnitude by the ion motion time towards the walls and the ion thermal velocities. Under our experimental conditions, plasma lifetime should have been of the order of some parts of a microsecond or, at least, of some microseconds, but not hundreds of microseconds.

This question was answered by Kadomtsev later. He studied the problem of losses on the basis of a model of turbulent convection in rarefied plasma.

According to this model, an intense mixing of the inner and outer regions, similar to the convection of a non-uniformly heated liquid in a field of gravity, occurs if the plasma is placed in a radially decreasing magnetic field and the vacuum chamber walls are metallic. Due to this mixing, there should be an almost uniform distribution of plasma over the entire trap cross-section up to the layer adjacent to the wall, where practically the whole density decrease is concentrated. The thickness of that layer is of the order of the ion Larmor radius. Each contact with the wall causes the plasma tongue to lose only a part of the particles contained and, therefore, the mean loss rate is significantly lower than when there are no conducting walls. The theory of turbulent convection of rarefied plasma, leads to the following expression for plasma confinement time in the trap:

$$\tau \simeq Ca\left[\left(1 + \frac{\omega_H^2}{\omega_0^2}\right)\frac{R_0 M}{\rho_i T_i}\right]^{\frac{1}{2}}, \qquad (11)$$

where C is the factor depending on that part of particles lost by the plasma tube at contact with the wall, a is the trap radius, ω_H and ω_0 the Larmor and Langmuir ion frequencies, R_0 the mean curvature radius of the lines adjacent to the wall, and M and T_i ion mass and temperature.

According to the turbulent convection model, one should expect that the transverse dimensions of the plasma tongues should be limited in the lower part by a Larmor-ion-radius order-of-magnitude and, in the upper part, by the transverse dimensions of the trap. Observations were made of the local pulsation of the plasma density using electric probes, so as to provide a qualitative verification of this model.

In order to measure the transverse dimension of the plasma tongues, negative biased probes were placed close to the side-wall in different points along the azimuth of the central cross-section. The contact of each plasma tongue with the wall is marked by a peak of ion current on the corresponding probe. Taking simultaneous oscillograms of the ion currents on two probes placed at varying distances from each other in azimuthal direction, according to the degree of correlation of the peaks, one can estimate the transverse dimensions of the plasma tongues adjacent to the wall.

The ion flux signal on one probe, placed close to the wall, consists of a large number of irregular peaks varying in duration and amplitude. The maximum duration of the peaks is 30 to 50 μs. These wide peaks are modulated by higher frequency pulsations. The shortest peak time is of the order of 2 - 3 μs.

Fig. 1 gives pair oscillograms of the currents on two probes placed at varying distances from each other in the azimuthal direction. It can be seen that at 1-cm interval between the probes the signal shapes nearly coincide.

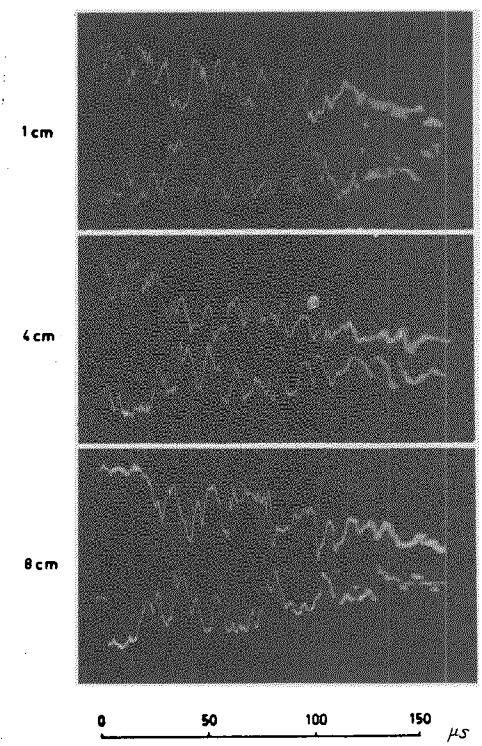

Fig. 1

Ion current correlation for probes at different distances along the azimuth, (the upper and lower oscilloscope traces are deflected in opposite directions).

By increasing the distance, the correlation between the most short-lived peaks is first affected and then that of lower-frequency pulsations.

The analysis of a number of oscillograms shows that with 4 cm between the probes, the correlation of peaks lasting 2-3 μs is strongly affected; at a distance of 8 cm, the correlation of peaks lasting from 8-10 μs is affected; at 12 cm distance, only the correlation between wide peaks, lasting more than 10 μs, remains unaffected (Fig. 2). At a distance between the probes corresponding to a 180° azimuth angle, there is no correlation of signals in both probes.

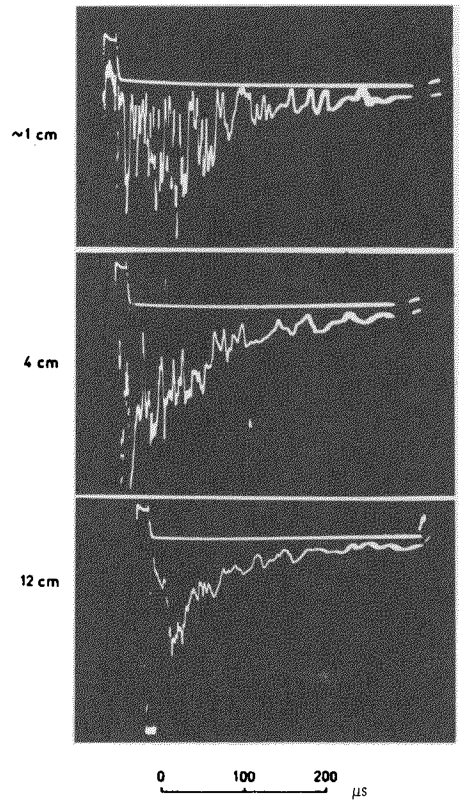

Fig. 2

Change of the signal shape of the ion current on a probe as a function of the distance from the wall (upper trace – accelerating pulse, lower trace – ion current)

It follows that the transverse dimensions of plasma tongues vary in close connection with the lifetime of the tongue. The shortest lived tongues (2-3 μs) have the smallest dimensions. Their width amounts to 3-4 cm, i.e. close to the mean Larmor ion diameter which is about 2.5 cm. The tongues lasting up to 10 μs are about 10-cm wide; and the tongues lasting more than 10 μs exceed 12 cm in width, i.e. their dimensions are comparable with those of the trap cross-section.

The character of plasma pulsations was also investigated with probes at varying distances from the wall. Fig. 2 shows oscillograms of ion current on a probe near the wall and at a distance of 4 to 12 cm from the wall. One can clearly see that the amplitude of high frequency pulsations sharply decreases in the deep plasma layers and that the current in the probe is smoother than that which is near the wall. This shows that small pulsations (with the dimension of a few centimetres and a lifetime of 2-3 μs) develop mainly in the layer near the wall. A more uniform plasma background modulated by large pulsations is found in the central regions of the trap.

All these results show that the space structure and the time factors of the plasma irregularities observed correspond to the physical picture of the convective plasma transfer across the magnetic field towards the wall.

Now, regarding the plasma lifetime, at a given magnetic field H and ion energy T, the expression (11) can be rewritten in the form of

$$\tau \simeq \text{constant} \times \left(\frac{B}{n} + 1\right)^{\frac{1}{2}}, \tag{12}$$

where $B = H^2/4\pi mc^2$. This relation describes the dependence of the plasma escape rate on the density n and can be experimentally verified. In the experiments performed for this purpose, the natural decrease of plasma density with time during its decay was used. We measured the rate of the fast ion density decrease at various moments of time at the end of the injection impulse. The maximum density corresponded to zero delay and the minimum density to a 850 μs delay.

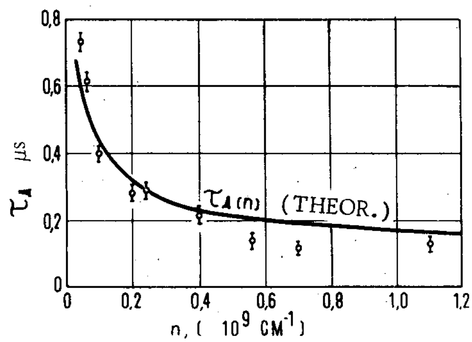

Fig. 3

Experimental and theoretical dependence of τ on the plasma density

Fig. 3 gives the results of the measurements of the τ dependence on the density for a density range from 5×10^7 to 1.1×10^9 cm^{-3}. The same figure gives, for comparison, the theoretical curve corresponding to expression (12). The constant is chosen so that the calculated curve would pass through one of the mean experimental points. It may be seen that the experimentally found τ dependence on the density is in satisfactory agreement with the theoretical predictions.

Unfortunately, the existence of plasma-flute instability was not confirmed in the other experiments at that time, in particular in the Livermore devices with adiabatic magnetic compression. Long plasma confinement with hot electrons covering a number of milliseconds was observed without any visible signs of instability. This led them to the conclusion that flute instability is not dangerous for mirror devices. As was found later, plasma stability was apparently due to poor vacuum conditions and the presence of cold plasma, which was discussed last time. Since in our experiments the plasma continued to escape from the trap, we had no other choice than to try and seek means to overcome the instability.

V. HYBRID TRAP WITH MINIMUM H

Taking into account that plasma-flute instabilities in mirror traps are caused by the radially decreasing magnetic field strength, it was obvious to start by searching for such magnetic configurations in which the field strength increases in all directions from the central region of the trap. Several minimum-H field configurations were proposed recently by Andreoletti, Furth and Taylor. We shall discuss in detail only the one configuration which was realized in our laboratory. It is a hybrid mirror trap with a magnetic field consisting of a combination of the usual field with a hyperbolic field generated by a system of straight currents. The basic principle of this system is shown in Fig. 4.

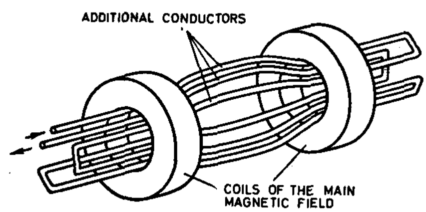

Fig. 4

Scheme of a hybrid magnetic mirror trap (usual field + hyperbolic field).

Let us consider in the first place the field of a linear conductor system which we shall call the stabilizing field.

Let this field be generated by currents flowing through n conductors in the form of long narrow bands distributed over the surface of a cylinder of unit radius. Each band occupies an arc of a circle subtending an angle α at the centre. The currents in the adjacent bands are directed in opposite directions (Fig. 5). The magnetic field of such a current system has only H_r and H_φ components, which are defined by the following expressions:

$$H_r = \frac{8I}{\alpha r} \sum_{k=0}^{\infty} \frac{\sin \frac{n\alpha}{4}(2k+1)}{2k+1} r^{\frac{n}{2}(2k+1)} \sin \frac{n\varphi}{2}(2k+1),$$

$$H_\varphi = \frac{8I}{\alpha r} \sum_{k=0}^{\infty} \frac{\sin \frac{n\alpha}{4}(2k+1)}{2k+1} r^{\frac{n}{2}(2k+1)} \cos \frac{n\varphi}{2}(2k+1),$$

where I is the total current through one band. An adequate degree of accuracy may be obtained, for practical purposes, if we limit ourselves to the two first terms. With r = 0.8 the third terms will give the following

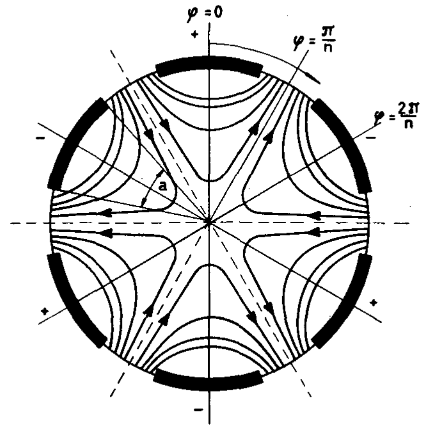

Fig. 5

Magnetic field lines of the stabilizing conductor system

corrections: with n = 4 – 3.5%, with n = 6 – 1.5%; with n > 6 the corrections will amount to even less. Taking into account the two first terms we obtain the expression for the absolute stabilizing field value

$$|H_\perp| = (H_r^2 + H_\varphi^2)^{\frac{1}{2}} = \frac{8I}{\alpha}\left(r^{\frac{n}{2}-1} \sin\frac{n\alpha}{4} + \frac{1}{3} r^{\frac{3n}{2}-1} \sin\frac{3n\alpha}{4} \cos n\varphi\right).$$

It is important to note that with $\alpha = (2/3)(2\pi/n)$, $\sin(3n\alpha/4) = 0$ and the second term disappears. In this case one may consider that $|H_\perp|$ does not depend on φ.

The hybrid field lines obtained from the sum of the basic mirror field and the stabilizing field have a complex space structure. The analytical consideration of the shape of these lines in a general form is very cumbersome, that is why we shall not deal with them now. The total field radial distribution for n = 6 is shown in Fig. 6.

The useful characteristic of the hybrid field of the type examined, is the so-called transverse mirror ratio which represents the relation between the maximum strength of the total field, close to the walls of the vacuum chamber ($H_\perp$) and the field in the centre of the trap ($H_{0\parallel}$).

$$R_\perp = \frac{(H_{0\parallel}^2 + H_\perp^2)^{\frac{1}{2}}}{H_{0\parallel}}.$$

The introduction of such a value is significant because the hybrid magnetic field lines, generally speaking, cross the side-walls of the vacuum

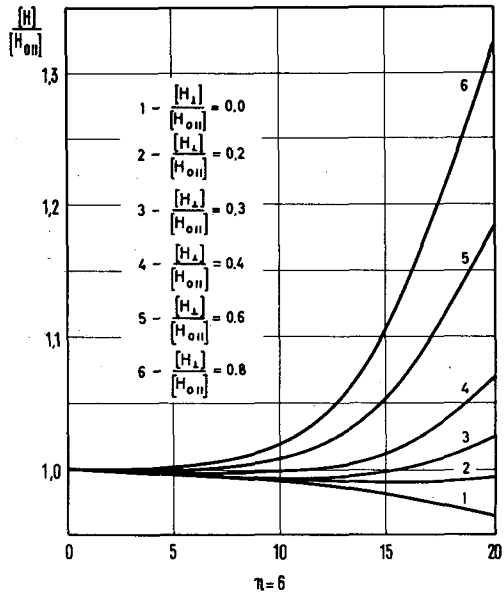

Fig. 6

Total field radial distribution

chamber. Only a very narrow tube of magnetic lines, close to the axis, passes through the entire trap without intersecting the side walls. The longer the system, the narrower the dimensions of such a tube. Thus, the particles on the lines of force which are at a sufficient distance from the axis, could easily reach the side wall moving along the lines of force. However, due to the stronger field next to the wall, they will be reflected from the strong field region as they are reflected from the longitudinal mirrors at the ends of the trap.

VI. THE STUDY OF PLASMA STABILITY IN THE PR5 DEVICE

Description of the device

The general view of the PR5 device is given in Fig. 7. The basic (longitudinal) field, constant in time, is generated by eight coils. The axial distribution of the longitudinal field is given in Fig. 8. The region G is the trap proper. On the left-hand side from G, we have the injection part of the device in which the plasma source is placed, as is also a differential pumping system evacuating neutral hydrogen from the source.

The maximum field strength in the centre of the trap (H_0) is 5000 Oe, the longitudinal mirror ratio $R_{\parallel}$ is 1.7. The distance between the mirrors is 120 cm.

Stabilizing windings are placed in the gap between the coils of the basic field and the vacuum chamber.

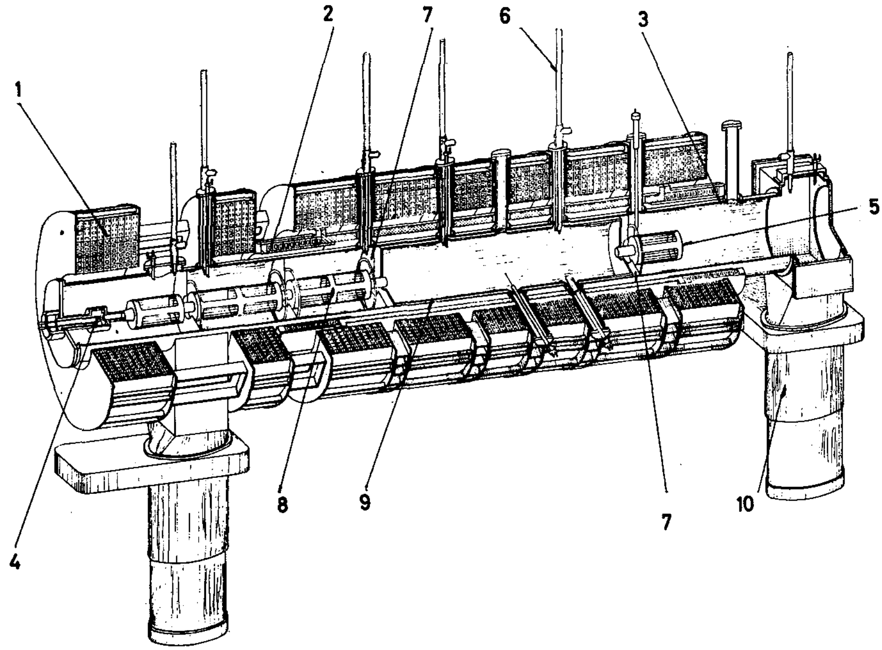

Fig. 7

General view of the PR-5

(1) Coils of the main (longitudinal) field;
(2) stabilizing winding;
(3) vacuum chamber;
(4) plasma source;
(5) target electrode;
(6) titanium evaporator;
(7) diaphragms;
(8) perforated screen;
(9) liquid nitrogen-cooled screen;
(10) pumps.

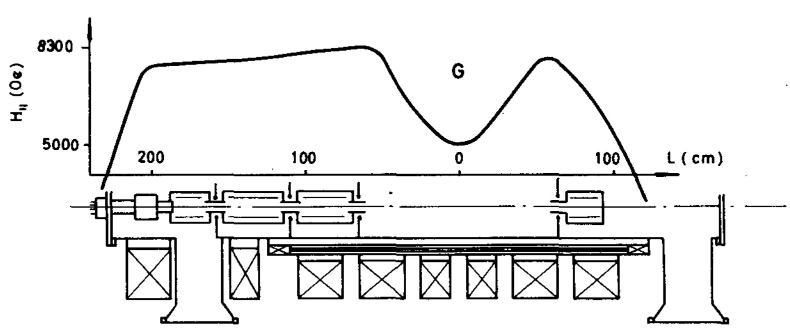

Fig. 8

Distribution of the main magnetic field along the axis of the installation

The maximum strength of the stabilizing field at the vacuum chamber wall ($H_\perp$) is 4500 Oe. The radial ("wall") mirror ratio $R_\perp$ in the central section of the trap for a maximum value for H_0 is equal to 1.4.

The 4-m long vacuum chamber of 40-cm diameter is made of stainless steel. The chamber is initially evacuated to a pressure of 1×10^{-6} torr by two oil-vapour pumps with liquid nitrogen traps. Sorption pumping with evaporated titanium is used to ensure a considerably higher vacuum. A cold plasma column is injected along the axis from the plasma source which is situated at the left-hand side of the device. This 2.0-cm diameter plasma column, passing through the trap, is neutralized on the receiving electrode which is at the same voltage as the anode of the plasma source. The plasma source works in a pulsed regime. Plasma density in the column: $n \simeq 10^{12}\ cm^{-3}$, $T_e = 10$ eV and $T_i \simeq 1$ eV. Pulsed magnetron injection is used for filling the trap with fast ions. A positive potential which is applied to the plasma source and to the connected column has the form of a square pulse of 30 kV amplitude and a duration of 20 - 30 μs (the pulse of the accelerating voltage). The radial electric field accelerates the ions from the column.

The pulse of the accelerating voltage is synchronized with the discharge pulse in the source in such a way that they terminate at the same time. When operating with a stabilized field, the discharge pulse is switched near the maximum of the stabilizing field. The plasma which fills the trap at the end of the injection has the following parameters if a stabilizing field is present: $n = 10^9 - 10^{10}\ cm^{-3}$, $T_i \simeq 5$ keV, $T_e \simeq 20$ eV.

VII. PLASMA LIFE-TIME MEASUREMENTS

The plasma life-time τ was measured against the decrease of the fast neutral-charge exchange particle flux after injection.

At constant pressure of neutral gas, the flux decrease rate is characterized by the mean life-time of the fast ions.

Fast neutral particles were detected by secondary electrons.

(a) Plasma life-time and stabilizing field dependence

The influence of the stabilizing field on the confinement properties of the trap can be observed very clearly if one measures plasma life-time dependence on the value $H_\perp$ at a constant basic field $H_{0\parallel}$. Fig. 9 gives the results of these measurements at 1.5×10^{-7} torr pressure and with several values of $H_{0\parallel}$. The radial mirror ratio $R_\perp$ for each of the $H_{0\parallel}$ values is plotted on the horizontal axis.

These curves show that the superposition of field $H_\perp$ induces a considerable increase of τ. The characteristic feature of this increase is that it is abrupt once $H_\perp$ reaches a given value. This value increases when $H_{0\parallel}$ increases; it also appears that the mirror ratio $R_\perp$ in the point of abrupt change remains the same for different values of $H_{0\parallel}$ and is approximately equal to 1.1.

The "abrupt" form of these curves allows us to distinguish between two confinement regimes: the regime with a large τ ($R_\perp > 1.1$), which we shall

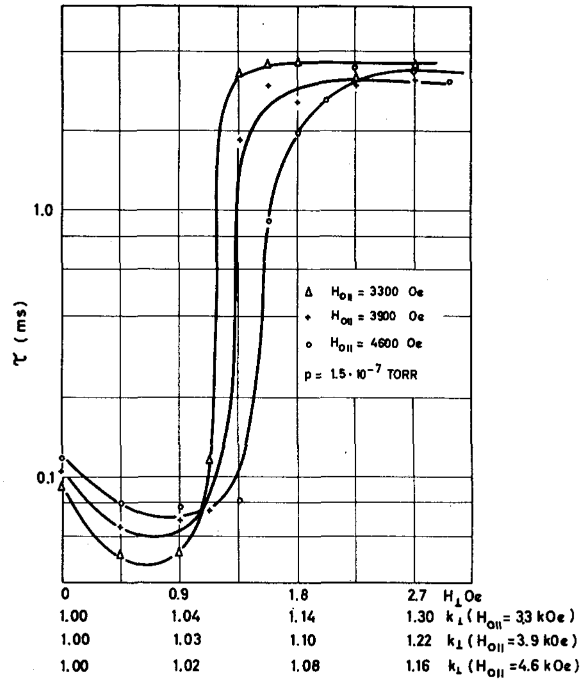

Fig. 9

Dependence of the plasma life-time on the stabilizing field strength

call stable, and a regime with a small τ ($R_\perp < 1.1$), which we shall call unstable.

In the series of measurements given in Fig. 9, τ in the non-stabilized regime is within 0.05 - 0.1 ms, which is in agreement with the data obtained earlier in the case of the ordinary trap. As was shown, in this case the life-time is defined by losses due to convective instability. In a stabilized regime, τ is about 3.5 ms. According to the evaluation made for these experimental conditions, the life-time is close to the ion charge exchange time.

Simultaneously with a rapid increase of τ the form of the oscillograms of the neutral particle flux also changes. Fig. 10 shows two oscillograms, one for the non-stabilized regime, the second for the stabilized regime. The first shows strong disordered oscillations which represent the fluctuations of the plasma density due to instability. The second emphasizes not only the increase of the decay constant, but also the absence of noticeable fluctuations.

The sharp transition in the confinement regimes is connected with the change of the radial distribution of the total magnetic field when $R_\perp$ increases. Fig. 11 gives the dependence of the total field on the radius in three sections of the trap for different $R_\perp$. As may be seen from the figure, at $R_\perp < 1.1$ the decrease along the radius of the basic field $H_{0\parallel}$ is not yet fully compensated along the whole length of the trap by the stabilizing field. It is only with the values of $R_\perp > 1.1$ that a region with a positive radial gradient of field appears

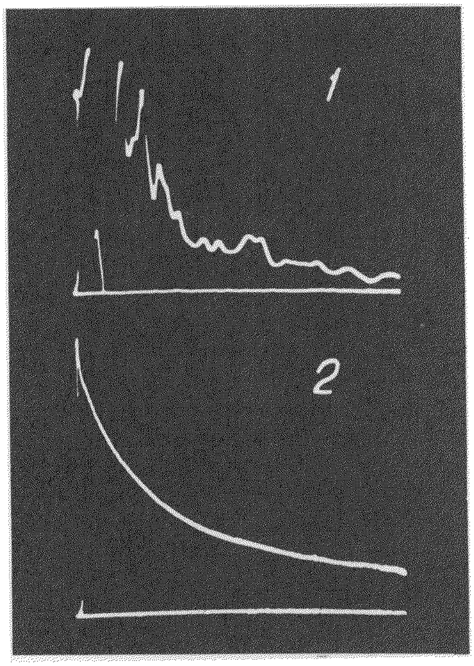

Fig. 10

Oscillograms of neutral particle current

(1) scanning time 500 μs;

(2) $R_\perp = 1.22$, time of scanning 10 ms; $H_{0\parallel} = 3300$ Oe; $p = 10^{-7}$ torr

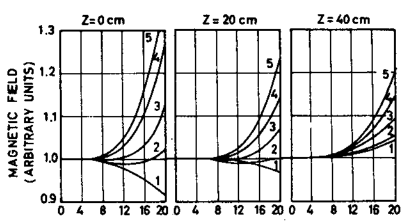

Fig. 11

Radial distribution of the total field in the three cross-sections of the PR-5 installation with different values of α

($R_\perp = 0$ corresponds to trap central cross-section)

(1) $R_\perp = 1.0$;

(2) $R_\perp = 1.07$;

(3) $R_\perp = 1.14$;

(4) $R_\perp = 1.22$;

(5) $R_\perp = 1.3$.

everywhere near the wall. When $R_\perp$ begins to exceed 1.1, the boundary of this region shifts to the axis of the trap.

Thus, the transition from the non-stabilized regime to the stabilized regime happens precisely when a radially growing field settles around the plasma along the whole length of the trap, i.e. when conditions are created for the suppression of the convective instability.

(b) Plasma life-time and pressure dependence

To elucidate the extent to which plasma instability is suppressed in a stabilized regime, measurements were made of τ in terms of the hydrogen pressure p in the trap.

If there is no instability, then practically the sole source of fast ion losses is the charge exchange (at the considered ion energies and densities, one can neglect the ion escape through the mirrors due to Coulomb scattering in comparison with the loss through charge exchange). In this case $\tau = 1/n_0\langle\sigma_c v_i\rangle$ (n_0 is the density of the neutral hydrogen, σ_c the charge exchange cross-section, and v_i the ion velocity) and consequently the dependence $1/\tau = f(p)$ diagram will be a straight line proceeding from the origin of the co-ordinates. If there are other losses in conjunction with charge exchange, those connected with plasma instability, then the straight line $1/\tau = f(p)$ will intersect the axis of the ordinate above zero in a point $1/\tau_0$, where τ_0 is the characteristic time of losses due to instability.

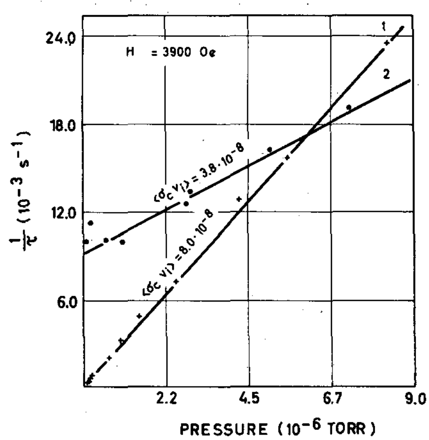

Fig. 12

Dependence of reciprocal fast ion life-time on pressure

$H_{0\parallel} = 3000$ Oe;

(1) $H_\perp = 2300$ Oe;

$\alpha_\perp = 1.15$ (stabilized regime);

(2) $H_\perp = 0$.

Fig. 12 gives the experimental dependence of $1/\tau$ from p with $R_\perp = 1.15$ for an interval of pressure from 6×10^{-8} to 1×10^{-5} torr (curve 1). For comparison we give the $1/\tau$ and p relationship, taken under the same conditions, but in the absence of a stabilizing field (curve 2). One may see from the figure that, in a stabilized regime, the experimental points are situated on a straight line which, after extrapolation to zero pressure, passes close to the origin of the co-ordinates.

The accuracy of the measurements and, hence, the accuracy of the extrapolations, allow us to assert that the characteristic time of non-charge-exchange losses, if such losses indeed occur, amounts to not less than 25 to 30 ms. The maximum value for τ, obtained from this series of measurements, is equal to 6 ms at $p = 6 \times 10^{-8}$ torr. Certain measurements, carried out at lower pressure values, give correspondingly higher τ values.

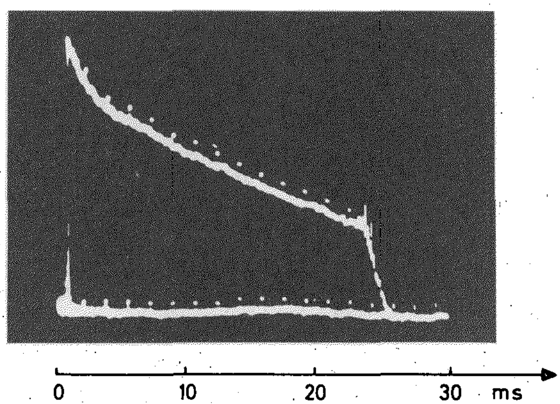

Fig. 13

Oscillogram of the re-charged neutral particle flux

Fig. 13 gives the oscillogram of the neutral particle flux for $p = 7 \times 10^{-9}$ torr, which is the lowest pressure obtainable in our device. In this case τ is around 60 ms which is also in close correspondence to the charge exchange time. The interruption of the signal before the end of the scanning time is determined by a decrease in the stabilizing field down to a magnitude which is insufficient to stabilize plasma in the trap. It should be pointed out, that the attempts to compare accurately the observed life times with those of charge exchange at sufficiently low pressure values lose their significance to a certain extent. In the presence of strong sorbing walls, one cannot speak about uniform neutral gas density in the volume of the trap if there are local sources of gas formation, and still less can one rely on the indications of the ion gauge when it is not placed directly in the vacuum chamber. Therefore, the presence or absence of non-charge-exhange losses can be determined only with the accuracy determined by the truly measured life times. The very fact that these times continue to grow up to such values as 60 ms with the decrease of pressure allows us to draw the conclusion that in a stabilized regime one observes in practice no plasma instability.

(c) The effect of the stabilizing field on the plasma density fluctuations

In addition to the plasma stability characteristics such as life-time τ, local plasma density fluctuations and the change in these fluctuations, due to the stabilizing field effect, are of interest. These data permit a more detailed picture to be obtained on the mechanism of the instabilities suppressed by a radially growing field.

The plasma density fluctuations were indicated by the change of ion current to the Langmuir probe placed in the plasma. The probe was located in the central cross-section and could be placed at varying distances from the axis. Oscillograms were taken during the plasma decay period for different values of the stabilizing field.

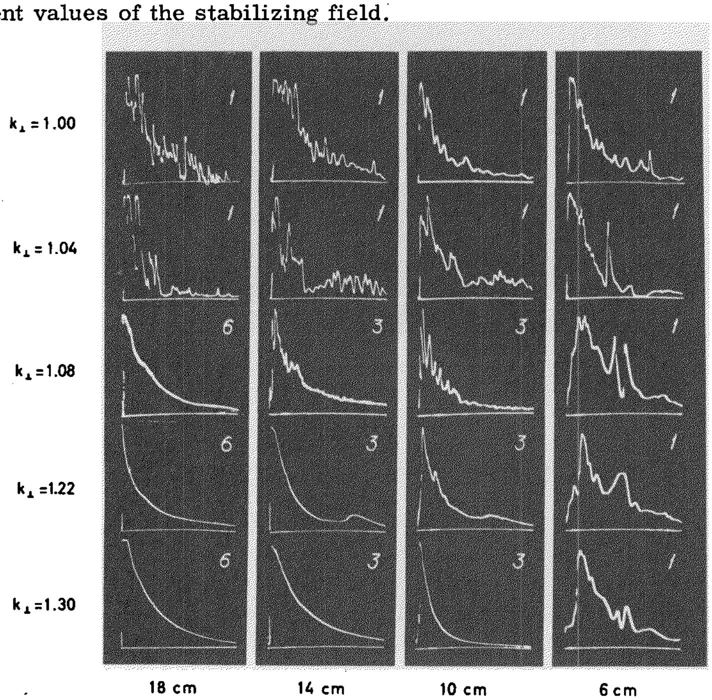

Fig. 14

Oscillogram of the ion current to the Langmuir probe at different distances from the axis of the ion trap (The time (ms) of screening is indicated in the upper right corner of each oscillogram.)

Figure 14 gives four oscillogram series corresponding to the following distances between the probe and the axis: 18, 14, 10 and 6 cm (the signal amplitudes in the oscillograms are given on an arbitrary scale).

At $R_\perp < 1.1$, strong fluctuations typical for unstable plasma are observed at all distances from the axis.

At $R_\perp > 1.1$, the fluctuations in the periphery (18, 14 and 10 cm) start decreasing sharply and disappear completely with the increase of the stabilizing field. It should be noted that the fluctuations first disappear in the vicinity of the wall and then, gradually, in deeper layers of the plasma. Near the axis strong fluctuations remain even at $R_\perp = 1.3$.

Comparing the oscillograms of Fig. 14 with the total magnetic field curves (Fig. 11) one may see that at a given $R_\perp$ fluctuations, and hence instability, are observed at those distances from the axis where the field either decreases radially or increases to a very slight extent. Fluctuations are absent in those regions where the field grows at a sufficiently high rate.

In this connection it is interesting to point out that a sharp increase of τ at $R_\perp \simeq 1.1$ and the transition to a stabilized regime are accompanied by a suppression of instability only in the external layer of the plasma. This alone is sufficient to eliminate entirely plasma escape from the trap. If $R_\perp$ exceeds 1.1, the stable zone at the periphery extends to the axis.

Thus, the experiment shows that the presence of instability in the inner regions of the trap does not prevent plasma confinement over a long period of time. The fact that at $R_\perp > 1.1$, the slowing down of the signal increases when the probe is removed from the wall, does not mean that plasma decays more rapidly in the inner regions than at the periphery. Simultaneous measurements of the neutral particle flux indicate that, by inserting the Langmuir probe into the trap, the life time of the entire plasma is decreased sharply. This is due to losses in the probe which are the greater, the deeper the insertion of the probe into the plasma.

(d) Plasma density distribution along the radius of the trap

The same Langmuir probe was used for measuring the density distribution along the radius. The measurements were carried out in the central section of the trap on the radius passing through the middle of the gap between the conductors of the stabilizing coil. The value of the ion current to the probe at the end of the injection pulse was adopted as the value characterizing plasma density. (The probe was biased to -80V with respect to the ground.)

Figure 15 gives the radial plasma density distribution for three values of the stabilizing field ($R_\perp = 1.08$, 1.15 and 1.3) and for distances from the axis starting at 5 cm. (Measurements could not be carried out at distances smaller than 5 cm due to probe breakdown.)

The curve taken in the absence of a stabilizing field reproduces satisfactorily the results of similar measurements carried out earlier in an ordinary trap. The smooth plasma distribution in an ordinary mirror trap is conditioned by the intense convective transfer of plasma across the magnetic field.

By introducing the stabilizing field, the radial density distribution is considerably deformed: plasma is "decompressed" from the wall and concentrates in the region of the trap near the axis; this "decompression" increases with the increase of $H_\perp$. It is evident that such a density redistribution results from the variation of the radial distribution of the magnetic field.

At the periphery, in those regions where the field increases radially, there is no longer any convective transfer. That is why plasma density decreases sharply there, and increases correspondingly in the region near

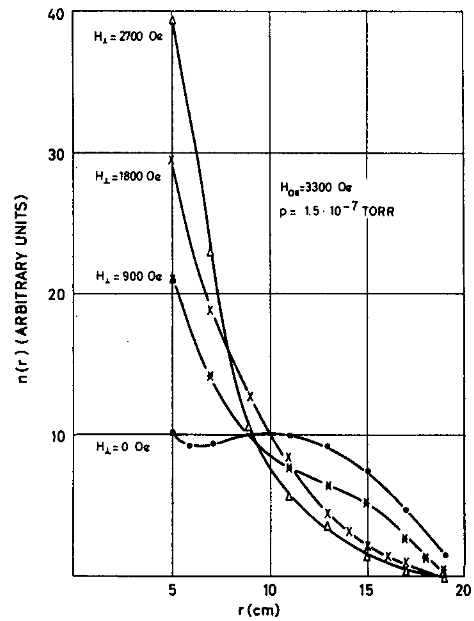

Fig. 15

Radial distribution of the plasma density in the trap

the axis. With the increase of $R_\perp$, the increasing field regions approach the axis and the radial density distribution narrows down.

It should be stressed that the curves of Fig. 15 can only serve as a very approximate qualitative description of the real density distribution. In the first place, as has already been noted, even a small probe in long plasma-confinement conditions causes a strong particle absorption. In the second place, the radial distribution in a trap with a combined field should not have a cylindrical symmetry. According to the structure of the combined field lines one can expect that the distribution will be more smeared at the azimuths corresponding to the centre of the gaps between the conductors of the stabilizing coil than in other directions. Unfortunately, the existing structure of the stabilizing coil makes measurements at different azimuths very complex.

Thus all results mentioned here show clearly enough that in a radially increasing magnetic field, plasma is confined for sufficiently long periods without any noticeable signs of instabilities. It should be stressed that these results concern only low density plasma with $\beta = \dfrac{n(T_i + T_e)}{H^2/8\pi} \approx 10^{-4}$. A further task should evidently be the carrying out of an investigation on the hybrid field properties at considerably stronger plasma densities.

PLASMA CONFINEMENT IN MAGNETIC WELLS

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I. SPECIAL EQUILIBRIA*

1. INTRODUCTION

In these articles we shall discuss the problem of confinement and stability of plasma in a special type of magnetic field. Fields of this type are variously known as "magnetic wells", "hybrid traps" or "minimum-B fields". They are of special interest because they combine the adiabatic containment properties of the simple mirror field with the stability of the cusp field.

The stability of the cusp is related to the fact that its field strength increases as one moves away from the centre, so that the class of fields we are interested in are those which have the basic feature that there is a region in which:

(a) The field is nowhere zero, so that adiabatic containment is possible.
(b) The magnetic field strength "increases outwards".

By this second property of $|B|$, "increasing outwards", one means that there exists a point, or in some cases a closed curve, which is a local minimum of B^2. In the neighbourhood of this point, or curve, the contours defined by B^2 = const. form a set of closed, nested surfaces and a surface of larger B^2 encloses those of smaller B^2. Since these surfaces are closed one can unambiguously refer to inside and outside; then one can say that the magnetic pressure is lower inside any given surface than outside it and it is in a region such as this that one hopes for stable plasma confinement. We shall find that it is possible to discuss the containment and stability of a special class of equilibria by using <u>only</u> the two properties (a) and (b). This means that our results can be applied to any magnetic well irrespective of its geometrical form.

It should first be emphasized that the surfaces of B^2 = const. (which may be termed magnetic isobars) are <u>not</u> flux surfaces. A line of force will generally cut a magnetic isobar twice (or not at all) and the points of intersection could, for example, form the turning points of particles contained on that line by the mirror effect.

2. MAGNETIC WELL CONFIGURATION - AN EXAMPLE

As an example of the type of magnetic field under discussion we may consider the configuration employed by Ioffe. Our object is merely to indicate some of the main features of this arrangement, particularly of its magnetic isobars.

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* This part is based upon the author's paper in Physics of Fluids <u>6</u>, see bibliography to Part I.

Near the centre of a mirror machine the field strength increases as one moves along the axis toward either mirror, but decreases as one moves radially away from the axis. A method of creating a field having the property that B^2 increases both axially and radially would therefore appear to be that one superimposes on the mirror a second field which increases as one moves from the axis but which is constant along the axis. Such a field is the "multipole" field provided by 2 l straight rods parallel to the axis of the machine, adjacent rods carrying current in opposite directions. Near the axis the multipole field is approximately

$$B_r = -\frac{lI^*}{R}\left(\frac{r}{R}\right)^{l-1}\cos l\theta,$$
$$B_\theta = +\frac{lI^*}{R}\left(\frac{r}{R}\right)^{l-1}\sin l\theta, \tag{2.1}$$

where R is the distance of the rods from the axis and I^* is a measure of the current in each rod. (The relationship of I^* to the actual current I depends on the way that the current is distributed over the cross-section of the rods and on the shape of this cross-section; for thin rods $I^* = 2I$.) The original mirror field can be approximately represented by

$$B_z = B_0[1 - \alpha I_0(2\pi r/L)\cos(2\pi z/L)],$$
$$B_r = -\alpha B_0 I_1(2\pi r/L)\sin(2\pi z/L), \tag{2.2}$$

where I_0 and I_1 are modified Bessel functions. The mirrors are situated at $z = \pm\frac{1}{2}L$ and the mirror ratio is

$$R_m = (1+\alpha)/(1-\alpha). \tag{2.3}$$

The formation of closed magnetic isobars of the required type can be illustrated easily when $l = 2$, for then near the centre of the machine, $z = 0$, $r = 0$, the field strength is given by

$$B^2 = B_0^2(1-\alpha)^2 + 4\pi^2 B_0^2\left\{\alpha(1-\alpha)\frac{z^2}{L^2} + \frac{r^2}{L^2}\left[\frac{I^{*2}L^2}{\pi^2 B_0^2 R^4} - \frac{\alpha(1-\alpha)}{2}\right]\right\}. \tag{2.4}$$

If the current in the multipole rods is small, so that

$$I^{*2} < (\pi^2 R^4/2L^2)\alpha(1-\alpha)B_0^2 \tag{2.5}$$

then the isobars form a family of hyperboloids. However, as the current in the multipole rods is increased so that

$$I^{*2} > (\pi^2 R^4/2L^2)\alpha(1-\alpha)B_0^2 \tag{2.6}$$

these magnetic isobars become closed (ellipsoidal) surfaces of the type we desire.

Before leaving this topic it is worth-while noting that the situation is not so simple when $l > 2$. If $l > 2$ then sufficiently near the axis the multipole field is always too weak to compensate for the radial decrease in the basic mirror field. In this case closed magnetic isobars are still formed but instead of a single minimum at $r = 0$, $z = 0$, there are $2l$ minima situated off the axis.

3. LOW-β EQUILIBRIA

We now consider the problem of plasma equilibrium. For equilibrium the pressure tensor $\vec{\vec{P}}$ must satisfy

$$\vec{j} \times \vec{B} = \vec{\nabla} \cdot \vec{\vec{P}}, \tag{3.1}$$

where $\vec{j}$ and $\vec{B}$ are connected by

$$\vec{\nabla} \times \vec{B} = 4\pi \vec{j} \tag{3.2}$$

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{3.3}$$

A full solution to the problem of equilibrium would involve solving these equations subject to boundary conditions such as the given currents in the external conductors. However, apart from the impracticability of such a programme, it is our present aim to derive general results independent of the detailed arrangement of conductors, and so applicable to all fields possessing properties (a) and (b) of section 1. We therefore seek low-β solutions (where β is the ratio of plasma pressure to magnetic pressure).

At zero β the magnetic field is the vacuum field due to external currents; this is easily calculated and will be considered as given. The first order perturbation in the field, due to plasma pressure, is given by:

$$\vec{j}_1 \times \vec{B}_0 = \vec{\nabla} \cdot \vec{\vec{P}}, \tag{3.4}$$

$$\vec{\nabla} \times \vec{B}_1 = 4\pi \vec{j}_1, \tag{3.5}$$

$$\vec{\nabla} \cdot \vec{B}_1 = 0, \tag{3.6}$$

where $\vec{j}_1$ is the plasma current density, $\vec{B}_0$ the original vacuum field, and $\vec{B}_1$ the perturbation in this field due to the presence of plasma.

Now it might appear that these equilibrium equations should have solutions $\vec{j}_1$ and $\vec{B}_1$ for any given plasma pressure $\vec{\vec{P}}$ and that there is, therefore, no problem. Indeed in axisymmetric configurations such as mirror or cusp this is true, but in general these equations will not possess a solution and our first task is to determine the conditions which $\vec{\vec{P}}$ must satisfy in order that a solution should exist.

This is perhaps most easily done as follows: Eqs. (3.5) and (3.6) are simply the magnetostatic equations which are known to have a solution if $\vec{j}_1$ exists and $\vec{\nabla} \cdot \vec{j}_1 = 0$. Our procedure therefore will be to solve Eq. (3.4) for $\vec{j}_1$ and then to examine under what conditions $\vec{\nabla}\cdot\vec{j}_1 = 0$. (As we shall be concerned only with $\vec{j}_1$ and $\vec{B}_0$ we may henceforth suppress all subscripts, provided we remember that $\vec{B}$ always denotes a vacuum field.)

To illustrate the argument consider the case of scalar pressure when Eq. (3.4) reduces to

$$\vec{j}\times\vec{B} = \vec{\nabla}p\,. \tag{3.7}$$

The first necessary condition on p is clearly

$$\vec{B}\cdot\vec{\nabla}p = 0 \text{ or } \partial p/\partial S = 0, \tag{3.8}$$

i.e. p is constant along a field line. Given that (3.8) is satisfied we can then solve (3.7) for $\vec{j}_\perp$ (the component of $\vec{j}$ perpendicular to $\vec{B}$),

$$\vec{j}_\perp = (-\vec{\nabla}p\times\vec{B})/B^2, \tag{3.9}$$

and therefore

$$\vec{j} = (-\vec{\nabla}p\times\vec{B})/B^2 + \lambda\vec{B}, \tag{3.10}$$

where λ is an arbitrary scalar.

The requirement $\vec{\nabla}\cdot\vec{j} = 0$ then gives

$$\vec{B}\cdot\vec{\nabla}\lambda = \vec{\nabla}\cdot(\vec{\nabla}p\times\vec{B}/B^2), \tag{3.11}$$

or

$$\vec{B}\cdot\vec{\nabla}\lambda = -2\vec{\nabla}B\cdot(\vec{\nabla}p\times\vec{B})/B^3. \tag{3.12}$$

Eq. (3.12) can be written

$$d\lambda/dS = -2\vec{\nabla}B\cdot(\vec{\nabla}p\times\vec{B})/B^4, \tag{3.13}$$

where S is measured along the line of force. A necessary condition for this equation to possess a unique single valued solution for λ is clearly

$$\oint\frac{\vec{\nabla}B\cdot(\vec{\nabla}p\times\vec{B})}{B^4}\,dS = 0, \tag{3.14}$$

where the integral is taken along any closed line of force. Newcomb has shown that this is also a sufficient condition.

In the case of scalar pressure, then, Eqs. (3.8) and (3.14) are the necessary and sufficient conditions which the pressure must satisfy if the plasma is to be in equilibrium. We now turn to the situation of immediate interest, namely when the pressure is anisotropic, as it must be in any mirror trap, and seek the analogous conditions on the pressure tensor.

Anisotropic pressure

In a co-ordinate system with the principal axis along the magnetic field the pressure tensor can be written

$$\overleftrightarrow{P} = p_\perp \overleftrightarrow{I} + (p_\parallel - p_\perp)\vec{n}\vec{n}, \tag{3.15}$$

where $\vec{n}$ is a unit vector in direction of $\vec{B}$ and $\overleftrightarrow{I}$ is the unit tensor.

The momentum balance equation is now

$$\vec{j}\times\vec{B} = \vec{\nabla}\cdot\overleftrightarrow{P} \tag{3.16}$$

and, from the parallel component of this equation, the first condition on $p_\perp$ and $p_\parallel$ is obtained;

$$\vec{n}\cdot\vec{\nabla}p_\perp + \vec{n}\cdot\vec{\nabla}\cdot\{(p_\parallel - p_\perp)\vec{n}\vec{n}\} = 0, \tag{3.17}$$

or

$$\frac{\partial p_\parallel}{\partial S} + \frac{(p_\perp - p_\parallel)}{B}\frac{\partial B}{\partial S} = 0, \tag{3.18}$$

where S is measured along the magnetic field. This condition specifies a relation between $p_\perp$ and $p_\parallel$ along a field line, replacing the simpler condition $\partial p/\partial S = 0$ of the scalar pressure theory. However, if Eq. (3.18) is satisfied then Eq. (3.16) can be solved for $\vec{j}_\perp$ as before,

$$\vec{j}_\perp = -\vec{\nabla}p_\perp\times\vec{B}/B^2 + \vec{B}\times\vec{\nabla}\cdot[(p_\parallel - p_\perp)\vec{n}\vec{n}]/B^2, \tag{3.19}$$

and so

$$\vec{\nabla}\cdot\vec{j}_\perp = 2\vec{\nabla}p_\perp\cdot(\vec{B}\times\vec{\nabla}B)/B^3 + \vec{\nabla}\cdot\{\vec{B}\times\vec{\nabla}\cdot[(p_\parallel - p_\perp)\vec{n}\vec{n}]/B^2\}. \tag{3.20}$$

It can be shown that because $\vec{B}$ is a vacuum magnetic field the last term can be transformed to give

$$\vec{\nabla}\cdot\{\vec{B}\times\vec{\nabla}\cdot[(p_\parallel - p_\perp)\vec{n}\vec{n}]/B^2\} = \vec{\nabla}(p_\parallel - p_\perp)\cdot(\vec{B}\times\vec{\nabla}B)/B^3. \tag{3.21}$$

Therefore we finally obtain

$$\vec{\nabla}\cdot\vec{j}_\perp = \vec{\nabla}(p_\perp + p_\parallel)\cdot(\vec{B}\times\vec{\nabla}B)/B^3. \tag{3.22}$$

Then, just as in the case of scalar pressure, the vanishing of $\vec{\nabla}\cdot\vec{j}$ requires

$$\vec{\nabla}\cdot\vec{j}_\parallel = \vec{B}\cdot\vec{\nabla}\lambda = -\vec{\nabla}\cdot\vec{j}_\perp, \tag{3.23}$$

so that

$$\vec{B}\cdot\vec{\nabla}\lambda = -\vec{\nabla}(p_\perp + p_\parallel)\cdot(\vec{B}\times\vec{\nabla}B)/B^3. \tag{3.24}$$

As before this can be written

$$\frac{d\lambda}{dS} = -\vec{\nabla}(p_\perp + p_\parallel)\cdot(\vec{B}\times\vec{\nabla}B)/B^4 \tag{3.25}$$

and, if the lines of force were closed, this would lead to the condition

$$\oint \vec{\nabla}(p_\perp + p_\parallel)\cdot\frac{(\vec{B}\times\vec{\nabla}B)}{B^4}\,dS = 0\,. \tag{3.26}$$

In the systems we are considering the lines of force are not closed within the plasma volume but leave the region of interest. In this case, provided the plasma is surrounded by a region in which no current flows, we must have

$$\int \vec{\nabla}(p_\perp + p_\parallel)\cdot\frac{(\vec{B}\times\vec{\nabla}B)}{B^4}\,dS = 0\,, \tag{3.27}$$

where the integral is taken from the point where the line of force first enters the plasma to the point where it first leaves it. (If this condition were not satisfied λ would not be zero when the line of force left the plasma and there would be currents flowing in the plasma free region.) Furthermore, it is clear that if this condition (3.27) is satisfied, a unique λ can always be constructed from (3.25). The condition (3.27) is therefore both necessary and sufficient.

Given anisotropic pressure conditions, then the necessary and sufficient conditions for equilibrium are (3.18) and (3.27). Before discussing some distributions satisfying these conditions we will first interpret these equilibrium constraints from the point of view of individual particle motions.

4. PARTICLE MOTION

The first constraint (3.18) is simply the requirement that the particles be in equilibrium along each field line considered individually. This is entirely consistent with the basic idea of adiabatic mirror containment; for if the magnetic moment of a particle

$$\mu = V_\perp^2/2B \tag{4.1}$$

is constant as it moves along a field line then

$$p_\perp \propto \int \rho(\mu,\epsilon)\frac{\mu B}{2}\,d\mu\,d\epsilon, \tag{4.2}$$

$$p_\parallel \propto \int \rho(\mu,\epsilon)(\epsilon - \mu B)\,d\mu\,d\epsilon, \tag{4.3}$$

where ρ is the local density of particles of specified magnetic moment μ and energy ϵ. This is proportional to (i) the number of such particles on the line = $f(\mu, \epsilon, L)$, (ii) to the density of lines = B, (iii) to the fraction of the time each particle spends near the point of interest,

$$dt \propto dl/(\epsilon - \mu B)^{\frac{1}{2}}. \tag{4.4}$$

Therefore, for particles contained by the mirror effect,

$$p_{\perp} = \int f(\mu, \epsilon, L) \frac{\mu B^2}{2(\epsilon - \mu B)^{\frac{1}{2}}} d\mu\, d\epsilon, \tag{4.5}$$

$$p_{\parallel} = \int f(\mu, \epsilon, L) B(\epsilon - \mu B)^{\frac{1}{2}} d\mu\, d\epsilon. \tag{4.6}$$

It can be verified by direct substitution that these expressions satisfy (3.18).

The second constraint (3.27) may be interpreted in terms of the guiding centre drifts of the particles on a field line. As is well known, the first order guiding centre drift of a particle in an inhomogeneous magnetic field is

$$\vec{v}_D = \frac{mc}{e} \frac{(\vec{B} \times \vec{\nabla} B)}{B^3} (\tfrac{1}{2} v_{\perp}^2 + v_{\parallel}^2), \tag{4.7}$$

where $v_{\perp}$ is the velocity perpendicular to the field and $v_{\parallel}$ that along it.

The total current associated with this drift is then

$$\vec{j}_D = [(\vec{B} \times \vec{\nabla} B)/B^3](p_{\perp} + p_{\parallel}), \tag{4.8}$$

and the divergence of this expression is

$$\vec{\nabla} \cdot \vec{j}_D = \vec{\nabla}(p_{\perp} + p_{\parallel}) \cdot (\vec{B} \times \vec{\nabla} B)/B^3, \tag{4.9}$$

so that the second condition for equilibrium can be written

$$\int (\vec{\nabla} \cdot \vec{j}_D) \frac{dS}{B} = 0. \tag{4.10}$$

The meaning of this is made clear if we consider not the integral along a field line but the integral over an infinitesimal flux tube. This can be obtained by multiplying (4.10) by BdA, when we have

$$\int_{\text{Flux tube}} (\vec{\nabla} \cdot \vec{j}_D) d\tau = 0, \tag{4.11}$$

so that the condition found for the existence of a solution to the magnetostatic fluid equations is equivalent to the statement that the divergence of the current associated with the guiding centre drifts should vanish when averaged over any flux tube. Of course, the current due to the guiding centre drifts is not the total current, but the difference can be expressed as the curl of the magnetization per unit volume, whose divergence vanishes identically. The constraint might therefore equally well be applied to the total current or to the drift current.

5. A CLASS OF EQUILIBRIA

Now let us consider some particular solutions of the equilibrium constraints (3.18) and (3.27), appropriate to the type of magnetic field under discussion. It should first be noted that the second constraint (3.27) is not serious in systems of axial symmetry such as the mirror or the spindle cusp. For in these systems the symmetry ensures that $\vec{\nabla}p$, $\vec{\nabla}\vec{B}$, and $\vec{B}$ are co-planar vectors (lying in the r, z, plane) so that the expression

$$\vec{\nabla}(p_\perp + p_\parallel)\cdot\vec{\nabla}B \times \vec{B} \tag{5.1}$$

vanishes identically. Similarly, in any cylindrically symmetric system, $\vec{\nabla}p$ and $\vec{\nabla}B$ are both radial and (5.1) again vanishes.

In other field configurations the constraint (3.27) can be a severe restriction; for example, the condition (3.27) (or rather (3.26) which is then the appropriate form) can never be satisfied by any confined plasma distribution within a circular torus. For in such a configuration, symmetry ensures that the integral (3.26) can only vanish if the integrand vanishes. As $(\vec{\nabla}B \times \vec{B})$ is in the direction parallel to the symmetry axis of the torus this means that p must be constant in this direction, thus the plasma is not confined. This, of course, is the well-known lack of equilibrium in a simple toroidal field.

If we leave aside for the moment the question of whether it represents contained plasma or not, a restricted class of solutions to the equilibrium constraints can always be found by demanding that (5.1) should vanish. This is certainly achieved if $(p_\perp + p_\parallel)$ is a function only of B, then, since the "parallel" equilibrium equation (3.18) gives $p_\perp$ in terms of $p_\parallel$, this will make $p_\perp$ and $p_\parallel$ individually functions of B alone. Making $p_\perp$ and $p_\parallel$ functions of B alone means the surfaces of constant B, the magnetic isobars, are also surfaces of constant $p_\perp$ and $p_\parallel$.

The significance of magnetic field configurations which possess closed magnetic isobars now becomes apparent. Equilibria in which $p_\perp$ and $p_\parallel$ are functions only of B exist in all field configurations, but only in those which possess closed magnetic isobars do these equilibria correspond to confined plasma configurations.

This special class of low-β equilibria, which have

$$p_\perp = p_\perp(B), \quad p_\parallel = p_\parallel(B) \tag{5.2}$$

and, from (3.18),

$$Bp'_\parallel = p_\parallel - p_\perp, \tag{5.3}$$

where the prime denotes differentiation with respect to B, is one whose stability will be proved in the next section.

An example of this class of equilibrium distribution is

$$\left.\begin{aligned} p_{\parallel} &= CB(B_0 - B)^n \\ p_{\perp} &= nCB^2(B_0 - B)^{n-1} \end{aligned}\right\} \quad \text{if } B < B_0, \tag{5.4}$$

$$p_{\perp} = p_{\parallel} = 0 \qquad\qquad \text{if } B > B_0,$$

where n and B_0 are arbitrary parameters. These equilibria correspond to plasma confined within the contour $B = B_0$ which, by the basic property of our fields, can be a closed contour.

Particle distribution functions corresponding to the equilibria (5.4) can also be written down in terms of the distribution in μ, ϵ space (see section 4). A particle distribution function which leads to the pressure distributions (5.4) is

$$\begin{aligned} f(\mu, \epsilon) &= (\mu B_0 - \epsilon)^{n-\frac{3}{2}} \cdot g(\mu), \qquad & \epsilon < \mu B_0 \\ f(\mu, \epsilon) &= 0, & \epsilon > \mu B_0 \end{aligned} \tag{5.5}$$

where $g(\mu)$ is an arbitrary function of the magnetic moment.

6. STABILITY OF THE SPECIAL EQUILIBRIA

To examine the stability of the equilibria described in the previous section let us first continue with a fluid description and consider the double adiabatic hydromagnetic energy principle derived by Bernstein et al.

According to this, the stability of a plasma configuration with anisotropic pressure is determined by the sign of the minimum of the energy integral.

$$\delta W_D = \int d\tau \Big\{ |\vec{Q}|^2 - \vec{j} \cdot \vec{Q} \times \vec{\xi} + \frac{5}{3} p_{\perp} (\vec{\nabla} \cdot \vec{\xi})^2 + (\vec{\nabla} \cdot \vec{\xi})(\vec{\xi} \cdot \vec{\nabla} p_{\perp}) + \frac{1}{3} p_{\perp} (\vec{\nabla} \cdot \vec{\xi} - 3q)^2$$

$$+ q \vec{\nabla} \cdot [\vec{\xi}(p_{\parallel} - p_{\perp})] - (p_{\parallel} - p_{\perp}) [\vec{n} \cdot (\vec{a} \cdot \vec{\nabla}) \vec{\xi} + \vec{a} \cdot (\vec{n} \cdot \vec{\nabla}) \vec{\xi} - 4q^2] \Big\}, \tag{6.1}$$

where

$$\vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}), \quad q = \vec{n} \cdot (\vec{n} \cdot \vec{\nabla}) \vec{\xi},$$
$$\vec{a} = (\vec{n} \cdot \vec{\nabla}) \vec{\xi} - (\vec{\xi} \cdot \vec{\nabla}) \vec{n}, \tag{6.2}$$

and $\vec{\xi}$ is an arbitrary displacement vector. δW_{min} should be positive for stability.

Examination of the energy integral shows that only the first term $|\vec{Q}|^2$ is independent of β so that at low β it must dominate (and so make δW positive) except for those displacements which themselves make $\vec{Q}$ zero. Physically these displacements are those which do not change the vacuum magnetic field - the so-called interchange modes.

Hence, at sufficiently low β we can determine stability by examining δW for displacements which satisfy

$$\vec{Q} = \vec{\nabla} \times (\vec{\xi} \times \vec{B}) = 0 \tag{6.3}$$

and, for these displacements,

$$q = \vec{\nabla} \cdot \vec{\xi} + \vec{\xi} \cdot \vec{\nabla} B/B, \quad \vec{a} = (\vec{\nabla} \cdot \vec{\xi} + \vec{\xi} \cdot \vec{\nabla} B/B)\vec{n}. \tag{6.4}$$

With the aid of (6.3) and (6.4) the energy integral can be greatly simplified. In fact

$$\delta W_D = \int d\tau \{3p_{\parallel} d^2 + dS(5p_{\parallel} - p_{\perp}) + S^2(p_{\perp} + 2p_{\parallel}) + d(\vec{\xi} \cdot \vec{\nabla} p_{\parallel}) + S[\vec{\xi} \cdot \vec{\nabla}(p_{\parallel} - p_{\perp})]\}, \tag{6.5}$$

where, for brevity, we have written

$$\vec{\nabla} \cdot \vec{\xi} \equiv d, \quad \vec{\xi} \cdot \vec{\nabla} B/B \equiv S.$$

So far this is quite general. For the equilibria found in section 5, namely those which have the properties

$$p_{\perp} = p_{\perp}(B), \quad p_{\parallel} = p_{\parallel}(B), \quad Bp'_{\parallel} = p_{\parallel} - p_{\perp}, \tag{6.6}$$

δW_D reduces to

$$\delta W_D = \int d\tau \left\{ \frac{1}{3p_{\parallel}} [3p_{\parallel}(d+S) - p_{\perp} S]^2 + S^2 \left[2p_{\perp} - \frac{p_{\perp}^2}{3p_{\parallel}} - Bp'_{\perp} \right] \right\}. \tag{6.7}$$

The first term is clearly non-negative so a sufficient criterion for stability according to the double adiabatic principle is

$$2p_{\perp} - (p_{\perp}^2/3p_{\parallel}) - Bp'_{\perp} > 0 . \tag{6.8}$$

Some explicit examples of equilibria were given by Eqs. (5.4). For these examples

$$Bp'_{\perp} = 2p_{\perp} - [(n-1)/n]\, p_{\perp}^2/p_{\parallel} \tag{6.9}$$

and a sufficient stability condition is $n > \frac{3}{2}$. (Note that this is also the condition for $f(\mu, \epsilon)$ in Eq. (5.5) to be continuous at $\epsilon = \mu B_0$.)

The small Larmor radius theory

The double adiabatic energy principle is open to two objections; firstly, that it is based on the assumption that in the plasma motion there is no heat flow along the lines of force, and secondly, that although a component of the displacement $\vec{\xi}$ along the lines of force is formally allowed, it is hard to see what is the real significance of this parallel displacement (since, in collisionless plasma, motion arises from $\vec{E}\times\vec{B}$ drifts).

An energy principle which is sufficient, though not necessary, for stability and which overcomes these objections was given by Kruskal and Oberman. This is based on the Boltzmann equation in the limit of small Larmor radius. In this case the appropriate energy integral can be written

$$\delta W_{k0} = \delta W_D - \int d\tau\{2p_\perp q(\vec{\nabla}\cdot\vec{\xi}) + (3p_\parallel - 2p_\perp)q^2\} + I, \tag{6.10}$$

where

$$I = \int d\tau\left\{\sum m_i \iint \frac{B}{v_\parallel} d\mu\, d\epsilon \cdot \left[\mu^2 B^2\left(\frac{\partial f_0}{\partial \epsilon}\right)(\vec{\nabla}\cdot\vec{\xi} - q)^2 - \frac{f^{*2}}{\partial f_0/\partial\epsilon}\right]\right\} \tag{6.11}$$

and

$$\frac{1}{2}v_\parallel^2 = \epsilon - \mu B. \tag{6.12}$$

In these expressions ϵ and μ are again the energy and magnetic moment as in section 5, and f^* is the perturbation in the particle distribution function. The quantity $f_0(\mu, \epsilon, L)$ is the unperturbed particle distribution, and in their derivation of the energy principle Kruskal and Oberman require that

$$\partial f_0/\partial\epsilon < 0. \tag{6.13}$$

The minimization of δW_{k0} has to be carried out over $\vec{\xi}$ and also over f^* subject to certain constraints. The minimization over f^* was carried out by Kruskal and Oberman but we will have no need of this in the present discussion.

It can be shown that the minimum of δW_{k0} is independent of $\xi_\parallel$ as it should be, so that $\xi_\parallel$ can be taken to be zero.

As before, at sufficiently low β we need only consider displacements which satisfy

$$\vec{\nabla}\times(\vec{\xi}\times\vec{B}) = 0 \tag{6.14}$$

so that Eqs. (6.4) are again valid. However as $\vec{\xi}$ is now perpendicular to $\vec{B}$ a further simplification can also be obtained. For (6.14) implies that

$$\vec{\xi}\times\vec{B} = \vec{\nabla}\varphi \tag{6.15}$$

and so $\vec{\xi}$ can now be written

$$\vec{\xi} = \vec{\xi}_\perp = \vec{B}\times\vec{\nabla}\varphi/B^2, \tag{6.16}$$

whence

$$\vec{\nabla}\cdot\vec{\xi} = -2\vec{\xi}\cdot\vec{\nabla}B/B. \tag{6.17}$$

With the aid of Eqs. (6.4) and (6.17) the energy integral may be reduced to

$$\delta W_{k0} = \int d\tau\{(2p_\perp - Bp_\perp')S^2\} + \int d\tau\left\{\sum m_i \iint \frac{B}{v_\parallel}\, d\mu\, d\epsilon \right.$$
$$\left. \times \left[\mu^2 B^2\left(\frac{\partial f_0}{\partial \epsilon}\right)S^2 - \frac{f^{*2}}{\partial f_0/\partial \epsilon}\right]\right\}. \tag{6.18}$$

This can be further simplified, for

$$p_\perp = \sum m_i \iint \frac{\mu B^2}{v_\parallel} f_0 d\mu\, d\epsilon, \tag{6.19}$$

and since

$$1/v_\parallel = \partial v_\parallel/\partial\epsilon, \tag{6.20}$$

a partial integration leads to

$$p_\perp = -\sum m_i \iint \mu B^2 v_\parallel \left(\frac{\partial f_0}{\partial \epsilon}\right) d\epsilon\, d\mu. \tag{6.21}$$

Then if $p \equiv p_\perp(B)$ differentiation with respect to B gives

$$Bp_\perp' = 2p_\perp + \sum m_i \iint \frac{\mu^2 B^3}{v_\parallel}\left(\frac{\partial f_0}{\partial \epsilon}\right) d\epsilon\, d\mu \tag{6.22}$$

and, using this result, the energy integral is finally reduced to

$$\delta W_{k0} = -\int d\tau \sum m_i \iint \frac{B}{v_\parallel} d\mu\, d\epsilon \left\{\frac{f^{*2}}{\partial f_0/\partial \epsilon}\right\}, \tag{6.23}$$

which is certainly positive if $(\partial f_0/\partial\epsilon) < 0$, a condition which is in any case required for the present energy principle to be valid.

According to the small Larmor radius theory, then, equilibria of the class (6.6) are stable if their corresponding particle distributions satisfy

$$\partial f_0/\partial\epsilon < 0. \tag{6.24}$$

Now, the specific examples (5.4) correspond to the particle distributions (5.5) and so are stable if

$$(\partial/\partial\epsilon)(\mu B_0 - \epsilon)^{n-\frac{3}{2}} < 0, \tag{6.25}$$

that is if $n > \frac{3}{2}$. In this case, therefore, the two energy principles lead to the same criterion.

7. DIRECT PROOF OF STABILITY

A direct proof of the stability of these special equilibria with $p_\perp = p_\perp(B)$ etc. can be developed by extending the "generalized entropy" arguments given in Dr. Rosenbluth's article (these Proceedings).

Let us consider a general particle motion in which the magnetic moment of a particle is invariant (as in small Larmor radius theory), then a general constant of the motion constructed from individual particle constants is

$$S = \int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau\, G(f, \mu). \tag{7.1}$$

Now consider a distribution function $f = f_0 + \delta f$, where f_0 is the initial equilibrium whose stability we want to discuss. Then we can write

$$\delta S = 0 = \int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau \left\{ G'(f_0, \mu)\delta f + G''(f_0, \mu)\frac{(\delta f)^2}{2} + \dots \right\}, \tag{7.2}$$

where

$$G'(f, \mu) \equiv \partial G/\partial f.$$

Now the equilibria we are considering have the property that $p_\perp$ and $p_\parallel$ are functions of B only and satisfy the parallel equilibrium equation. It can be seen from Eqs. (4.5) and (4.6) that such equilibria correspond to particle distribution functions which depend only on μ and ϵ (i.e. $f(\epsilon, \mu, L)$ is independent of the particular flux line L). For these equilibria, therefore, the function G can be chosen so that

$$G'(f_0, \mu) \equiv \epsilon \tag{7.3}$$

(at least if $\partial f_0/\partial\epsilon$ is monotonic) and with this choice for G Eq. (7.2) becomes

$$\int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau\, (\epsilon\delta f) = -\int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau\, \frac{(\delta f)^2}{2(\partial f_0/\partial\epsilon)} + \dots, \tag{7.4}$$

which may be written

$$\delta K = -\int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau\, \frac{(\delta f)^2}{2(\partial f_0/\partial\epsilon)} + \dots, \tag{7.5}$$

where K is the total kinetic energy of the particles,

$$K = \int \frac{B}{v_\parallel} d\mu\, d\epsilon\, d\tau\, (\epsilon f).$$

If now $\partial f_0/\partial\epsilon < 0$, it is clear that to second order in δf, $\delta K > 0$ so that any change δf in f around f_0 will increase the kinetic energy. Furthermore if the equilibrium has no electric fields and is of such low β that the magnetic field is a vacuum field, then any perturbations can also only increase the field energies. As the total energy is constant it is clear that δf cannot grow indefinitely and in particular cannot grow exponentially. Therefore the system is stable.

Thus it has been shown that any low-β equilibrium with f_0 a function only of μ and ϵ is stable against all perturbations in which the magnetic moment is an invariant. This is certainly sufficient to demonstrate stability against hydromagnetic motions, and may also ensure stability against certain micro-instabilities.

8. CONCLUSIONS

The existence of a set of closed magnetic isobars in a "magnetic well" has enabled us to construct a special class of confined plasma distributions, those with $p_\perp$ and $p_\parallel$ functions of B alone, which satisfy the conditions for equilibrium. These equilibria are stable against interchanges according to both the double-adiabatic energy principle and the more complete small-Larmor-radius theory. A direct proof of stability against all motions in which the magnetic moment of a particle is an invariant has also been given.

It is easily shown that these equilibria have the property that $j_\parallel = 0$, which ensures that they are also stable against several forms of "drift" instability. One concludes therefore that "magnetic wells" do indeed offer the possibility of stable plasma confinement.

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II. GENERAL EQUILIBRIA*

1. INTRODUCTION

In the previous part we considered a special class of equilibria in magnetic wells; those with $p_\perp \equiv p_\perp(B)$, $p_\parallel \equiv p_\parallel(B)$ and $Bp_\parallel' = p_\parallel - p_\perp$. Alternatively we can say that these special equilibria are those which correspond to particle distribution functions $f(\mu, \epsilon, L)$ which are independent of L so that

* This part is based on the authors paper in Phys. Fluids 7, see Bibliography to part II.

$f \equiv f(\mu, \epsilon)$. In this part we will examine the general equilibria in magnetic wells, or indeed in any mirror trap. For the discussion it is convenient to use the "guiding centre drift" description of a low-β plasma instead of the fluid description which we used throughout much of the first part.

First we note that in part I it was convenient to employ an approach in which equilibrium and stability could be discussed first; only later did one discuss confinement. In other words instead of taking a confined (localized) distribution and applying a stability criterion one considered stable distributions, with $\vec{p}$ expressed as a function of $|\vec{B}|$, and then applied a confinement criterion. The advantage of this approach is that the actual form of the magnetic field only enters the problem through the confinement criterion, and furthermore it enters in a very simple way, e.g. in part I it reduced to the question "do the surfaces $|\vec{B}|$ = constant form a closed nested set?". In the present lecture we adopt a similar viewpoint but we now employ the "particle-drift" description of a low-β plasma instead of a fluid description.

The instantaneous drift velocity is well known, but a more relevant concept is the average drift motion over several oscillations between mirrors. First we will show that the equations for this average motion can be put in a simple form if the appropriate coordinates are used; this is because the adiabatic invariants $\mu = mv_{\perp}^2/B$ and $J = \oint v_{\parallel} dS$ are constant during the drift motion. The key to the problem is to note that if one specifies the line of force on which the particle is moving (by coordinates α, β) and also specifies μ and J, then its energy is determined. This allows one to regard the energy of a particle, not as an independent variable but as a known function of μ, J, α and β. Then this function $K(\mu, J, \alpha, \beta)$ plays the role of a Hamiltonian.

As a consequence of this, any equilibrium distribution can be expressed in the form

$$F_{eq}/(\mu, J, \alpha, \beta) \equiv F[\mu, J, K(\alpha, \beta, \mu, J)], \tag{1.1}$$

where one is not regarding K as a variable but as a known function of α, β, μ and J.

Next we still study stability and show that once the general equilibrium has been expressed in the special form (1.1) we can obtain a simple criterion for stability against interchanges (i.e. motions in which flux tubes are interchanged and the magnetic field is unaltered - these are the most important of the possible instabilities at low-β). In fact the system is stable against interchanges if

$$\partial F(\mu, J, K)/\partial K < 0. \tag{1.2}$$

2. AVERAGE GUIDING CENTRE DRIFT

When the Larmor radius is small compared to the scale of the variations in magnetic field the motion of a particle can be regarded as a rapid gyration about a guiding centre. In the course of this motion the magnetic moment $\mu = mv_{\perp}^2/B$ is a constant and as a result the particle is confined, between magnetic mirrors, to the region where $\mu B < E$ (where E is the particle energy).

In this event the guiding centre itself oscillates rapidly along a line of force between the two mirror points and at the same time it "drifts" more slowly across the field. The instantaneous drift velocity is well known, being given by

$$v_d = \frac{c\vec{n}}{B} \times \left\{ \vec{\nabla}\phi + \frac{\mu\vec{\nabla}B}{e} + \frac{m}{e} v_{\shortparallel}^2 \frac{\partial\vec{n}}{\partial S} \right\}, \tag{2.1}$$

where $\vec{n}$ is the unit vector along $\vec{B}$. In view of the rapid oscillation along the line of force this instantaneous drift velocity is of less significance than the average guiding centre drift over a period of the oscillation between mirrors. The equations for this average drift can be put in a particularly simple (canonical) form which was first given by Northrop and Teller. They considered the drift process in detail and actually constructed the average of (2.1) over the period of an oscillation between mirrors - a rather lengthy procedure. However, the canonical equations can be derived directly, without recourse to (2.1), by a canonical transformation.

The simplicity of the final form of the drift equations is made possible by using a representation of the magnetic field which allows the field lines to be used as one element of a co-ordinate grid. One writes

$$\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta ,$$

then clearly α and β are constant along a field line so that they can be regarded as the co-ordinates of that field line. More specifically if we consider any surface S cut by field lines and draw on this surface the lines α = const., β = const., then these lines form a co-ordinate grid which the lines of force are located by the values of α and β at their intersection with S. The scale of the co-ordinates α and β can be chosen so that the flux through any part ΔS of the surface S is numerically equal to

$$\iint_{\Delta S} d\alpha \, d\beta .$$

In terms of this representation of the field, the vector potential can be written

$$\vec{A} = \alpha\vec{\nabla}\beta .$$

Once the field lines have been specified by the (α, β) co-ordinate system, any point P in space can be located by coordinates (α, β, χ), where α and β are the co-ordinates of the field line on which P lies and χ is the magnetic potential along that field line from P to the reference surface S, i.e.

$$\chi = \int_S^P \vec{B} \cdot \overrightarrow{dS} .$$

We now return to the problem of describing the average guiding centre motion. As the magnetic moment of the particle is constant, the guiding centre moves as if it were a particle of charge e, mass m, and magnetic moment $\mu\vec{n}$. The Lagrangian for such a particle is

$$\mathcal{L} = \frac{1}{2}mv^2 + \left(\frac{e}{c}\right)\vec{v}\cdot\vec{A} - e\phi - \mu B, \tag{2.2}$$

and the total energy is

$$E = \frac{1}{2}mv^2 + e\phi + \mu B\,. \tag{2.3}$$

We now express the Lagrangian in the (α, β, χ) co-ordinate system and use the corresponding $\alpha\vec{\nabla}\beta$ representation of $\vec{A}$, then

$$\mathcal{L} = \epsilon + \left(\frac{m\dot{\chi}^2}{2B^2}\right) + \left(\frac{e}{c}\right)\alpha\dot{\beta} - e\phi(\alpha, \beta, \chi) - \mu B(\alpha, \beta, \chi), \tag{2.4}$$

where ϵ is the kinetic energy associated with the transverse drift and is negligible compared to the other terms in (2.4) when the drift velocity is small compared to the actual particle velocity. The conjugate momenta to to α, β and χ are then

$$p_\alpha = 0, \quad p_\beta = \frac{e\alpha}{c}, \quad p_\chi = \frac{m\dot{\chi}}{B^2}, \tag{2.5}$$

and we note that there is, in fact, no momentum conjugate to α, instead $e\alpha/c$ is itself conjugate to β. The Hamiltonian function is then

$$H = (B^2/2m)p_\chi^2 + e\phi(\alpha, \beta, \chi) + \mu B(\alpha, \beta, \chi). \tag{2.6}$$

In order to obtain the equations of motion for the average drift we should solve the equations of motion in the χ direction and then average the transverse $(\dot{\alpha}, \dot{\beta})$ equations over this motion, as was done by Northrop and Teller. However, we can obtain the same result directly if we eliminate χ from the Hamiltonian by an appropriate canonical transformation to action-angle variables. To do this we introduce as a co-ordinate the action conjugate to χ, i.e.,

$$J = \oint p_\chi d\chi = \oint [2m(H - e\phi - \mu B)]^{\frac{1}{2}}\, ds, \tag{2.7}$$

where the integral is along a particular (α, β) field line and is over one period of the oscillation between mirrors.

This equation implicitly defines H as a function of the new variables α, β, J and μ, and also preserves the form of Hamilton's equations of motion. When the Hamiltonian (energy) is expressed in terms of α, β, J, and μ through (2.7) we denote it by K. Then recalling that $e\alpha/c$ is conjugate to β and that $K(\alpha, \beta, J, \mu)$ is now the Hamiltonian function, we can write

$$\dot{\alpha} = -\frac{c}{e}\frac{\partial K(\alpha, \beta, J, \mu)}{\partial \beta}, \tag{2.8}$$

$$\dot{\beta} = \frac{c}{e}\frac{\partial K(\alpha, \beta, J, \mu)}{\partial \alpha} \tag{2.9}$$

and

$$\dot{J} = 0. \tag{2.10}$$

It is important to note that these simple equations are only true when the motion is expressed in the (α, β) co-ordinate system and when K is expressed in terms of α, β, μ and J by means of (2. 7), i. e.

$$J = \oint [2m\{K - e\phi(\alpha, \beta, s) - \mu B(\alpha, \beta, s)\}]^{\frac{1}{2}}\, ds.$$

3. EQUILIBRIUM DISTRIBUTION

Once the equations of motion have been put into the canonical form the construction of the equilibrium distribution is obvious. Let $F[\alpha, \beta, J, \mu, t]$ be the particle density in (α, β, J, μ) space, then

$$-\frac{\partial F}{\partial t} = \frac{\partial (F\dot{\alpha})}{\partial \alpha} + \frac{\partial (F\dot{\beta})}{\partial \beta}, \tag{3.1}$$

and, using the values of $\dot{\alpha}$ and $\dot{\beta}$ given by (2. 8) and (2. 9), a stationary state exists if and only if

$$\frac{\partial K}{\partial \beta}\frac{\partial F}{\partial \alpha} - \frac{\partial K}{\partial \alpha}\frac{\partial F}{\partial \beta} = 0, \tag{3.2}$$

that is if F is a function of α and β only through the quantity $K(\alpha, \beta, J, \mu)$. Any equilibrium can therefore be written

$$F_{eq} = F\{\mu, J, K(\alpha, \beta, \mu, J)\}. \tag{3.3}$$

As μ, J, and K are all constants of the motion this merely says that the equilibrium distribution is a function of the constants of the motion - a well-known result. It should be remembered that $F\{\mu, J, K\}$ is defined so that

$$F\{\mu, J, K(\alpha, \beta, J, \mu)\}\, d\mu\, dJ\, d\alpha\, d\beta \tag{3.4}$$

is the number of particles in the element $(d\mu\, dJ\, d\alpha\, d\beta)$ and not as if

$$F\{\mu, J, K\}\, d\mu\, dJ\, dK \tag{3.5}$$

were the number in $(d\mu\, dJ\, dK)$. The difference arises because of the existence of the surfaces of constant (μ, J, K). The function F is constant over such a surface so that one of the space co-ordinates does not really enter into the specification of F. (This is the general analogue of the fact that for equilibrium distributions in an axisymmetric system the azimuthal angle is redundant.)

4. STABILITY

In a low-β system in which the magnetic field is the vacuum field due to external conductors, "interchanges" of flux tubes are the most important form of instability. Indeed these are the only quasi-hydrodynamic (i.e. adiabatic) instabilities possible at low-β. It is an important result therefore, that a very simple criterion can now be obtained for the stability of equilibria such as (3.3) against these interchanges.

We suppose that the equilibrium is stationary and that there is no electric field in the equilibrium state. Then K is defined as a function of J, μ, α and β by

$$J = \oint [2m\{K - \mu B(\alpha, \beta, s)\}]^{\frac{1}{2}}\, ds. \tag{4.1}$$

An "interchange" motion is one in which particles initially on a given flux tube remain on that flux tube. It results from the "$\vec{E} \times \vec{B}$" drift associated with an electric field transverse to the magnetic field. We will consider a possible interchange in which particles on a flux tube (α_1, β_1) are interchanged with those on an equivalent flux tube (α_2, β_2). In this motion the invariants μ and J of each particle are conserved, but its energy may alter.

The total energy of the particles on the two flux tubes concerned before the interchange was

$$W_i = \int d\mu dJ\{F(1)K(1) + F(2)K(2)\}, \tag{4.2}$$

where

$$F(1) \equiv F\{\mu, J, K(\mu, J, \alpha_1, \beta_1)\} \tag{4.3}$$

and

$$K(1) \equiv K(\mu, J, \alpha_1, \beta_1), \tag{4.4}$$

and K(2) and F(2) are similarly defined.

After the interchange the particles which were on (α_1, β_1) have moved to (α_2, β_2) and so have energy K(2) and vice versa. The energy after the interchange is therefore

$$W_f = \int d\mu\, dJ\{F(1)K(2) + F(2)K(1)\}. \tag{4.5}$$

The change in energy resulting from the interchange is thus

$$(W_f - W_i) = -\int d\mu\, dJ\{[F(2) - F(1)] \times [K(2) - K(1)]\}. \tag{4.6}$$

It may be noted that we have so far made no restriction that the change $[F(2) - F(1)]$ need be small but we now make the usual assumption that the displacements are infinitesimal and calculate the energy change to second order in displacement. If the displacement of the flux tubes is measured by $\delta\alpha$ and $\delta\beta$ we have

$$\delta^2 W = -\int d\mu\, dJ \left(\frac{\partial F}{\partial\alpha}\,\delta\alpha + \frac{\partial F}{\partial\beta}\,\delta\beta\right) \times \left(\frac{\partial K}{\partial\alpha}\,\delta\alpha + \frac{\partial K}{\partial\beta}\,\delta\beta\right). \tag{4.7}$$

However, since F depends on α and β only through K, this becomes

$$\delta^2 W = -\int d\mu\, dJ \left(\frac{\partial K}{\partial\alpha}\,\delta\alpha + \frac{\partial K}{\partial\beta}\,\delta\beta\right)^2 \left(\frac{\partial F}{\partial K}\right)_{\mu J}. \tag{4.8}$$

It is now apparent that $\delta^2 W$ must be positive for all $\delta\alpha, \delta\beta$ if

$$\partial F/\partial K < 0,$$

therefore a criterion which is sufficient for stability against "interchanges" is

$$\left(\frac{\partial F}{\partial K}\right)_{\mu J} < 0 \tag{4.9}$$

for all μ, J, K.

Criteria which are both necessary and sufficient can be obtained in terms of the appropriate averages of $\partial F/\partial K$. Thus if

$$\lambda_{\alpha\alpha} \equiv \int d\mu\, dJ \left(\frac{\partial K}{\partial\alpha}\right)^2 \left(\frac{\partial F}{\partial K}\right), \tag{4.10'}$$

$$\lambda_{\beta\beta} \equiv \int d\mu\, dJ \left(\frac{\partial K}{\partial\beta}\right)^2 \left(\frac{\partial F}{\partial K}\right), \tag{4.10''}$$

$$\lambda_{\alpha\beta} \equiv \int d\mu\, dJ \left(\frac{\partial K}{\partial\alpha}\right)\left(\frac{\partial K}{\partial\beta}\right)\left(\frac{\partial F}{\partial K}\right), \tag{4.10'''}$$

then a necessary and sufficient set of conditions is

$$\lambda_{\alpha\alpha} < 0, \quad \lambda_{\beta\beta} < 0, \quad [\lambda_{\alpha\beta}]^2 < \lambda_{\alpha\alpha}\lambda_{\beta\beta}. \tag{4.11}$$

The simple condition (4.9) demands that F should decrease with increasing K while confinement of plasma requires that F should decrease toward the periphery of the system so that (4.9) and confinement are compatible only if K has the general form of a "potential-well" in the (α, β) space, that is if $K(\alpha, \beta)$ possesses a minimum within the region of interest. If the magnetic field itself possesses a minimum then $K(\alpha, \beta)$ will possess a minimum for a wide range of μ, J so that many classes of stable equilibria can be constructed in these "minimum-B" fields. Among these are the equilibria discussed in part I which do indeed satisfy (4.9). In fact the special equilibria discussed then correspond to distribution functions $F(\mu, J, K)$ which are chosen to be independent of J and for which $\partial F / \partial K < 0$. Such distributions represent confined plasma only in fields which possess a minimum in B.

It must again be emphasized that the simplicity of the result (4.9) arises solely from the correct choice of variables – it is only correct when F is expressed in the form

$$F\{\mu, J, K(\mu, J, \alpha, \beta)\}.$$

No such simple expression could be obtained if, for example, one had expressed F in the more usual variables (μ, K, x).

5. EXAMPLE OF METHOD

The actual form of the magnetic field has not been mentioned in the theory given above. This is because it enters the problem only in the determination of K. As K is defined by the single integral (4.1) it is not difficult to compute K once the field is given, and an example is illustrated below. (This was compiled by F. M. Larkin at Culham Laboratory.) In this example the field is an elementary form of the configuration used in Ioffe's stabilized mirror experiments and is produced by two circular coils and four infinite straight conductors, Fig. 1. The coils are of radius R and separation 2R and carry a current I/2. The straight conductors are distant $R\sqrt{2}$ from the common axis of the two circular coils and adjacent conductors carry a current I in opposite directions. The field is thus a superposition of an orthodox mirror and an $l = 2$ multipole cusp.

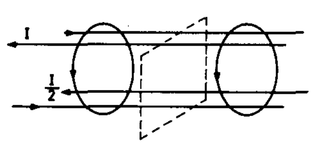

Fig. 1

Coil arrangement.

The (α, β) plane for this calculation was chosen to be the midplane of the system, perpendicular to the common axis of the circular coils; then, because of the symmetry of the conductors, the $K(\alpha, \beta, J, \mu)$ contours have eightfold symmetry (Fig. 2, Table I; Figs. 3 and 4). Only one quadrant of the (α, β) plane is shown. The figures cover only the central part of the α, β plane out to a radius of about $\frac{1}{3}R$.

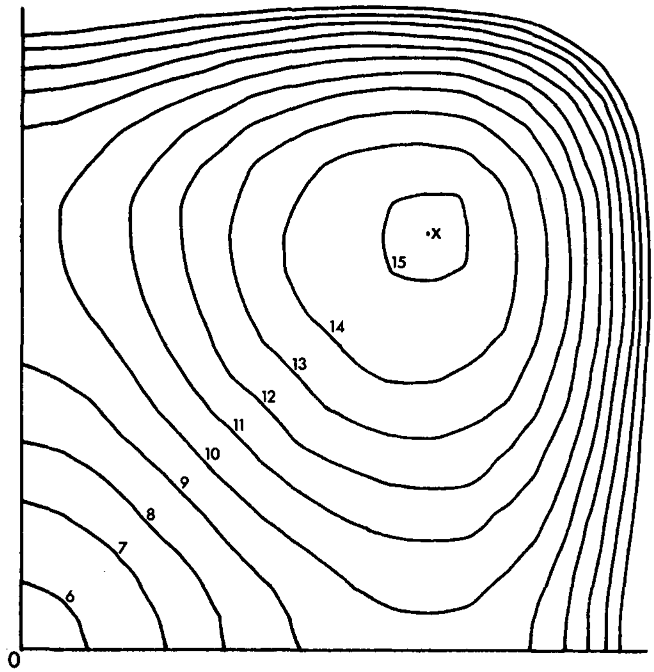

Fig. 2

Constant K/μ contours for $(J/\mu^{\frac{1}{2}}) = 0.616$.
(see Table I)

TABLE I

CONTOURS CORRESPONDING TO VALUES OF K/μ IN ARBITRARY UNITS

Contour	1	3	5	7
K/μ	0.492	0.496	0.500	0.504
Contour	9	11	13	15
K/μ	0.508	0.512	0.516	0.520

One may interpret these diagrams somewhat as one interprets contour heights on a geographic map. For example one may note such items as the following: (i) As the K surfaces are also particle drift surfaces one sees immediately where the particles drift, but also from Eqs. (1.1) and (1.2) one gets a picture of the speed of drift from the separation between contours (just as one pictures the gradient from the separation between height contours on a map). (ii) As the K surfaces are surfaces of constant F_{eq} they can also be visualized as density contours for this function. One can also see some more important points concerning stability. Thus (iii) the example

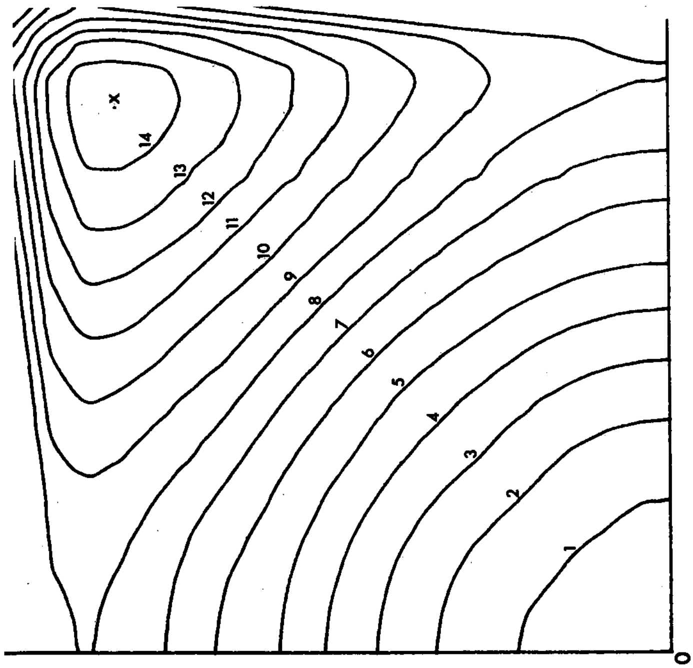

Fig. 4
Constant K/μ contours for $(J/\mu^{\frac{1}{2}}) = 0.305$.

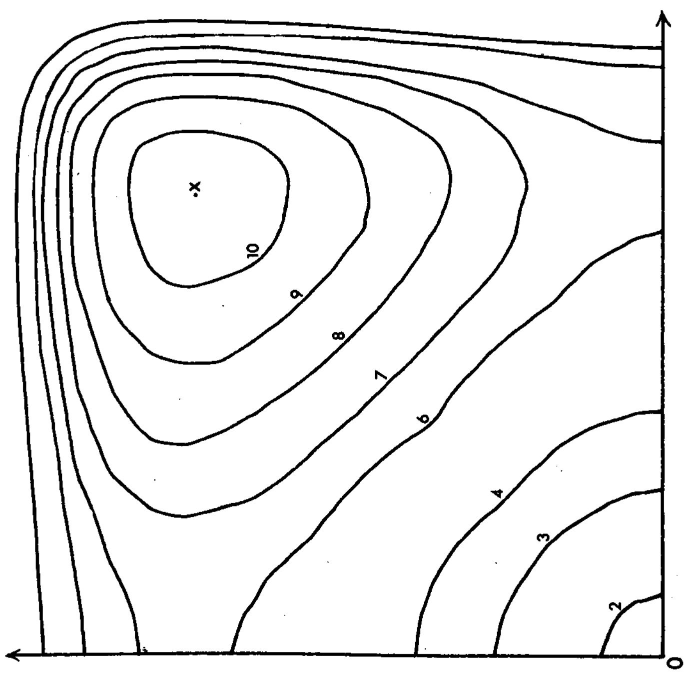

Fig. 3
Constant K/μ contours for $(J/\mu^{\frac{1}{2}}) = 0.464$.

shown has a minimum in K at the centre of the system so that confined distributions stable against interchange by (4. 9) can be set up in this region of the field. However one can also see that the region of such stable confinement is small (remember the diagram shows only the central part of the system out to about one third of a coil radius). (iv) There are also other closed K-contours centred about a point X on the 45° axis (this is the axis passing through one of the straight conductors) so that other confined equilibria exist in this region. However, as the point X correponds to a maximum rather than a minumum in K, such equilibria cannot satisfy the criterion (4. 9).

The existence of a minimum in K, which ensures the existence of stable confined equilibria, is a general feature of fields in which $|\vec{B}|$ itself possesses a minimum as in the present example.

6. CONCLUSIONS

It is clear from the example that the discussion of equilibrium, stability, and confinement of low-β plasma in adiabatic mirror traps, is, indeed, much simplified if the problem is approached in the way described in this part. Far reaching results can often be obtained with little effort. The method involves using the field lines themselves as co-ordinates (α, β) and expressing the particle distribution function in the phase space of α, β, μ, and J, where μ and J are the two adiabatic invariants. The energy K is not treated as an independent variable but is defined by

$$J = \oint [2m\{K - \mu B(\alpha, \beta, s)\}]^{\frac{1}{2}} ds. \tag{6. 1}$$

(Note that this is the reverse of the usual procedure, in which K is regarded as a variable and J is defined by (6. 1).)

In terms of these variables the equilibrium distribution function is of the form

$$F = F\{\mu, J, K(\alpha, \beta, \mu, J)\} \tag{6. 2}$$

and a sufficient condition for stability is

$$\left(\frac{\partial F}{\partial K}\right)_{\mu J} < 0. \tag{6. 3}$$

Necessary and sufficient conditions are given by (4. 11).

The confinement (localization) of the distribution is determined by the topology and location of the K = const. contours which are easily computed. Confined equilibria satisfying the stability criterion (6. 3) can always be found if the $K(\alpha, \beta)$ function possesses a minimum in the region of interest. This is the case if $|\vec{B}|$ itself possesses a minimum. Thus in a "minimum-B" field one always has "minimum-K". However one can have a minimum in K (for many values of J, μ) without having a minimum in B.

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III. STABILITY AT FINITE PRESSURE *

1. INTRODUCTION

In the preceding parts we have discussed the stability of low-β plasma in magnetic wells. Now we turn to the discussion of stability at finite-β. That is to say: we know that plasma in magnetic wells is stable at low-β, how far can we raise the plasma pressure and still retain stability?

As in the low-β case it is convenient to discuss first the stability of the special class of equilibria with $p_\perp = p_\perp(B)$, $p_\parallel = p_\parallel(B)$, $Bp_\parallel' = p_\parallel - p_\perp$.

First we should note that we can have equilibria of this form at finite β and not merely at low-β (see Bibliography to part III, paper by Northrop and Whiteman). Furthermore these equilibria retain the property that $j_\parallel \equiv 0$. Unlike the low β case, however, (when we could immediately construct the special equilibria merely by calculating the vacuum fields) we have still to solve a "self-consistent" problem in order to find the field. Nevertheless this is still much easier than the general equilibrium calculation.

We shall discuss the question of stability by means of the usual energy principle, that is we examine the sign of the potential energy $\delta W(\xi, \xi)$ involved in a small arbitrary displacement ξ. If this is negative for any ξ the system is unstable: if it is always positive the system is stable.

2. THE ENERGY PRINCIPLE

There are several possible forms for the energy integral δW. The simplest of these, the scalar pressure MHD principle, is not applicable since there can be no equilibrium with scalar pressure in magnetic wells. The Double-Adiabatic (or Chew-Godlberger-Low) Energy principle can be used but a better form is that given by Kruskal and Oberman which is exact in the limit of small Larmor radius. This is the form we shall use in this lecture.

The appropriate energy functional for the Kruskal-Oberman theory is

$$\Delta W = \frac{1}{2}\int \delta W d\tau,$$

---

* This part is based on work by R. J. Hastie and the author.

where

$$\delta W = \vec{Q}^2 - \vec{j}\cdot\vec{Q}\times\vec{\xi} + (2p_\perp + c)(\vec{\nabla}\cdot\vec{\xi} - q)^2 + (\vec{\xi}\cdot\vec{\nabla}p_\perp)(\vec{\nabla}\cdot\vec{\xi} - q)$$
$$+ (\vec{\xi}\cdot\vec{\nabla}p_\parallel)q - (p_\parallel - p_\perp)[\vec{n}\cdot(\vec{a}\cdot\vec{\nabla})\vec{\xi} + \vec{a}\cdot(\vec{n}\cdot\vec{\nabla})\vec{\xi} - q^2 - q\vec{\nabla}\cdot\vec{\xi}]$$
$$- \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \epsilon}\langle v_\parallel^2 q + \mu B(\vec{\nabla}\cdot\vec{\xi} - q)\rangle^2 d\mu\, d\epsilon, \quad (2.1)$$

where

$$\vec{Q} = \vec{\nabla}\times(\vec{\xi}\times\vec{B}), \qquad \vec{a} = (\vec{n}\cdot\vec{\nabla})\vec{\xi} - (\vec{\xi}\cdot\vec{\nabla})\vec{n},$$

and

$$q = \vec{n}\cdot\vec{a}. \quad (2.2)$$

The displacement $\xi_\parallel$, parallel to $\vec{B}$, can be taken to be zero, since δW can be shown to be independent of this component of $\vec{\xi}$. Henceforth therefore $\vec{\xi}$ always denotes a vector which is perpendicular to $\vec{B}$. The average appearing in (2.1) is defined by

$$\langle g\rangle = \int \frac{dl}{v_\parallel} g \Big/ \int \frac{dl}{v_\parallel} \quad (2.3)$$

and

$$C = \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \epsilon}(\mu B)^2 d\mu\, d\epsilon, \quad (2.4)$$

where $f(\mu, \epsilon, \vec{x})$ is the particle distribution expressed in terms of the magnetic moment μ, the energy ϵ, and the parallel velocity $v_\parallel$ is given by $\frac{1}{2}v_\parallel^2 = \epsilon - \mu B$. We shall restrict ourselves to distributions for which $\partial f/\partial\epsilon < 0$; this is in any case necessary for the derivation of the energy principle given by Kruskal and Oberman. A great simplification of the ensuing algebra can be achieved if we first transform δW into a new form. We introduce the notation

$$\frac{\vec{\xi}\cdot\vec{\nabla}B}{B} \equiv s$$

(because we know that $\vec{\nabla}B$ is an important factor in stability) and express as much as possible of δW in terms of s. Then δW becomes

$$\delta W = \vec{Q}_\perp^2\left(1+\frac{p_\perp - p_\parallel}{B^2}\right) + Q_\parallel^2\left(1+\frac{2p_\perp + C}{B^2}\right) - j_\parallel(\vec{n}\cdot\vec{Q}_\perp \times \vec{\xi})\left(1+\frac{p_\perp - p_\parallel}{B^2}\right)$$
$$+ q[\vec{\xi}\cdot\vec{\nabla}p_\parallel + (p_\perp - p_\parallel)s] - \left[\vec{\xi}\cdot\vec{\nabla}p_\perp - (2p_\perp + C)s\right]\left(\frac{2Q_\parallel}{B} + s\right)$$
$$- \sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \epsilon}\left\langle v_\parallel^2 q - \mu B\left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu\, d\epsilon . \tag{2.5}$$

We will also need two relations (derived from the equilibrium equations and the definition of C) for the gradients of $p_\perp$ and $p_\parallel$ along the lines of force. From the earlier parts we know that in equilibrium

$$\frac{\partial p_\parallel}{\partial s} = -\frac{(p_\perp - p_\parallel)}{B}\frac{\partial B}{\partial s} \tag{2.6}$$

and by taking the derivative of $p_\perp$ along the lines of force, (using the expression given in the first part for $p_\perp$ in terms of $f(\mu, \epsilon, x)$) we find

$$\frac{\partial p_\perp}{\partial s} = \frac{(C + 2p_\perp)}{B}\frac{\partial B}{\partial s}. \tag{2.7}$$

3. STABILITY OF SPECIAL EQUILIBRIA

Using the form (2.5) for δW it is now easy to determine the stability of the special equilibria $p_\perp = p_\perp(B)$, etc. For these equilibria it is easy to show that (2.6) and (2.7) lead to

$$\vec{\xi}\cdot\vec{\nabla}p_\parallel + (p_\perp - p_\parallel)s = 0,$$
and
$$\vec{\xi}\cdot\vec{\nabla}p_\perp - (2p_\perp + C)s = 0, \tag{3.1}$$

so that two of the terms in δW vanish identically. Furthermore the special equilibria possess the property that $j_\parallel \equiv 0$; hence δW reduces to

$$\delta W = \vec{Q}_\perp^2\left(1+\frac{p_\perp - p_\parallel}{B^2}\right) + Q_\parallel^2\left(1+\frac{2p_\perp + C}{B^2}\right)$$
$$- \sum m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \epsilon}\left\langle v_\parallel^2 q - \mu B\left(\frac{Q_\parallel}{B} + s\right)\right\rangle^2 d\mu\, d\epsilon . \tag{3.2}$$

It is apparent that sufficient conditions for stability are

$$1 + \frac{p_\perp - p_\parallel}{B^2} > 0, \tag{3.3}$$

and

$$1+\frac{2p_\perp + C}{B^2} > 0. \tag{3.4}$$

However these conditions are also necessary for stability because it is always possible to find a displacement ξ for which the term $Q_\perp^2$ dominates all others and for this displacement δW would be negative unless (3.3) were satisfied. Similarly one can find another displacement for which $Q_\parallel^2$ is the dominant term and so (3.4) is also necessary. Hence (3.3) and (3.4) are exact, necessary and sufficient conditions for the stability of the special equilibria.

The first of these conditions is the "firehose" stability criterion and is rarely of interest in magnetic wells. We shall not consider it any further. The second condition (3.4) is related to the "mirror" instability criterion but a more useful form can be obtained by using (2.7) to write it as

$$B+\frac{dp_\perp}{dB} > 0. \tag{3.5}$$

In this form it clearly represents a restriction on the maximum pressure gradient we are allowed in magnetic wells and so by integration we can find the maximum pressure we are allowed. This is

$$p_\perp^{max} = \frac{(B_1^2 - B_0^2)}{2}, \tag{3.6}$$

where B_1 is the field strength on the plasma boundary and B_0 the field strength at the "minimum-B" point of the well. In other words the maximum pressure is equal to the depth of the well if this is measured in terms of its magnetic pressure.

4. STABILITY OF GENERAL EQUILIBRIA

Now we turn to the stability of general equilibria at finite-β. Here we cannot hope for exact results so we proceed immediately to an approximation – the "shallow-well" approximation.

To explain the idea let us first suppose that we were to expand δW in power of β as

$$\delta W = \delta W_0 + \beta \delta W_1 + \beta^2 \delta W_2 + \dots , \tag{4.1}$$

and then, if we examine the minimum of δW, we find that min. $\delta W_0 = 0$ and min. $\delta W_1 > 0$. At sufficiently small β, terms in β^2 etc. are negligible and the system is stable. Instabilities can arise when β is large enough for succeeding terms $\beta^2 \delta W_2$, etc. to become comparable with $\beta \delta W_1$, and one method of determining the critical β might be to evaluate δW_2. However, there are practical and logical objections to this. For example, if $\beta^2 \delta W_2$ is comparable with $\beta\delta W_1$, then one must expect that $\beta^3\delta W_3$ is also comparable with $\beta\delta W_1$, and the β-expansion is apparently invalid.

This objection can be overcome, and the problem much simplified if the expansion of δW is made not just in the single small quantity β but in several small quantities, which are then grouped appropriately. One can then obtain an expansion of δW in which stability is again determined by the sign of the first non-zero term in the expansion, rather than by comparison of several terms.

To perform this sort of expansion we regard the magnetic field as being made up of a number of constituents. The basic zero-order magnetic field is a uniform field $\vec{B}_0$ in the z-direction and a magnetic well is created by the addition of a "mirror" component $\vec{B}_m$ and a "stabilizing" component $\vec{B}_s$. This terminology is chosen because the superposition of a mirror field and a multipole (stabilizing) field is a well known way of producing the desired form of minimum-B field. However the significant properties of the component which we call "mirror" and "stabilizing" are that the "mirror" component is principally parallel to $\vec{B}_0$ while the "stabilizing" component is purely perpendicular to $\vec{B}_0$. A further contribution to the field is produced by the plasma itself and is denoted by $\vec{B}_\beta$ so that

$$\vec{B} = \vec{B}_0 + \vec{B}_m + \vec{B}_s + \vec{B}_\beta, \tag{4.2}$$

where $\vec{B}_m$, $\vec{B}_s$, $\vec{B}_\beta$ are all small compared to $\vec{B}_0$. This approximation corresponds to considering the stability of plasma in a "shallow" magnetic well. The pressure tensor

$$\vec{\vec{P}} = p_\perp \vec{\vec{I}} + (p_\parallel - p_\perp)\vec{n}\,\vec{n} \tag{4.3}$$

is likewise treated as a small quantity compared to the magnetic pressure $B_0^2/2$, and we therfore have to consider a number of small quantities

$$\frac{B_m}{B_0} = \mu, \quad \frac{B_s}{B_0} = \sigma, \quad \frac{B_\beta}{B_0} = \beta, \quad \frac{2p_\perp}{B_0^2} = \beta_\perp, \quad \frac{2p_\parallel}{B_0^2} = \beta_\parallel . \tag{4.4}$$

In principle we should now expand δW in powers of all these small quantities and then group terms together to obtain a stability criterion. However it is more convenient to do the grouping first, by attributing relative orders of magnitude to the small quantities σ, μ, β, $\beta_\perp$, $\beta_\parallel$ in terms of a single expansion parameter λ and then to expand in powers of this single quantity.

Our first task is therefore to assign the relative orders of the small quantities. The appropriate ordering is that in which all the relevant quantities appear in the eventual stability criterion, but one can anticipate this ordering as follows. The earlier work shows that $\vec{\nabla}|\vec{B}|$ plays an important role in determining stability at low β, so that we want $\vec{B}_s$, $\vec{B}_m$, $\vec{B}_\beta$ all to contribute to $|\vec{B}|$ in the same order in λ. Because $\vec{B}_s$ is perpendicular to $\vec{B}_0$ it contributes to $|\vec{B}|$ only as $\vec{B}_s^2$ and must therefore be chosen of lower order in λ than $\vec{B}_m$ or $\vec{B}_\beta$. In fact the appropriate ordering of the field components is

$$B_s \approx \lambda B_0, \quad B_m \approx \lambda^2 B_0, \quad B_\beta \approx \lambda^2 B_0. \tag{4.5}$$

The remainder of the ordering is determined by consideration of the equilibrium equations and one finds that

$$j_\perp \approx \lambda^2, \quad j_\parallel \approx \lambda^4, \quad p_\perp \approx \lambda^2, \quad p_\parallel \approx \lambda^4. \tag{4.6}$$

We also have to attribute an order to the non-fluid terms such as C. This can be done through Eq. (2.7) which indicates that as $\partial p_\perp/\partial s$ and $\partial B/\partial s$ are both of second order we must regard C as a zero-order quantity. Finally the displacement $\vec{\xi}$ which minimizes δW will depend on λ so it too must be expanded in λ and so

$$\xi = \xi_0 + \lambda\xi_1 + \lambda^2\xi_0 + \dots \quad . \tag{4.7}$$

All that remains to be done now is to expand all the equilibrium quantities which appear in δW in powers of λ, taking note of the orders of the various terms, and then to collect up terms in λ^0, λ^1, λ^2 etc. We find that it is necessary to go to fourth order in λ which leads to a large amount of algebra - that is why it is most important to start with the form (2.5) for δW rather than (2.1). Using (2.5) many terms do not contribute until λ^4 order so the calculation of lower orders is greatly simplified.

We find, for example, that in zero order

$$\delta W_0 = [Q_\perp^0]^2 + \left(1 + \frac{C}{B_0^2}\right)[Q_\parallel^0]^2 - \sum m_i \iint \frac{B_0}{v_\parallel} \frac{\partial f}{\partial \epsilon} \mu^2 \langle Q_\parallel^0 \rangle^2 \, d\mu \, d\epsilon \,. \tag{4.8}$$

Because $\dfrac{\partial f}{\partial \epsilon} < 0$, this is certainly non-negative if

$$\left(1 + \frac{C}{B_0^2}\right) > 0. \tag{4.9}$$

However, even if this condition is satisfied δW_0 can still be minimized to zero by displacements such that,

$$\vec{Q}_0 = \vec{\nabla} \times (\vec{\xi}_0 \times \vec{B}_0) = 0 \,. \tag{4.10}$$

Hence (4.9) is necessary for stability but it is not sufficient. To get sufficient conditions we must proceed to higher orders. We find then that $\delta W_1 = 0$ and

$$\delta W_2 = [\vec{Q}_\perp^1]^2 + \left(1 + \frac{C}{B_0^2}\right)[Q_\parallel^1]^2 - \sum m_i \iint \frac{B_0}{v_\parallel} \mu^2 \frac{\partial f}{\partial \epsilon} \langle Q_\parallel^1 \rangle^2 d\mu \, d\epsilon . \tag{4.11}$$

Again, provided (4.9) is satisfied, this is non-negative, but it can still be made zero so we must continue further. Then $\delta W_3 = 0$ and

$$\delta W_4 = \vec{Q}^2 - \left(2\frac{Q_\parallel}{B} + s\right)(\vec{\xi}_0 \cdot \vec{\nabla} p_\perp) + C\left(\frac{Q_\parallel}{B} + s\right)^2$$

$$-\sum m_i \iint \frac{B_0^3}{v_\parallel}\frac{\partial f}{\partial \epsilon}\mu^2 \left\langle \frac{Q_\parallel}{B} + s \right\rangle^2 d\mu\, d\epsilon\,, \tag{4.12}$$

where the subscripts on $\vec{Q}_2 \equiv \vec{Q}$ and $s_2 \equiv s = \dfrac{\vec{\xi} \cdot \vec{\nabla} B}{B_0}$ have been suppressed.

This expression (4.12) leads to several sufficient criteria for stability which can be obtained by writing it in the form

$$\delta W_4 = \vec{Q}_\perp^2 + (1 + C/B^2)\left[Q_\parallel + \frac{Cs - \vec{\xi}_0 \cdot \vec{\nabla} p_\perp}{B(1 + C/B^2)}\right]^2$$

$$-\sum_i m_i \iint \frac{B}{v_\parallel}\frac{\partial f}{\partial \epsilon}(\mu B)^2 \left\langle \frac{Q_\parallel}{B} + s \right\rangle^2 d\mu\, d\epsilon$$

$$+\frac{(Cs - \vec{\xi}_0 \cdot \vec{\nabla} p_\perp)[\vec{\xi}_0 \cdot \vec{\nabla}(p_\perp + \frac{1}{2}B^2)]}{B^2 + C}\,. \tag{4.13}$$

For example it is clearly sufficient for stability if the last term in (4.13) is positive, i.e. if

$$[\vec{\xi}_0 \cdot \vec{\nabla}(p_\perp + \tfrac{1}{2}B^2)]\,[C(\vec{\xi}_0 \cdot \vec{\nabla} B) - \vec{\xi}_0 \cdot \vec{\nabla} p_\perp] > 0. \tag{4.14}$$

So that two sufficient conditions are (4.14) and

$$\left(1 + \frac{C}{B^2}\right) > 0. \tag{4.15}$$

For many equilibria these conditions can be further simplified. If we use (2.7) we can write (4.15) as

$$\vec{\nabla}_\parallel B \cdot \vec{\nabla}_\parallel (p_\perp + \tfrac{1}{2}B^2) > 0, \tag{4.16}$$

while if we write the last factor in (4.14) in terms of the distribution function the corresponding criterion can be reduced to the form

$$k^2 \vec{\nabla}_\perp B \cdot \vec{\nabla}_\perp (p_\perp + \tfrac{1}{2}B^2) > 0, \tag{4.17}$$

(where k^2 is a quantity which depends on $f(\mu, \epsilon, x)$ and is positive for cases of interest).

Hence for general equilibria we can regard (4.16) and (4.17) as replacing the single condition we found for the special equilibria, namely

$$\frac{dp_\perp}{dB} + B > 0 .$$

As before, (4.16) and (4.17) impose restrictions on the maximum permissible pressure gradient and by integration we can find the maximum pressure. From (4.16) one finds

$$p_\perp^{max} \ll \Delta_\parallel \frac{B^2}{2}, \tag{4.18}$$

where $\Delta_\parallel \frac{B^2}{2}$ measures the depth of the magnetic well in directions along the lines of force. From (4.17) one finds similarly

$$p_\perp^{max} \ll \Delta_\perp \frac{B^2}{2},$$

where $\Delta_\perp \frac{B^2}{2}$ is the depth of the well in directions transverse to lines of force.

We should also note that (4.17) has a simple interpretation in terms of the curvature of the lines of force for

$$\frac{1}{B^2} \vec{\nabla}_\perp (p_\perp + \tfrac{1}{2}B^2)$$

is just equal to the radius of curvature of the lines of force. Hence violation of (4.17) occurs when the plasma diamagnetic currents have so modified the vacuum field that the relative signs of $\vec{\nabla}_\perp p_\perp$ and the curvature have been reversed compared to the zero-β situation. In the low-β limit the lines of force in a minimum-β system are everywhere convex toward the plasma – that is have "stable" curvature. If one now adds some plasma to the centre of the system then, being diamagnetic, it causes the lines of force to "bulge out", that is it tends to create an unstable curvature.

5. SUMMARY

To summarize then: at finite-β we have an exact calculation of the critical pressure for the special equilibria. This critical pressure is equal to the depth of the well as measured by its magnetic pressure. We also have an approximate estimate of the critical pressure for more general equilibria namely that the maximum pressure is equal to the transverse depth of the well or to the longitudinal depth, which ever is least.

As a final point we should note that the depth of the well is to be measured in the presence of the plasma. Because the plasma is diamagnetic its pressure increases the depth of the original vacuum well by an amount which depends on its shape. For a long (in the direction of the field), thin plasma this effect is large and this would appear to give an advantage to such system. However this advantage is offset by the difficulty of creating a significant "transverse depth" in a long system before the plasma is added. In fact geometrical arguments, based upon the "field curvature" interpretation of the stability criteria, indicate that these two effects almost exactly compensate one another with the result that there is no special advantage, from this point of view, in wells of any particular aspect ratio.

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IV

PLASMA TURBULENCE

MICROINSTABILITIES

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I. GENERAL CONSIDERATIONS

A confined plasma is of necessity a non-equilibrium plasma since the only equilibrium distribution is the well-known Maxwellian one. In the presence of static electric and magnetic fields this has the form

$$f = e^{-H/kT}, \tag{1}$$

where $H = \frac{1}{2}mv^2 + e\varphi$. This depends on position only through the term $e\varphi$ which, however, can be useful for confining either electrons or ions, but not both. Hence any system in which the plasma is confined away from walls is out of equilibrium. The question of confinement is therefore the question of the rate at which equilibrium must be approached. One mechanism for the attainment of equilibrium is binary collisions which lead to a Maxwellian distribution and, more slowly, to a diffusion of plasma across the magnetic field. However, we know that these are relatively slow at high temperature, the rates being proportional to $T^{-\frac{3}{2}}$; and containment would be adequate for many purposes such as fusion if this were the only equilibration mechanism. However, one has to be a little careful here as it is not necessarily correct that growth rates in collision-dominated plasmas are proportional to collision frequency. For example, there exist some so-called resistive instabilities which grow like $\nu^{\frac{1}{3}}$ under certain circumstances, where ν is the collision frequency [1]. These are discussed in the paper by Furth. Collisions may also lead to instability of the drift mode as discussed by Sagdeev.

We will confine ourselves here to the collisionless situation and enquire as to the mechanism and rate at which a collisionless confined system would proceed towards equilibrium. Here it is customary to subdivide the problem into two parts — hydrodynamic instabilities and microinstabilities. Broadly speaking a hydrodynamic instability is one which is derived from the fluid equations and in which the plasma moves as a whole, frozen to the field lines by the well-known law

$$\vec{E} + \frac{1}{c}\vec{v}\times\vec{B} = 0, \tag{2}$$

where $\vec{E}$ is the electric field, $\vec{B}$ the magnetic field and $\vec{v}$ the fluid velocity. In general such motions are slow, having frequencies much smaller than the ion gyrofrequency Ω_i and the plasma frequency ω_p. These are also of relatively long wave length, $ka_i \ll 1$, where k is the wave number and a_i

the ion gyroradius. A distinguishing feature is that the plasma moves as a whole and we know by analogy to ordinary hydrodynamics that motions resulting from a hydrodynamic instability will be large scale, rapid and chaotic leading to swift destruction of confinement. This has been well verified in many experiments with pinches, mirror machines, etc.

All other instabilities, not derivable from the fluid equations but only from the microscopic equations, i. e. the collisionless Boltzmann equation, are referred to as microinstabilities.

We know very well how to discuss the stability of a confined plasma in the hydromagnetic approximation from the energy principle [2, 3, 4] and we know that it is not so difficult to provide a macroscopically stable situation. In the first instance this arises from the requirement (2) i. e. the plasma is frozen to the magnetic field lines. This imposes great topological constraints on the possible plasma motions since the plasma and the field must move as a whole. In particular it makes for relatively simple hydromagnetic stabilization of the plasma by shear of the field lines so that systems like the stellarator and levitron, etc. are hydromagnetically stable.

On the other hand, the perfect conductivity law (2) is a relatively weak one easily violated by such small effects as the finite Larmor radius, finite resistivity, the presence of a small electric field along the field lines, etc.; and while the experimental evidence is not absolutely conclusive one begins to feel quite uneasy about relying on this form of stabilization. Thus it is in fact necessary to examine more complex forms of motion, namely the microinstabilities. These are much more widespread and difficult to eliminate than the hydrodynamic instabilities, precisely because two powerful constraints on the motion are no longer present – the requirement (2) as discussed above and the much stronger constraint that the magnetic moment is an adiabatic invariant of the motion. This adiabatic invariance leads to the very strong result that all low frequency disturbances are stable in a minimum-B geometry as discussed by TAYLOR [5]. Thus we are led to a consideration of the microinstabilities.

Here the subject again splits into two parts – (a) the question of the linear stability of a confined equilibrium, and (b) assuming that the plasma is unstable the question of the rapidity of the resulting motion of the plasma. i. e. the non-linear and turbulence problems. We shall here be concerned with the first question.

To date we do not know for certain whether any confined plasma can be made linearly stable against all possible microinstabilities. Perhaps not, in which case the non-linear behaviour is of paramount interest. We will content ourselves here with the remark that the non-linear behaviour of a microinstability must be quite different from that of a hydrodynamic instability. For example only a small group of resonant particles may be involved as compared to the whole plasma in the hydrodynamic case. Perhaps only one or the other charged species is involved in the motion, leading rapidly to an imbalance of the charge, thereby modifying the motion. Moreover the microinstabilities are often of short wave-length so that the amplitude of motion quickly exceeds the wave-length. Beyond this point we would expect the linear theory not to hold. Similarly at frequencies of order Ω_i, the velocity of the plasma $\approx \omega a$, where ω is the frequency and a the amplitude of the disturbance, and the kinetic energy of motion would be of order $(\omega a)^2$ so that at quite small amplitudes the energy of motion exceeds the available energy and one must expect a qualitative deviation from the linear behaviour, perhaps manifesting itself as a slow anomalous diffusion. In short, with hydrodynamic instabilities one may expect a rapid mass motion away from confinement; with microinstabilities, on the other hand, a rather complex turbulence and enhanced diffusion should result. In a few cases it has been possible to show that this diffusion must be quite small. In many cases no definite answer is available. Thus we cannot be sure whether all known microinstabilities may be avoided or whether we have catalogued all possibilities, nor if the resultant effect is catastrophic. In some cases they may be even controllable and useful for such purposes as plasma heating.

Now I would like to turn briefly to some general questions before discussing specific types of plasma instabilities. First we know there are many types of stable waves in plasmas — hydrodynamic, Alfvén, plasma oscillations, etc. These are always calculated, with some sort of idealizations or approximations, e. g. in a uniform infinite medium or using such approximations as small Larmor radius, $\vec{E}+\vec{v}\times\vec{B}/c = 0$, etc. Suppose that with these approximations we find the waves are stable, i.e. ω is real, the time dependence of the wave being of the form exp $i\omega^0 t$. The exact calculation will give

$$\omega = \omega^0 + \epsilon\,\delta\omega, \tag{3}$$

where $|\epsilon\,\delta\omega/\omega^0| \ll 1$ if our approximations are good. However, if $\delta\omega$ were complex then even a small change would imply instability in which case the approximate calculation would have given us no useful information regarding the stability of the system.

A simple argument due to LOW [6] assures us that this cannot happen. There exists a constant of motion, namely the energy H, such that

$$\frac{dH}{dt} = 0. \tag{4}$$

Suppose the system exists in an equilibrium state with energy H_0. We describe the perturbed motion by a displacement $\vec{\xi}$. If in equilibrium a particle is at $\vec{r}, \vec{v}$, t, then after the perturbation, its co-ordinates would be

$$\vec{r}+\vec{\xi},\ \vec{v}+\frac{d\vec{\xi}}{dt},\ t, \tag{5}$$

where $\vec{\xi}(\vec{r},\vec{v},t)$ is the displacement. The energy of the system can be expressed completely in terms of $\vec{\xi}$ and $\partial\vec{\xi}/\partial t = i\omega\vec{\xi}_0$. Note that if the particle displacements are given we can calculate the field energies. Thus, correct to second order, we can write

$$H = H_0 + H_1(\vec{\xi}) + H_2(\vec{\xi}, \vec{\xi}). \tag{6}$$

Since we started from a state of equilibrium, H_1 must vanish. For H_2 we can write, quite generally

$$H_2 = \int d^3\vec{r}\, d^3\vec{r}'\, d^3\vec{v}\, d^3\vec{v}'\, \vec{\xi}(\vec{r}, \vec{v}, t) K(\vec{r}, \vec{v}, \vec{r}', \vec{v}') \vec{\xi}(\vec{r}', \vec{v}', t). \tag{7}$$

Such an H_2 can be defined for the approximate system as well as the exact system. The approximate system was stable so that

$$H_2^0 > 0, \tag{8}$$

where we use a superscript to denote quantities referring to the approximate system. Further, since $dH^0/dt = 0$, H_2^0 must be a constant. Now we can write

$$\vec{\xi}(\vec{r}, t) = \vec{\xi}(\vec{r}) e^{i\omega t}, \tag{9}$$

and since H_2^0 is constant and non-zero, this must mean that the time dependence of $\vec{\xi}^0$'s in Eq. (7) cancels out. This is possible if ω is real where the relative phases of the kinetic and potential energy are such as to keep the energy time independent.

Now let us consider the exact system. The functions characterizing the exact system can be written in terms of those of the approximate system

$$\vec{\xi} = \vec{\xi}^0 + \epsilon d\vec{\xi}, \quad K(\vec{r}, \vec{r}') = K^0(\vec{r}, \vec{r}') + \epsilon \delta K \tag{10}$$

and

$$\omega = \omega^0 + \epsilon \delta\omega. \tag{11}$$

The second order energy is

$$H_2 = \int d^3\vec{r}\, d^3\vec{r}'\, \vec{\xi}(\vec{r}) K(\vec{r}, \vec{r}') \vec{\xi}(\vec{r}'). \tag{12}$$

The perturbations in the $\vec{r}$ and $\vec{v}$ dependence of $\vec{\xi}$ and K will only make a change $0(\epsilon)$ in H_2 but if $\delta\omega$ were complex, then there would appear a factor $\exp(-2\epsilon\delta\omega_i t)$ outside the integral on the right-hand side of Eq. (12), where $\delta\omega = \delta\omega_r + i\,\delta\omega_i$. In that case

$$\frac{dH_2}{dt} \neq 0, \tag{13}$$

which is incompatible with the fact that the energy is a constant of motion. Therefore ω_i must vanish, and must in fact vanish to all orders in ϵ.

Hence an instability can arise only if either:

(a) the modifications are sufficiently large (ϵ finite) so as to significantly alter $\vec{\xi}$ so that the time independent part of H_2 is reduced to zero. This implies a finite threshold for instability. For example, in the two-stream instability the current must exceed a threshold value to cause instability; or

(b) a completely new type of wave may arise. For example, in the case of inhomogeneous plasmas the drift mode or universal instability may arise which has no analogue for homogeneous plasmas.

For the most part, in attacking the linear instability problem it has been necessary simply to hunt for various types of unstable modes which might arise in specific situations. It is easy to identify the potential sources of instability, namely different forms of deviation from equilibrium, and to study their effect on the known plasma waves, but by and large the procedure has been a matter of trial and error. It is often easy to find some unstable mode and calculate its properties. But if none is found it is very difficult to be sure that all possibilities have been examined and that some unthought-of mode might not be unstable. In other words, it is hard to give a positive proof of stability, but this may be done in a particular case which we now describe.

Let us consider a uniform plasma in a static externally-produced magnetic field $\vec{B}$ and assume that there are no electric fields present, $\vec{E} = 0$, i.e. the fields are in the lowest possible energy state consistent with external coils. Moreover the velocity distribution of the plasma is assumed to be uniform isotropic and a monotonically decreasing function of v^2, i.e.

$$f = f(v^2), \quad \frac{\partial f}{\partial v^2} < 0. \tag{14}$$

We do not assume $f(v^2)$ to be a Maxwellian distribution.

In phase space all plasma motions are incompressible because they are governed by the Liouville equation. Hence we define a constant of motion

$$S = \int G(f)\, d^3\vec{x}\, d^3\vec{v}, \tag{15}$$

with $\partial S/\partial t = 0$, since the volume of phase space occupied by any value of f is not changed by the motion. G can be chosen to be any function of f and this leads to a class of constants of motion. We now wish to find the lowest possible energy

$$H = \int \frac{1}{2} mv^2 f d^3\vec{x} d^3\vec{v} + \frac{1}{8\pi}\int B^2 d^3\vec{x} + \frac{1}{8\pi}\int \mathscr{E}^2 d^3\vec{x} \tag{16}$$

which a plasma can reach subject to the constraints

$$N = \int f d^3\vec{x} d^3\vec{v} = \text{constant} \tag{17}$$

and

$$S = \int G(f) d^3\vec{x} d^3\vec{v} = \text{constant}. \tag{18}$$

For the moment we leave G undetermined.

Using the Lagrangian multipliers α and β, the variation of H leads to

$$\frac{1}{2}\, mv^2 + \alpha + \beta\, \frac{\partial G}{\partial f} = 0 \tag{19}$$

or

$$\frac{1}{2}\, mv^2 + \alpha + \beta\, \frac{\partial G}{\partial v^2}\, \frac{\partial v^2}{\partial f} = 0. \tag{20}$$

If $\partial f/\partial v^2 \neq 0$ and since $f(v^2)$ is a known function, one can solve Eq. (20) to obtain a G(f) which is appropriate to show that the state $f(v^2)$ is indeed the lowest accessible energy state. The positive proof of the stability of this distribution demonstrates at least that this slightly non-equilibrium situation is stable and that collective motions alone do not invariably proceed directly to equilibrium in such situations, for example, as a uniform plasma with $T_e \neq T_i$.

So far it has not been possible to extend this line of reasoning very far into more complex situations. In particular, for non-uniform plasmas it is always possible to lower the energy by expanding the plasma towards uniformity. In order to proceed further, it would be necessary to find additional constants of the motion to restrict further the possible changes of energy. For some types of motions, particularly those at low frequencies, such constants do indeed exist. For example, if we assume particles frozen to the field lines together with the conservation of entropy we recover the results of Oberman and Kruskal on hydromagnetic stability; if we use the invariance of μ we recover Taylor's proof of minimum-B stability.

It is useful, however, even where this cannot be done, to proceed a little further and develop the notion of free energy. We may consider some equilibrium situation and calculate the lowest energy state which is available to it subject to the constancy of some $\int G(f)\, d^3\vec{x}\, d^3\vec{v}$ e. g. $G = f \ln f$. Except for the special case outlined above this energy will be lower than that of the original state. The difference represents a free energy possibly available for driving an instability. Again we may emphasize that this does not necessarily imply instability but only the possibility of it. Having identified the possible free energy for driving an instability, it remains to search for specific mechanisms for the motions which may ensue. We find the following main sources of free energy:

1. Kinetic energy of drifts in the plasma. These might include particle beams, currents parallel to the magnetic field $\vec{B}$, plasma mass motion in equilibrium, or diamagnetic currents which are always present in a plasma.

2. Magnetic energy. The field energy itself may be raised well above the vacuum value and free energy may be gained by relaxing the field. For low values of β, this free energy is of order β^2.

3. Anisotropies in the distribution function. In confinement schemes involving magnetic mirrors there must always exist anisotropy in the velocity

distribution perpendicular and parallel to the static magnetic field. For example, at the mirror point there is no parallel energy. Free energy may always be gained by relaxing towards isotropy.

4. Expansion energy. By expanding the plasma towards uniformity, the energy will be lowered. The available energy $E \simeq (V_0/V)^{2/3}$, where V_0 and V are the initial and final volumes respectively occupied by the plasma.

Of these possible sources of free energy, the first type is usually quite small, since the energy of diamagnetic currents $\approx m_e$, the second type is small at low β, the third type is present in all open-ended confinements but not in toroidal systems, while the fourth type always represents a large source of free energy. It is the last source which, of course, feeds the hydrodynamic instability. However, it is an open question how much of this free energy is available in hydrodynamically stable situations where the plasma is unable to expand in a uniform way.

II. LOSS-CONE INSTABILITY

We will now discuss a specific calculation due to POST and ROSENBLUTH [8] in which instability feeds on the anisotropy of the distribution function. We will go through this in some detail both as an illustration of the methods involved and because of its intrinsic interest. We consider a case in which the ion distribution function is characterized by a loss cone as would be brought about by magnetic mirrors. Then

$$f = f(v_\perp^2, v_\parallel^2), \tag{21}$$

where

$$f = 0 \qquad \text{for } v_\perp^2 < \gamma v_\parallel^2, \tag{22}$$

where γ is some constant which depends on the mirror ratio. We will now demonstrate that any such confinement is unstable. We make the following idealizations:

1. We assume the plasma to be uniform and infinite immersed in a uniform magnetic field $\vec{B} = B_0\hat{e}_z$. Later on we apply the results to finite geometry.
2. For the sake of simplicity we assume the electrons are at zero temperature. The only important modification which results for finite electron temperature is the possibility of electron Landau damping if $\omega < k_\parallel v_e^{th}$, where $k_\parallel = \vec{k} \cdot \hat{e}_z$ and v_e^{th} is the electron thermal velocity. It turns out that this does not occur for $T_e < T_i$.
3. The ratio of plasma to magnetic pressure, β, is small. For this case any unstable perturbation must be electrostatic i. e. $\vec{E} = -\vec{\nabla}\varphi$. This is true because a non-electrostatic disturbance would imply a perturbed magnetic field. This would change the magnetic field energy, in fact increase it, since the uniform field has the lowest possible field energy. Since β is low, this would more than compensate any possible decrease in particle energy.

We first give a physical argument as to why we expect such a distribution to be unstable. From the Penrose criterion as discussed by Professor

Simon we know that a necessary condition for the 2-stream instability is that

$$N(v_y^2) = \int f(v_x^2 + v_y^2, v_z^2) dv_x dv_z \tag{23}$$

be a non-monotonic function of v_y^2; i.e. the number of particles with a given component of velocity must increase somewhere with velocity. Now consider

$$\left.\frac{\partial N(v_y^2)}{\partial v_y^2}\right|_v = \int \frac{\partial f}{\partial v_\perp^2} dv_y dv_x dv_z \delta(v_y - v)$$

$$= \frac{1}{2} \int \frac{\partial f}{\partial v_\perp^2} dv_\perp^2 d\varphi dv_z \delta(v_\perp \cos\varphi - v)$$

$$= \frac{1}{2} \int \frac{\partial f}{\partial v_\perp^2} dv_\perp^2 dv_z \frac{1}{v_\perp \sin\varphi}$$

$$= \frac{1}{2} \int dv_z \int_v^\infty dv_\perp^2 \frac{\partial f}{\partial v_\perp^2} \frac{1}{(v_\perp^2 - v^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2} \int dv_z \int \frac{df}{(v_\perp^2 - v^2)^{\frac{1}{2}}}. \tag{24}$$

For a fixed value of v_z, the plot of f <u>versus</u> $v_\perp^2$ has the form shown in Fig. 1.

In the integral over df in Eq. (24) note that since f must be zero both at $v_\perp^2 = \gamma^2 v_z^2$ and at ∞, df has both a positive range from 0 to $v_{\perp max}^2$ and a numerically equal negative range from $v_{\perp max}^2 \to \infty$. If $v > v_{\perp max}$ then only the negative range is sampled and $\partial N/\partial v^2 < 0$. For small v, however, the contribution from positive df will predominate as the square root is smaller for small $v_\perp$. Hence N <u>versus</u> v^2 has the shape shown in Fig. 2.

We shall now discuss this problem in detail by solving the Boltzmann and the Poisson equations:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right) \cdot \vec{\nabla}_v f = 0, \tag{25}$$

and

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho, \tag{26}$$

where

$$\rho = \sum_{j=e,i} e_j \int f_j \, d^3\vec{v}. \tag{27}$$

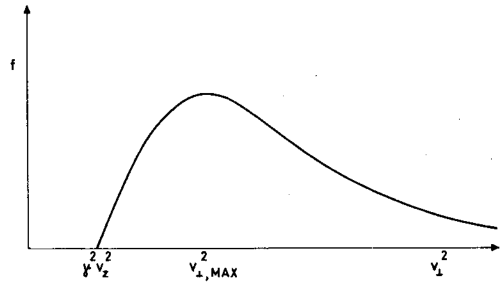

Fig. 1

Plot of ion distribution function versus $v_\perp^2$.

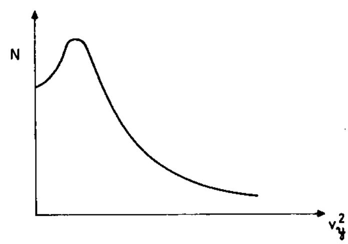

Fig. 2

Plot of $N(v_y^2)$ versus v_y^2.

In equilibrium, $f = f_0$, $\vec{E} = 0$, $\vec{B} = B_0\hat{e}_z$. In the perturbed state $f = f_0 + f_1$, $\vec{E} = \vec{E}_1 = -\vec{\nabla}\varphi_1$, and $\vec{\nabla} \cdot \vec{E}_1 = 4\pi\rho_1$. The linearized equations now are:

$$\frac{\partial f_1}{\partial t} + \vec{v} \cdot \vec{\nabla} f_1 + \frac{e}{m} (\vec{v} \times \vec{B}_0) \cdot \vec{\nabla}_v f_1 = -\frac{e}{m} \vec{E} \cdot \vec{\nabla}_v f_0. \tag{28}$$

We solve this equation by the method of characteristics. Note that Eq. (28) can be written as

$$\frac{d}{dt} f_1 = -\frac{e}{m} \vec{E}_1 \cdot \vec{\nabla}_v f_0, \tag{29}$$

where d/dt is the total time derivative as we follow the unperturbed trajectory of the particle in the space of $\vec{x}$ and $\vec{v}$. The formal solution of Eq. (29) can be written down readily:

$$f_1(\vec{x}, \vec{v}, t) = \frac{e}{m} \int_{-\infty}^{t} \vec{\nabla}\varphi_1(\vec{x}', t') \cdot \vec{\nabla}_{v'} f_0 dt'. \tag{30}$$

Here $(\vec{x}', \vec{v}', t')$ are the co-ordinates of a particle which at time t was at $\vec{x}, \vec{v}$. We now assume that

$$(f_1, \varphi_1) = (\tilde{f}, \tilde{\varphi}) \exp i(\omega t + k_z z + k_y y). \tag{31}$$

Noting that

$$\vec{k} \cdot \vec{\nabla}_{v'} f_0 = 2\left[k_y v_y' \frac{\partial f_0}{\partial v_\perp'^2} + k_z v_z' \frac{\partial f_0}{\partial v_z'^2} \right], \tag{32}$$

Eq. (30) leads to

$$\tilde{f}(\vec{v}) = \frac{e}{m} \tilde{\varphi} \int_{-\infty}^{t} dt' \left\{ \exp i[\omega(t' - t) + k_z(z' - z) + k_y(y' - y)] \right\}$$

$$\times 2i \left[k_y v_y' \frac{\partial f_0}{\partial v_\perp^2} + k_z v_z' \frac{\partial f_0}{\partial v_z'^2} \right]. \tag{33}$$

Note that as usual in such problems the integral converges if $IP(\omega) < 0$. For $IP(\omega) \geqslant 0$ one must analytically continue.

The particle trajectories are (putting $\tau = t' - t$):

$$v_y' = v_\perp \cos(\Omega\tau + \Psi), \tag{34}$$

$$v_z' = v_z, \tag{35}$$

which upon integration give

$$y' - y = \frac{v_\perp}{\Omega} \left[\sin(\Omega\tau + \Psi) - \sin \Psi \right], \tag{36}$$

$$z' - z = v_z \tau. \tag{37}$$

$v_\perp^2$ and v_z^2 are themselves constants of motion and, therefore,

$$\frac{\partial f_0}{\partial v_\perp^2} \quad \text{and} \quad \frac{\partial f_0}{\partial v_z^2}$$

are constants of motion and can be taken out of the integral in (33). Making use of the identity

$$e^{i\gamma \sin\theta} = \sum_{n=-\infty}^{n=+\infty} J_n(\gamma) e^{in\theta}, \tag{38}$$

Eq. (33) can be written as

$$\tilde{f} = \frac{2ie}{m}\tilde{\varphi}\int_{-\infty}^{0} d\tau\, e^{i\omega\tau} \sum_{n,m} J_n(k_\perp v_\perp/\Omega) J_m(k_\perp v_\perp/\Omega)$$

$$\times e^{i(n-m)\psi + in\Omega\tau} \left[k_z v_z \frac{\partial f_0}{\partial v_z^2} + \frac{\partial f_0}{\partial v_\perp^2} k_\perp v_\perp \right.$$

$$\left. \times \frac{1}{2}\left(e^{i(\Omega\tau+\psi)} + e^{-i(\Omega\tau+\psi)} \right) \right] e^{ik_z v_z \tau}. \tag{39}$$

The time integrations are now trivial and if we use the identity

$$\frac{2n}{\gamma} J_n(\gamma) = J_{n-1}(\gamma) + J_{n+1}(\gamma), \tag{40}$$

we obtain

$$\tilde{f} = \frac{2e}{m}\tilde{\varphi}\sum_{n,m} \frac{J_n(k_\perp v_\perp/\Omega) J_m(k_\perp v_\perp/\Omega)}{\omega + k_z v_z + n\Omega} e^{i(n-m)\psi}$$

$$\times \left[k_z v_z \frac{\partial f_0}{\partial v_z^2} + n\Omega \frac{\partial f_0}{\partial v_\perp^2} \right]. \tag{41}$$

The charge density is given by (cf. Eq. (27)):

$$\tilde{\rho} = \int \tilde{f} d^3\vec{v} = \frac{1}{2}\int \tilde{f} d\Psi\, dv_\perp^2 dv_z. \tag{42}$$

Substituting this into the Poisson equation (26), we obtain

$$k^2\tilde{\varphi} = 2\pi \sum_{e,i} \int \tilde{f} d\Psi\, dv_\perp^2 dv_z. \tag{43}$$

Finally, on substituting for f from Eq. (41) into (43) and carrying out the Ψ integration, one gets the dispersion relation:

$$k^2 = 8\pi^2 \sum_{j=e,i} \frac{e_j^2}{m_j} \int dv_\perp^2 dv_z \sum_n \frac{J_n^2(k_\perp v_\perp/\Omega)}{\omega + k_z v_z + n\Omega} \left[k_z v_z \frac{\partial f_0}{\partial v_z^2} + n\Omega \frac{\partial f_0}{\partial v_\perp^2} \right]. \tag{44}$$

At this point we must consider separately the evaluation of the ion and electron terms and make some further approximations appropriate to the specific case we are considering.

For the electrons we have taken zero temperature so only the term $n = 0$ need be considered. Further, $J_0^2 = 1$ and we consider the case $\omega \gg k_z v_z$.

Thus

$$\int dv_\perp^2 dv_z \frac{1}{\omega + k_z v_z} k_z v_z \frac{\partial f_0}{\partial v_z^2}$$

$$= \int dv_\perp^2 dv_z\, k_z v_z \frac{1}{\omega}\left(1 - \frac{k_z v_z}{\omega} + \frac{k_z^2 v_z^2}{\omega^2} - \dots\right)\frac{\partial f_0}{\partial v_z^2}\,. \tag{45}$$

The first term vanishes by symmetry since f_0 is an even function of v_z^2 and the second term may be evaluated by parts, and using the fact that

$$n_0 = \pi \int f_0\, dv_\perp^2 dv_z, \tag{46}$$

we obtain for the electron density

$$\tilde{\rho}_e = n_0 \frac{e^2}{m} \frac{k_z^2}{\omega^2} \tilde{\varphi}\,. \tag{47}$$

Thus from Eqs. (44) and (47) we get

$$1 = \frac{\omega_{pe}^2}{\omega^2}\frac{k_z^2}{k^2} + \frac{8\pi^2}{k^2}\frac{e^2}{M_i}\int dv_\perp^2 dv_z \sum_{n=-\infty}^{\infty} \frac{J_n^2(k_\perp v_\perp/\Omega)}{\omega + k_z v_z + n\Omega} \times \left[k_z v_z \frac{\partial f_0}{\partial v_z^2} + n\Omega \frac{\partial f_0}{\partial v_\perp^2}\right]. \tag{48}$$

Now from the fact that the electron and ion contributions to the charge density are inversely proportional to the mass we may guess that at low number density the waves will be dominated by the electron contribution. We will, therefore, seek a solution of the form that the electrons determine the frequency of the wave while the "residue" from the ion term yields the imaginary part. From the electron term of (48) we obtain

$$\omega = \omega_{pe} \frac{k_z}{k} = \omega_{pe} \cos\theta, \tag{49}$$

where θ is the angle which the propagation vector $\vec{k}$ makes with the direction of the magnetic field. To analyse the problem in general would require the details of the distribution function and would involve complicated analysis.

However, to demonstrate the existence of instabilities and their general properties in such systems we note that to achieve a large ion residue term we would like to have $\omega + n\Omega \simeq 0$. From Eq. (48) it is clear that this may only be achieved if

$$\omega_{pe}^2 > n^2\Omega^2, \tag{50}$$

since $k_\parallel^2 < k^2$. This provides a limiting density above which such resonances are possible. We now investigate the ion term in Eq. (48) under the condition that $\omega + n\Omega = 0$, choosing only the imaginary part arising from the pole of the resonant term when the integration over v_z is performed, and coming from $v_z = 0$. We thus obtain for the imaginary part

$$IP = i\pi \frac{8\pi^2}{k^2} \frac{e^2}{M} \frac{n\Omega}{k_{\parallel}} \int dv_{\perp}^2 J_n^2\left(\frac{k_{\perp}v_{\perp}}{\Omega}\right) \frac{\partial f_0}{\partial v_{\perp}^2}\bigg|_{v_{\parallel}=0} \tag{51}$$

$$= i\pi \frac{8\pi^2}{k^2} \frac{e^2}{M} \frac{n\Omega}{k_{\parallel}} \int J_n^2(k_{\perp}v_{\perp}/\Omega) df_0. \tag{52}$$

We recall that (see Fig. 1) f must vanish at $v_{\perp}^2 = \gamma v_z^2$ and at infinity, so that df is positive for small values of $v_{\perp}^2$ and negative for large $v_{\perp}^2$, while J_n^2 has the form shown in Fig. 3 going asymptotically as $\Omega/2\pi k_{\perp}v_{\perp}$. Thus in performing

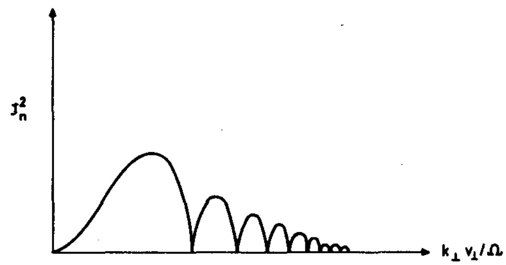

Fig. 3

Plot of J_n^2 versus $k_{\perp}v_{\perp}/\Omega$.

the integral in Eq. (52) we note that positive contributions come from small values of $v_{\perp}^2$ and negative contributions from large values. If $k_{\perp}v_{\perp max}/\Omega \ll n$ then most of the integral will occur on the rising portion of the curve and the integral (52) will be negative. Conversely for $k_{\perp}v_{\perp max}/\Omega >> n$, the integral will be positive. Using (46), we may approximate

$$IP \approx i\pi \frac{\omega_{pi}^2}{k^2} \frac{n\Omega}{|k_{\parallel}|} \frac{\delta(k_{\perp}v_{\perp max}/\Omega)}{v_i^3}, \tag{53}$$

where δ is a function with properties sketched in Fig. 4, reaching a maximum of order of magnitude unity.

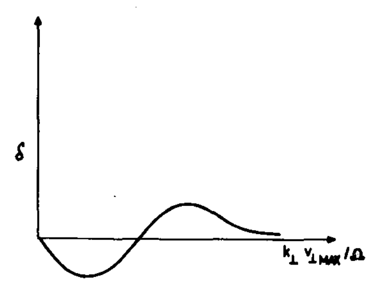

Fig. 4

Plot of δ versus $k_{\perp}v_{\perp max}/\Omega$.

Finally on substituting (53) into (48) and remembering that $\omega_{pe}^2 k_\parallel^2/k^2 \simeq n^2\Omega^2$ we find

$$\omega^2 = n^2\Omega^2 + \frac{i\omega_{pi}^2\, n\Omega\, \delta\omega^2}{k^2 k_\parallel v_i^3} \tag{54}$$

or

$$\omega = -n\Omega - \frac{i}{2}\frac{\omega_{pi}^2\, \delta n^2\Omega^2}{k^2 k_\parallel v_i^3}\,. \tag{55}$$

Remembering that the time dependence of the waves is of the form $e^{i\omega t}$, it becomes apparent that for any positive δ the system will be unstable. This means that for $k_\perp v_{\perp max}/\Omega > n$, the waves will be unstable with a peak growth rate obtained by putting $k_\perp v_\perp/\Omega \simeq n$, and roughly given by

$$\gamma \simeq \frac{\omega_{pi}^2\, \omega_{pe}}{n^2\Omega^2}\,. \tag{56}$$

It is to be noted that in this entire discussion we have used no properties of the ion distribution function other than the fact that it contains a loss cone. It should be noted that oscillations near the ion cyclotron frequency have been observed in various experiments, e.g. ALICE, DCX-2, Phoenix etc., where sufficient densities were attained roughly as given by condition (50).

Under conditions where the electron temperature does not strictly vanish, the density limitations are a bit more severe than those required in the absence of electron Landau damping. Thus

$$\begin{aligned}\Omega_i \simeq \omega > k_z v_{z,e} &= \frac{k\Omega_i}{\omega_{pe}}\, v_e \\ &= \frac{\Omega_i^2 v_e}{\omega_{pe} v_i}\frac{k v_{i\perp}}{\Omega_i} > \frac{\Omega_i^2}{\omega_{pe}}\frac{v_{e\parallel}}{v_{i\perp}}\end{aligned} \tag{57}$$

or

$$\omega_{pe} > \Omega_i\, \frac{v_{e\parallel}}{v_{i\perp}}\,. \tag{58}$$

If we examine the growth rate given by Eq. (56) we see that as the density increases such that $\omega_{pe}\omega_{pi}^2 > \Omega^3$ then the growth rate becomes comparable to the frequency. This implies of course that our selection of a single resonant term in Eq. (48) is not valid and that instead many values of n will contribute and then the series must be summed.

The identification with the 2-stream instability [cf. Eq. (24)] is more evident at high densities where as already mentioned the growth rate exceeds the cyclotron frequency for the ions. So we proceed to the case

$$I P(\omega) > \Omega_i\,. \tag{59}$$

In this case the gyration of the ions is not important and one could arrive directly at the dispersion relation by using the rectilinear orbits of a freely moving particle in Eqs. (34) to (37). However, we may also proceed from the general form of Eq. (48). At high densities, it turns out that $k_\parallel^2/k_\perp^2 \ll 1$, since $\omega_{pe}^2/\omega^2 \gg 1$ and we may neglect $k_z v_z$ in the ion term in Eq. (48). The ion term is proportional to

$$\int dv_\perp^2 dv_z \sum_n \frac{J_n^2(k_\perp v_\perp/\Omega)}{\omega + n\Omega} n\Omega \frac{\partial f_0}{\partial v_\perp^2}$$

$$= -\omega \int dv_\perp^2 dv_z \sum_n \frac{J_n^2(k_\perp v_\perp/\Omega)}{\omega + n\Omega} \frac{\partial f_0}{\partial v_\perp^2}, \quad (\text{since } \Sigma J_n^2 = 1 \text{ and } \int df = 0) \tag{60}$$

$$= -i\omega \int dv_\perp^2 dv_z \sum_n \int_{-\infty}^{0} d\tau\, e^{i(\omega + n\Omega)\tau} J_n^2(k_\perp v_\perp/\Omega) \frac{\partial f_0}{\partial v_\perp^2} \tag{61}$$

$$= -i\omega \int dv_\perp^2 dv_z \int_{-\infty}^{0} d\tau\, e^{i\omega\tau} J_0\left[\sqrt{2}\, \frac{k_\perp v_\perp}{\Omega} \left(1 - \cos \Omega\tau\right)^{\frac{1}{2}}\right] \frac{\partial f_0}{\partial v_\perp^2}\, d\tau, \tag{62}$$

where in writing the last equation use has been made of the addition rule for Bessel functions [9]. Since $IP(\omega) \gg \Omega$, we may expand

$$1 - \cos \Omega\tau \simeq \frac{1}{2}\Omega^2\tau^2, \tag{63}$$

as the later cycles of the ion oscillation will be suppressed by the factor $e^{i\omega\tau}$. The right hand side of Eq. (62) now reduces to

$$-i\omega \int dv_\perp^2 dv_z \int_{-\infty}^{0} d\tau\, e^{i\omega\tau} J_0(k_\perp v_\perp \tau) \frac{\partial f_0}{\partial v_\perp^2}\, d\tau \tag{64}$$

$$= -\omega \int dv_\perp^2 dv_z' \frac{1}{(\omega^2 - k_\perp^2 v_\perp^2)^{\frac{1}{2}}} \frac{\partial f_0}{\partial v_\perp^2}. \tag{65}$$

Note that the ion gyro-frequency has disappeared from this term and that we have recovered the form of Eq. (24). On substituting this into the dispersion relation (48) we obtain

$$1 = \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} - \frac{8\pi^2 e^2}{k^2 M} \int dv_\perp^2 \frac{1}{(1 - k_\perp^2 v_\perp^2/\omega^2)^{\frac{1}{2}}} \frac{\partial g_0}{\partial v_\perp^2} \tag{66}$$

where $g_0(v_\perp^2) = \int_{-\infty}^{\infty} f_0(v_\perp^2)dv_z$ and $g_0(0) = g_0(\infty) = 0$, or

$$1 = \frac{\omega_{pe}^2}{\omega^2}\frac{k_\parallel^2}{k^2} - \frac{8\pi^2 e^2}{k^2 M} \int \frac{dg_0}{(1 - k_\perp^2 v_\perp^2/\omega^2)^{\frac{1}{2}}}. \tag{67}$$

Again we argue as before that dg_0 can be either positive or negative so that it is possible for the imaginary part of (67) to have either sign and that when $k_\perp \bar{v}_\perp/\omega > 1$ the sign is such as to cause instability ($\bar{v}_\perp$ is the mean ion velocity.)

The basic wave involved is again an electrostatic plasma oscillation with $\omega = \omega_{pe} k_z/k$. Before proceeding further let us turn to consideration of, instead of an infinite medium, a confined plasma of finite length such as a mirror machine and see the effect of such instabilities. These plasma oscillations are running waves. If a wave packet at frequency ω is launched at a point in the field the wave will run along the field lines with velocity $d\omega/dk_z = \omega_{pe}/k = \omega/k_z$ either growing or decaying in amplitude depending on whether the time dependent problem is unstable or stable. When the wave reaches the end of the plasma $\omega_{pe} \to 0$ and $k_\parallel$ tends to become quite large. This in turn means that Landau damping of electrons should occur. In general when $k_z \to 0$ reflection is implied, when $k_z \to \infty$ absorption is implied. In our case the wave should be absorbed at the end of the plasma and hence the question of prime interest is how much it will have grown in travelling the length of the plasma. If it has e-folded by 10-20 times we may expect a serious disruption of the plasma with effective violent instability and scattering of ions into the loss cone.

It is thus appropriate instead of solving the dispersion relation for ω to solve for k_z, the spatial dependence of the wave. Thus consider again Eq. (67):

$$k_\parallel^2 = \frac{k^2\omega^2}{\omega_{pe}^2} + \frac{8\pi^2 e^2}{M}\frac{\omega^2}{\omega_{pe}^2}\int dv_\perp^2 \frac{\partial g_0/\partial v_\perp^2}{(1 - k_\perp^2 v_\perp^2/\omega^2)^{\frac{1}{2}}}. \tag{68}$$

Again using Eq. (46) we can write this roughly as

$$k_\parallel^2 = \frac{k^2\omega^2}{\omega_{pe}^2} - i\,\frac{m}{M}\frac{\omega^2}{\bar{v}^2}\,\delta\left(\frac{\omega}{k_\perp \bar{v}}\right), \tag{69}$$

where δ will be positive so as to cause growing waves if $k_\perp \bar{v}/\omega > 1$. Solving Eq. (69), we get

$$k_\parallel = \frac{k\omega}{\omega_{pe}} - i\,\frac{\delta}{2}\frac{\omega_{pe}}{\bar{v}}\frac{m}{M}\frac{\omega}{k\bar{v}}; \tag{70}$$

$\bar{v}$ is the mean thermal speed for the ions.

Thus the most rapidly growing waves will occur for the maximum value of $(\omega/k\bar{v})(1/2)\delta(\omega/k\bar{v}) = \gamma$, a numerical constant depending on the details of the distribution function and of order of magnitude unity. For a distribution corresponding to collisional equilibrium in a mirror machine of mirror ratio 2, we have found $\gamma \simeq 0.1$. For very extreme distributions such as those resulting from δ functions (as in a neutral injection mechanism), $\gamma \gg 1$. Thus if L is the length of the plasma column, the effective growth of the wave is given by the number of e-foldings.

$$N \simeq k_{z,i} L \simeq \left(\frac{m}{M}\right)^{\frac{1}{2}} \frac{\gamma L}{\lambda_{D,i}}, \tag{71}$$

where $\lambda_{D,i}$ is the ion Debye length. Numerically,

$$N \simeq 10^{-5}\, \gamma \left(\frac{n}{T}\right)^{\frac{1}{2}} L, \tag{72}$$

where T is the ion temperature in keV, n the ion density in cm^{-3}. At very high densities $\omega_{pe} > \Omega_e$ this rate is reduced by a factor $1/(1+\omega_{pe}^2/\Omega_e^2)^{\frac{1}{2}}$. This arises from the higher order Bessel functions in Eq. (44) for electrons which we have neglected.

Since the cause of these instabilities is $\partial f_0^{ions}/\partial v_\perp^2 > 0$, they may arise in other situations as well as from the loss-cone distribution. For example, a neutral injection system at a fixed non-zero energy will produce instability and even isotropic distributions peaked around non-zero energy may be quite unstable.

Similar sorts of instabilities affect other waves, such as Alfvén and whistler modes, but tend to be weaker effects in the limit of low β, since resonance is only possible for very fast particles, and are not quite universal in the sense of applying to all loss-cone distributions.

III. DISPERSION RELATION FOR MICROINSTABILITIES OF INHOMOGENEOUS PLASMA

In this section we proceed essentially as before except that now we wish to consider the effects of a slight inhomogeneity on the plasma stability. We consider the following situation: a plasma of low β, in a uniform field, B_z, with gravitational potential gx has a Maxwellian velocity distribution with uniform temperature, T, and a spatial density which is weakly dependent on x and independent of y. The constants of particle motion in this system are

$$E = \frac{mv^2}{2} + mgx$$

and by symmetry

$$P_z = mv_z, \quad P_y = mv_y + eA_y/c,$$

where $\bar{A}$ is the vector potential giving the constant B_z. Taking $A_y = 0$ at $x = 0$ then allows P_y to be written $P_y = mv_y + \Omega mx$.

Then since any function of the constants of the motion is a steady state solution of the Vlasov equation we can take

$$f_0 = n_0 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left[1 + \epsilon\left(x + \frac{v_y}{\Omega}\right)\right] \exp - \left[\frac{mv^2}{2kT} + \frac{mgx}{kT}\right]. \tag{73}$$

where then $\frac{1}{n}\frac{dn}{dx} \simeq \epsilon - \frac{mg}{kT}$ will be taken to be small so that $\left(\frac{1}{n}\frac{dn}{dx}\right)R_L \ll 1$. This is the simplest solution of the Vlasov equation that gives the desired density dependence. In order to have charge neutrality $\frac{1}{n}\frac{dn}{dx}$ must be the same for electrons and ions so that at least near $x = 0$

$$\epsilon_i - \frac{Mg}{kT} = \epsilon_e - \frac{mg}{kT} \simeq \frac{1}{n}\frac{dn}{dx}.$$

We now proceed as before. Assume a perturbation of the form

$$\vec{E} = -\vec{\nabla}\varphi$$
$$\varphi = \varphi(x)\, e^{i[K_y Y + K_z Z + \omega t]}, \tag{74}$$

solve the perturbed Vlasov equation by the method of characteristics as before and substitute into Poisson's equation. New terms of course arise due to the term $\epsilon v_y/\Omega$ in f_0, and moreover there is an essential complication due to the fact that φ is now also a function of x and hence appears as an unknown in the equation

$$f_1 = \frac{e}{m}\int_{-\infty}^{t} \vec{\nabla}\varphi \cdot \vec{\nabla}_v f_0 \, dt.$$

Thus strictly speaking we are left with an integral equation for $\varphi(x)$. This may be reduced to a differential equation by expanding

$$\varphi(x') = \varphi(x) + \varphi'(x)(x' - x) + \varphi''(x)(x' - x)^2/2 + \ldots,$$

where $(x' - x)^n$ is given simply in terms of the orbit of a spiralling particle in a constant magnetic field. In practice we will be concerned with $K \langle R_L \rangle$ sufficiently small that the terms in $(x' - x)^n$ may by neglected. Later we refine the discussion to show how these terms may be included. For the moment we write down the dispersion relation for $\varphi(x) = \hat{\varphi} = \text{constant}$.

From (73) and (74) we find, dropping small terms of order ϵ^2,

$$\vec{\nabla}_v f_0 = f_0\left[-\frac{m\vec{v}}{kT} + \frac{\epsilon}{\Omega}\hat{y}\right] \tag{75}$$

$$\vec{E} \cdot \vec{\nabla}_v f_0 = f_0\left[-\frac{m}{kT}\vec{v} \cdot \vec{\nabla}\varphi + \frac{\epsilon i K_y}{\Omega}\varphi\right] \tag{76}$$

Then using

$$\frac{d}{dt}\varphi(\vec{r}(t), t) = \frac{\partial\varphi}{\partial t} + \vec{v} \cdot \vec{\nabla}\varphi = i\omega\varphi + \vec{v} \cdot \vec{\nabla}\varphi, \quad \varphi(t = -\infty) = 0$$

and the fact that f_0 is a constant of the motion in the unperturbed fields to integrate by parts gives

$$f_1 = \frac{e}{m} f_0 \left[-\frac{m}{kT}\varphi + i\left[\frac{\omega m}{kT} + \frac{\epsilon K_y}{\Omega}\right] \int_{-\infty}^{t} \varphi(\vec{r}', t')dt' \right]. \tag{77}$$

In the presence of a gravitational field there is a particle drift in the $\hat{y}$ direction of g/Ω in addition to the uniform circular motion so that

$$y' - y = \frac{v_\perp}{\Omega}\sin(\Omega t' + \psi) + g/\Omega t' - \frac{v_\perp}{\Omega}\sin\psi,$$

where ψ is an initial phase. The orbits are otherwise the same as Eq. (34-37). Then from $\exp i\gamma \sin\psi = \sum_n J_n(\gamma)\exp in\psi$ we write as before

$$\varphi(\vec{r}'(t'), t') = \hat{\varphi} \exp i\left(K_\parallel v_\parallel t' + K_y \frac{v_\perp}{\Omega}\sin(\Omega t' + \psi) + K_y \frac{g}{\Omega} t' + \omega t'\right) - i\frac{K_y v_\perp}{\Omega}\sin\psi$$

$$= \left\{\hat{\varphi} \exp it'\left(K_\parallel v_\parallel + K_y \frac{g}{\Omega} + \omega\right)\right\}$$

$$\times \sum_n J_n\left(\frac{K_y v_\perp}{\Omega}\right)\exp in(\Omega t' + \psi) \sum_m J_m\left(\frac{K_y v_\perp}{\Omega}\right)\exp(-im\psi). \tag{78}$$

As usual, f_1 is assumed to have the same space-time dependence as φ, i.e.

$$f_1 = \hat{f}_1 \exp(iK_y y + iK_z z + i\omega t).$$

After substituting (78) into (77) and doing the time integration and letting $x = 0$,

$$\hat{f}_1 = \hat{\varphi} n_0 \left(\frac{e}{m}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left\{ -\frac{m}{kT} + \left[\frac{\omega m}{kT} + \frac{\epsilon K_y}{\Omega}\right]\right.$$

$$\left. \times \sum_{m,n} \frac{J_n\left(\frac{K_y v_\perp}{\Omega}\right) J_m\left(\frac{K_y v_\perp}{\Omega}\right)}{\omega + K_\parallel v_\parallel + n\Omega + K_y g/\Omega} e^{i(n-m)\psi} \right\} e^{-\frac{mv^2}{2kT}}\left[1 + \frac{\epsilon v_y}{\Omega}\right]. \tag{79}$$

To obtain the dispersion relation we must first evaluate

$$\rho = e\int d\psi \frac{dv_\perp^2}{2} dv_\parallel f_1. \tag{80}$$

Because of

$$\int_0^{2\pi} d\psi e^{i(n-m)\psi} = 2\pi\delta_{m,n}$$

the angle integration gives, after summing over m,

$$\int_0^{2\pi} d\psi \hat{f}_1 = \hat{\phi} 2\pi n_0 \left(\frac{e}{m}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left\{ -\frac{m}{kT} + \left[\frac{\omega m}{kT} + \frac{K_y \epsilon}{\Omega}\right]\right.$$

$$\left. \times \sum_n \left[1 + \frac{n\epsilon}{K_y}\right] \frac{J_n^2\left(\frac{K_y v_\perp}{\Omega}\right)}{\omega + K_\parallel v_\parallel + n\Omega + K_y g/\Omega}\right\} \exp\left[-\frac{m(v_\perp^2 + v_\parallel^2)}{2kT}\right]. \tag{81}$$

For our purposes $(n\epsilon/K_y) \ll 1$ in all applications and we neglect it henceforth. Integrating over $dv_\perp^2/2$ and using the identity

$$\int_0^\infty J_n^2(\alpha x)\, e^{-x^2} dx = e^{-\alpha^2/2} I_n(\alpha^2/2),$$

where $I_n(x) = i^{-n} J_n(ix)$, gives

$$\int_0^{2\pi} d\psi \int_0^\infty \frac{dv_\perp^2}{2} \hat{f}_1 = \hat{\phi} 2\pi n_0 \left(\frac{e}{m}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left\{ -1 + \left[\omega + \frac{kTK_y\epsilon}{m\Omega}\right]\right.$$

$$\left. \times \sum_n \frac{I_n(K_y^2 kT/\Omega^2 m) \exp - (K_y^2 kT/\Omega^2 m)}{\omega + K_\parallel v_\parallel + n\Omega + K_y g/\Omega}\right\} \exp - \frac{m v_\parallel^2}{2kT}$$

or rearranging terms

$$\int_0^{2\pi} d\psi \int_0^\infty \frac{dv_\perp^2}{2} \hat{f}_1 = \hat{\phi} 2\pi n_0 \left(\frac{e}{m}\right)\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \left\{ -1 + \left[1 + \frac{kTK_y\epsilon}{m\omega\Omega}\right]\right.$$

$$\times \sum_n \left(\frac{\omega}{\omega + n\Omega + K_y g/\Omega}\right) \left[\exp - \left(\frac{K_y^2 kT}{\Omega^2 m}\right)\right]$$

$$\left. \times I_n\left(\frac{K_y^2 kT}{\Omega^2 m}\right)\left[1 - \frac{K_\parallel v_\parallel}{\omega + n\Omega + K_y g/\Omega + K_\parallel v_\parallel}\right]\right\} \exp - \frac{m v_\parallel^2}{2kT}.$$

The integration over $v_\parallel$ in equation (80) must still be performed and involves the function

$$W(x) = -\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(\exp - y^2) y\, dy}{x + y} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(\exp - y^2) dy}{(x+y)^2} = \frac{1}{2} Z'(-x).$$

The function $Z(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(\exp - y^2) dy}{y - x}$ was discussed in paper by Dr. Simon and has been tabulated by CONTE and FRIED [10].

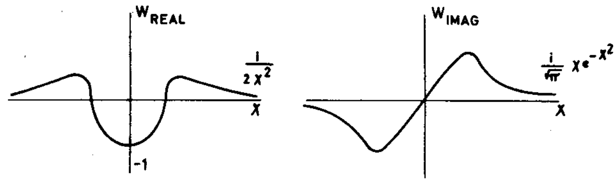

Fig. 5

Properties of W function for real x.

The properties of W are summarized in Fig. 5, for real x. Thus we have

$$\hat{\rho} = e \int_0^{2\pi} d\psi \int_0^{\infty} \frac{dv_\perp^2}{2} \int_{-\infty}^{\infty} dv_\parallel \hat{f}_1$$

$$= \hat{\phi}\left(\frac{2\pi n_0 e^2}{m}\right)\left(\frac{m}{2\pi kT}\right)\left\{-1+\left[1+\frac{kTK_y\epsilon}{m\omega\Omega}\right]\sum_n \frac{\omega \exp - (K_y^2 kT/\Omega^2 m)}{\omega + n\Omega + K_y g/\Omega}\right.$$

$$\left.\times\, I_n\left(\frac{K_y^2 kT}{\Omega^2 m}\right)\left[1+W\left(\frac{\omega+n\Omega+K_y g/\Omega}{K_\parallel (2kT/m)^{1/2}}\right)\right]\right\}.$$

Then substituting in Poisson's equation,

$$K^2\hat{\phi} = 4\pi \sum_{j=e,i} \hat{\rho}_j$$

gives the dispersion relation we have been seeking:

$$K_y^2 = \sum_{j=e,i} \frac{1}{\lambda_{D_j}^2}\left\{-1+\left[1+\frac{kT_j K_y}{m\Omega\omega}\left(\frac{m_j g}{kT_j}+\frac{1}{n_j}\frac{dn_j}{dx}\right)\right]\right.$$

$$\left.\times \sum_n \frac{\omega e^{-Z_j} I_n(Z_j)}{\omega + n\Omega_j + K_y g/\Omega_j}\left[1+W\left(\frac{\omega+n\Omega_j+K_y g/\Omega_j}{K_\parallel (2kT_j/m_j)^{1/2}}\right)\right]\right\} \tag{82}$$

where

$$\lambda_D = (kT/4\pi n_0 e^2)^{1/2} \quad \text{and} \quad Z \equiv K_y^2 \langle R_L^2 \rangle = K_y^2 kT/m\Omega^2.$$

In the absence of g, $\epsilon kT/m\Omega = \langle R_L^2\rangle \Omega\epsilon = v_{drift}$, where v_{drift} is the average velocity determined by the diamagnetic current, and $\epsilon = (1/n)(dn/dx)$. (Note $v_{de} = -v_{di}\, T_e/T_i$.) In what follows, we sometimes use R_i, R_e for the electron Larmor radii.

This dispersion relation will be applied to the following cases:

Case I — $K_\parallel = 0$, $\omega \ll \Omega$.

This case gives the flute instability and the finite Larmor radius effects.

Case II — $v_{th,e} > (\omega/K_{\parallel}) > v_{th,i}$, $\omega \ll \Omega$.

This case gives the drift instability.

Case III — $\omega \simeq \Omega_i = K v_{drift}$.

This is the Mikhailovsky-Timofeev instability.

Before proceeding further with the application of Eq. (82) to these specific cases I would like at this point to make a few remarks about how the procedure must be modified to consider the x-dependence of φ. To do that, as we have mentioned earlier, it would have been necessary to include $\varphi(x')$ in the orbit integration. Using the appropriate orbital values for $(x' - x)$ and proceeding by expansion of φ, it is easily found that the only change in Eq.(82) is to replace K_y^2 where it appears (on the left hand side, and in the expression for Z) by $(K_y^2 - d^2/dx^2)$. Neglecting $K^2\lambda_D^2$ for simplicity we see that an infinite order differential equation for φ would result from expansion of $e^{-Z}\mathrm{Im}(Z)$. However, each factor $(d^2/dx^2)^n$ is multiplied by a factor $(R_i^2)^n$. If the Larmor radius is small, in a sense to be discussed, we may keep only the lowest order term to obtain

$$\frac{d^2\varphi}{dx^2} + \frac{1}{R_i^2} D(x, \omega)\varphi = 0,$$

where $D(x, \omega)$ is, except for a possible multiplicative factor, the usual "localized" dispersion relation, Eq. (82). If the x-dependence of D is weak, i. e. if the equilibrium quantities which appear in D do not vary much over the distance R_i, we may expand around some value x_0. For example in Eq.(82), since we have neglected $K^2\lambda_D^2$, the only spatially varying term is $(1/n)(dn/dx)$. Let x_0 be a point where $(1/n)(dn/dx)$ has an extremum, and let ω_0 be determined by $D(x_0, \omega_0) = 0$. Then we may expand $\omega = \omega_0 + \delta\omega$, $x = x_0 + \delta x$ to find

$$\frac{d^2\varphi}{dx^2} + \frac{1}{R_i^2}\left\{\frac{\partial D}{\partial \omega}\bigg|_{\substack{x=x_0\\ \omega=\omega_0}} \delta\omega + \frac{1}{2}\frac{\partial^2 D}{\partial x^2}\bigg|_{\substack{x=x_0\\ \omega=\omega_0}} (\delta x)^2\right\}\varphi = 0.$$

This is of course just Hermites equation permitting well behaved localized solutions if $\partial^2D/\partial x^2 < 0$. Actually the question is not quite so simple since $\partial^2D/\partial x^2$ is in general complex, not real. None the less, solutions may be obtained which behave asymptotically as $\exp\left[\pm \frac{(\delta x)^2}{2}\frac{1}{R_i}\left(-\frac{1}{2}D''\right)^{\frac{1}{2}}\right]$ so that if D varies over a scale large compared to R_i (i. e. if $R_i^2 D'' < 1$), and if D'' is not close to a positive real number, the eigenfunction φ will only be large in a small region close to x_0 where our expansion of D is valid. Note also that the higher order terms $(R_i)^{2n}d^{2n}\varphi/dx^{2n}$ will be small in this region.

Using the well-known eigenvalues of Hermite's equation we may also determine the frequency shift of a spectrum of discrete eigenvalues (n = 0, 1, 2 etc.)

$$\delta\omega = \left(n + \frac{1}{2}\right) \frac{R_i\left(-2\,\frac{\partial^2 D}{dx^2}\right)^{\frac{1}{2}}}{\frac{\partial D}{\partial \omega}} .$$

It will turn out that the cases we wish to consider all satisfy these conditions so that the localized eigenvalues are a good approximation except for certain cases involving a sheared magnetic field in which case $k_\parallel$ depends on x and as the dispersion relation is quite sensitive to $k_\parallel$ the approximation breaks down. We discuss these cases later.

IV. SPECIAL CASES OF MICROINSTABILITIES OF HOMOGENEOUS PLASMAS

The first application of our general dispersion relation Eq. (82) to be considered is the familiar flute or hydromagnetic interchange instability [11]. Properly speaking, this is not really a microinstability but a hydrodynamic one, since it involves mass motion of both electrons and ions frozen to magnetic field lines in the limit of zero Larmor radius. In this limit the instability in fact reduces to the well-known Rayleigh-Taylor instability induced by a density gradient in the presence of an unfavourable gravitational force. We discuss this case in order to show how these results are modified by microscopic effects. Needless to say the force $m\vec{g}$ is really only a convenient representation of the forces which arise in an inhomogeneous magnetic field so that $m\vec{g} \equiv -\mu\nabla B + mv_\parallel^2/R$, where R is the radius of curvature of a field line and μ the particle magnetic moment. These forces give rise to guiding centre drifts which enter the equation in the same way as the gravitational drift we have used.

The flute instability is characterized by the fact that the potential is constant along a magnetic field line. This means that in Eq. (82) we choose $k_\parallel = 0$, $W = 0$. In distinction to the drift instability this implies that there is no finite length condition which must be met since no problem arises of fitting a wave-length of the instability into the plasma, unless the field line is shorted at the ends by conducting end plates. Such shorting would not occur in an open-ended system in the absence of cold isotropic plasma, as the hot plasma would then not be in electrical contact with the ends. We also consider that the frequency is very low compared to the cyclotron frequency so that only the term n = 0 need be considered in Eq. (82). Making these approximations we find that the dispersion relation simplifies to

$$K_y^2 = \sum_{j=e,i} \frac{1}{\lambda_{D_j}^2} \left\{ -1 + \left[1 + \frac{K_y k T_j}{m_j \Omega_j} \frac{1}{n_j} \frac{dn_j}{dx} \frac{1}{\omega + K_y g_j/\Omega_j}\right] e^{-Z_j} I_0(Z_j) \right\} \tag{83}$$

We may now solve directly for ω. To simplify this a bit further we assume $n_e = n_i$, $(1/n_e)(dn_e/dx) = (1/n_i)(dn_i/dx)$, $Z_e = 0$, i.e. neglecting the electron Larmor radius, and putting $m_e g_e = m_i g_i (T_e/T_i)$, since the actual forces are proportional to particle energy. In what follows we omit the subscripts, as all quantities refer to the ions. We write the dispersion realtion in dimensionless form introducing

$$\omega = \frac{K_y g}{\Omega}\left(Y + \frac{T_e}{T_i}\right) \quad \text{and} \quad 1 + \frac{T_e}{T_i} = \tau$$

to obtain a quadratic equation for Y:

$$Y^2 + Y[\tau + \alpha\{1 - e^{-Z} I_0(Z)\}] + \alpha\tau = 0, \tag{84}$$

where

$$\alpha = \frac{kT}{mg}\frac{1}{n}\frac{dn}{dx}\frac{1}{K^2\lambda_D^2 + \{1 - e^{-Z} I_0(Z)\}}. \tag{85}$$

Solving for Y we find

$$Y = \frac{-\{\tau + \alpha[1 - e^{-Z} I_0(Z)]\} \pm \sqrt{\{\tau + \alpha[1 - e^{-Z} I_0(Z)]\}^2 - 4\alpha\tau}}{2}.$$

Instability will obtain if the argument of the square root is negative. The most interesting case, and also the most unstable one, is the case of small Larmor radius, $Z \ll 1$, in which case we may put $1 - e^{-Z} I_0(Z) = K^2 R_i^2$ and obtain for the growth rate

$$\omega_i = \frac{K_y g}{\Omega}\left(\alpha\tau - \frac{1}{4}\{\tau + \alpha K^2 R_i^2\}^2\right)^{\frac{1}{2}}, \tag{86}$$

where, in this approximation,

$$\alpha = \frac{kT}{mg}\frac{1}{n}\frac{dn}{dx}\frac{1}{K_y^2}\frac{1}{\lambda_D^2 + R_i^2}.$$

The usual hydrodynamic result is obtained by neglecting the stabilizing term in (86) since $1/K_Y^2(\lambda_D^2 + R_i^2) \gg 1$ and putting $R_i^2 \gg \lambda_D^2$. Substituting $R_i^2 = kT/m\Omega^2$, we obtain directly the hydrodynamic result

$$\omega_i \approx \left(g\,\frac{1}{n}\frac{dn}{dx}\,\tau\right)^{\frac{1}{2}}.$$

In order to study the effect of the stabilizing terms let us apply Eq. (86) to the usual open-ended confinement (mirror) system. Since the longest wave lengths are most unstable we will set $K_Y \approx 1/r = (1/n)(dn/dx)$, the smallest possible wave number. Here r is the plasma radius. Similarly $kT/mg \approx R$ where R is the radius of curvature of the magnetic field lines. Generally $R \gg r$ and $R > 0$ for mirror-type fields, $R < 0$ for stabilized fields.

For stability, we require from Eq. (86)

$$\tau + \frac{R}{r}\frac{\epsilon - 1}{\epsilon} > 2\sqrt{\tau}\,\frac{(Rr)^{\frac{1}{2}}}{R_i}\left(\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{2}}, \tag{87}$$

where $\epsilon = 1 + 4\pi n M c^2/B^2$, being the usual plasma dielectric constant.

We see that at high densities ($\epsilon \to \infty$) the equilibrium will be stable if $R_i/r > 2\sqrt{\tau}\,(r/R)^{\frac{1}{2}}$. This is the so-called finite Larmor radius stabilization [12] and shows that if the radius of curvature is quite large, as is usually the case, its destabilizing effect may be neutralized even by a Larmor radius considerably smaller than the plasma radius. The physical origin of this effect is that the $\vec{E}\times\vec{B}/B^2$ drift which is characteristic of hydrodynamic motion must in fact be averaged over the orbit of a particle. Because of their finite Larmor radius the ions see a slightly different perturbed electric field than the electrons, causing a slight out of phase motion which may lead to stabilization. Unfortunately, in a cylindrical system there is one possible unstable mode, the mode with azimuthal wave number m = 1, which corresponds to a uniform sideways displacement of the plasma (i.e. a constant electric field) and is not stabilized by this mechanism. However it does apply to higher modes of instability for which the stability condition may be seen to be $(R_i/r) > (2\sqrt{\tau}/m)\,(r/R)^{\frac{1}{2}}$.

At low densities the system also becomes stable, as seen from Eq.(87), at a density such that $\lambda_{Di} > (2/\sqrt{\tau})(Rr)^{\frac{1}{2}}$ [13]. This effect has been clearly observed in neutral-injection experiments where the density builds up smoothly with time as more and more particles are injected until such time that the density is sufficient to violate the above condition. After this, flutes are observed and the density is not able to increase above this critical value since the flutes carry plasma to the wall as rapidly as it is being injected. The physical mechanism for stabilization is simply that at very low densities the collective interactions become very weak, so that the characteristic time for oscillation becomes longer than the drift time $K_y g/\Omega$, after which stabilization occurs. It will be noted that if $R_i < r$, as we have assumed, then there is always an intermediate range of density which is unstable.

Next we discuss the so-called drift instability [14, 15, 16, 17]. This is a low frequency mode with $\omega \ll \Omega$, so we need consider only n = 0 and we also choose $v_{th,e} = \left(\frac{kT_e}{m_e}\right)^{\frac{1}{2}} > \frac{\omega}{k_\parallel} > v_{th,i} = \left(\frac{kT_i}{m_i}\right)^{\frac{1}{2}}$. We thus set $W_i = 0$; and $W_e(x) = -1 + (i/\sqrt{\pi})x$. We will also set $g \equiv 0$ for this application. It is seen therefore that the drift instability does not arise from gravitational destabilization but rather from the diamagnetic currents inherent to a confined plasma. For this reason it has sometimes been called the "universal" instability, somewhat of a misnomer as we shall see. We will also neglect $K_y^2\lambda_D^2$ and $K_y^2R_e^2$ compared to unity, and put $T_e = T_i$ for algebraic simplicity. With these approximations our basic dispersion relation Eq. (82) reduces to

$$2 = \left[1 + \frac{K_y v_d}{\omega}\right] e^{-Z_i}\, I_0(Z_i) + i\left[1 - \frac{K_y v_d}{\omega}\right]\frac{1}{\sqrt{\pi}}\,\frac{\omega}{|k_\parallel|(2T/m_e)^{\frac{1}{2}}}\,, \qquad (88)$$

where v_D the diamagnetic drift velocity has been defined as

$$v_d = \frac{1}{n}\frac{dn}{dx}\frac{kT}{m\Omega} \approx R_i\,\frac{1}{n}\frac{dn}{dx}\, v_{th,i}\,.$$

This may now be easily solved for ω, regarding the imaginary terms as a small correction. We find

$$\omega = K_y v_d \frac{e^{-Z} I_0(Z)}{2 - e^{-Z} I_0(Z)} - \frac{2i}{\sqrt{\pi}} \frac{(K_y v_d)^2}{|k_\parallel| \sqrt{\frac{2T_e}{m_e}}} \frac{e^{-Z} I_0(Z)[1 - e^{-Z} I_0(Z)]}{\{2 - e^{-Z} I_0(Z)\}^3}, \tag{89}$$

so that the mode is always unstable. The basic frequency $\omega = K_y v_d$ can be seen to arise from the requirement of charge neutrality in a simple model where the ions move across the field with velocity E_y/B while the electrons adjust their density along the lines according to $\exp(-e\varphi/kT)$. These different modes of behaviour arise from the different relationships between the phase velocity of the wave and the thermal velocities of the two species.

The imaginary part of ω arises from a competition between two Landau damping type terms — the ordinary stabilizing term arising from electron motion along field lines and a destabilizing term arising from the change in resonant electron density due to the initial density gradient, as discussed by Dr. Simon.

A slight modification of these results is obtained if there exists in the equilibrium a drift velocity u [18], parallel to the magnetic field, between the electrons and ions. In this situation in the derivation ω must be interpreted as the frequency in the rest frame of the species being considered. Thus in Eq. (88), in the imaginary term, which arose from the electrons, ω should be replaced by $\omega - k_\parallel u$, giving rise for $Z_i = 0$ to the frequency $\omega \approx K_y v_d \left(1 - \frac{i}{\sqrt{\pi}} \frac{u}{v_{th,e}}\right)$. Fairly large drifts may occur in such situations as Ohmic heating and there appears to be good reason to suspect that these modes may be responsible for stellarator pump-out. Other modes would be obtained if we had considered $\nabla T \neq 0$ [14].

The growth rates determined from Eq. (88) are quite small except in the limit of very short wave length perpendicular to the magnetic field and very long wave length parallel to the magnetic field.

We note in fact that the requirement $\omega > k_\parallel v_{th,i}$, necessary to avoid ion Landau damping, leads to a fairly severe critical length requirement since, for lengths less than

$$L_{cr} = \frac{\pi}{k_\parallel} = \pi \frac{v_{th,i}}{\omega} = \frac{\pi v_{th,i}}{K_y R_i v_{th,i} \frac{1}{n}\frac{dn}{dx}} \frac{(2 - [\exp - K_y^2 R_i^2] I_0(K_y^2 R_i^2))}{\exp[- K_y^2 R_i^2] I_0(K_y^2 R_i^2)},$$

the instability cannot occur. For wave lengths long compared to the ion gyro radius this is quite restrictive while even in the limit $K_y R_i \to \infty$ we obtain $L_{cr} \approx (2\pi)^{\frac{3}{2}} \Big/ \frac{1}{n}\frac{dn}{dx}$, a fairly long thin plasma. Waves at the frequencies predicted for the drift instability, have been observed in Cs plasmas.

The drift instability may also be stabilized by a shear in the magnetic field. Thus if $B_z = B_0$, $B_y = B_0 Sx$, where S^{-1} is the field shearing distance, we have $k_{\parallel} = k_y Sx$. The requirement $\omega = k_y v_d > k_{\parallel} v_{th,i}$ to avoid ion Landau damping over the entire plasma, i.e. for $x \approx \left(\frac{1}{n}\frac{dn}{dx}\right)^{-1}$ leads to the stability criterion

$$S > R_i \left(\frac{1}{n}\frac{dn}{dx}\right)^2, \tag{90}$$

a rather modest amount of shear.

It should also be evident that, since this drift instability occurs at very low frequency and conserves the particle magnetic moment μ, it must also be stable for distributions of the form $f(\mu, \epsilon)$ (in the case of magnetic wells), as discussed by Dr. Taylor. A direct calculation starting with this type of distribution function shows in fact that this is so.

We now turn to another type of instability characteristic of an inhomogeneous plasma — this time a high frequency instability in which particle magnetic moment is not conserved [19]. Again we return to Eq. (82) under the circumstances that $k_{\parallel} = 0$, $g = 0$, $Z_i >> 1$, $Z_e < 1$ and $\omega \ll \Omega_e$, so only $n = 0$ need be retained for electrons, and $\omega \approx \Omega_i$, so only $n = -1$ need be retained for ions. For simplicity we put $T_e = T_i$, $n_e = n_i$. Thus Eq. (82) reduces, using the asymptotic result, $\lim_{Z \to \infty} e^{-Z} I_0(Z) = \frac{1}{(2\pi Z)^{\frac{1}{2}}}$, to

$$\left(1 + \frac{k_y v_D}{\omega}\right)\left(-1 + \frac{\omega}{\omega - \Omega}\,\frac{1}{(2\pi Z_i)}\right) = K_y^2(\lambda_D^2 + R_e^2) \tag{91}$$

The right hand side of Eq. (91) may be regarded as composed of small correction terms so that we might expect that for most choices of parameters the dispersion relation will yield two waves, a drift wave $\omega = -K_y v_D$ and an ion cyclotron wave $\omega = \Omega/[1 - (\sqrt{2\pi}|K_y|R_i)^{-1}]$. However in the event that K_y is chosen properly so that both requirements may be satisfied simultaneously, then it becomes possible to obtain complex roots.

Reducing Eq. (91) to a quadratic we obtain

$$\omega^2\left[\left(1 - \frac{1}{(2\pi Z_i)^{\frac{1}{2}}} + K_y^2(\lambda_D^2 + R_e^2)\right)\right]$$

$$+\omega\left[K_y v_D\left(1 - \frac{1}{(2\pi Z)^{\frac{1}{2}}}\right) - \Omega\left(1 + K_y^2(\lambda_D^2 + R_e^2)\right)\right] - K_y v_D \Omega = 0.$$

Thus the stability condition is given by

$$\left[K_y v_D\left(1 - \frac{1}{(2\pi Z)^{\frac{1}{2}}}\right) - \Omega\left(1 + K^2(\lambda_D^2 + R_e^2)\right)\right]^2 + 4K_y v_D \Omega\left(1 + K_y^2(\lambda_D^2 + R_e^2) - \frac{1}{(2\pi Z_i)^{\frac{1}{2}}}\right) > 0$$

or

$$\left[K_y v_D\left(1 - \frac{1}{(2\pi Z)^{\frac{1}{2}}}\right) + \Omega\left(1 + K_y^2(\lambda_{D_e}^2 + R_e^2)\right)\right]^2 + 4K_y v_D \Omega K_y^2(\lambda_D^2 + R_e^2)\,\frac{1}{(2\pi Z)^{\frac{1}{2}}} > 0. \tag{92}$$

Since both $K_y^2(\lambda_D^2+R_e^2)$ and $1/\sqrt{Z_i}$ are small it is evident that the wave will be stable unless the first term almost vanishes, in which case the second term will permit an instability with growth rate (setting $-k_y v_D \approx \Omega$)

$$\omega_i \approx \Omega\left(\frac{1}{\sqrt{2\pi}} K_y \frac{(\lambda_D^2+R_e^2)}{R_i}\right)^{\frac{1}{2}} \approx \frac{1}{4\sqrt{2\pi}}\,\Omega\left(\frac{\lambda_D^2+R_e^2}{R_i^2}\,\frac{1}{R_i\frac{1}{n_i}\frac{dn_i}{dx}}\right)^{\frac{1}{2}}. \tag{93}$$

However it is not possible in all cases to choose K_y so as to make the first term vanish and the condition that the first term does not vanish for any real K_y yields the final condition for stability

$$\lambda_D^2+R_e^2 > \frac{1}{4}\frac{v_d^2}{\Omega^2} = \frac{1}{4}R_i^2R_i^2\left(\frac{1}{n}\frac{dn}{dx}\right)^2,$$

or

$$\left(\frac{B^2}{4\pi nMc^2}+\frac{m}{M}\right)^{\frac{1}{2}} > R_i\,\frac{1}{n}\frac{dn}{dx}\,. \tag{94}$$

Thus plasmas with a sufficiently gentle gradient $R_i\,\frac{1}{n}\frac{dn}{dx} < \sqrt{\left(\frac{m}{M}\right)}$ or of low density with $\epsilon \approx 1$ are not affected by this mode. The experiments of Dr. Ioffe for example appear to be right at the borderline of predicted instability. Since this mode has such an extremely short wave length one might hope it leads to rather small diffusion and perhaps also may be easily stabilized by shear.

I trust that it has at least become clear at this point that these micro-instability calculations, even in their linear phase, are based on very highly idealized models and simplifications so that considerably more work is required to apply the results with confidence to inhomogeneous laboratory plasmas. Ultimately theory must depend on much detailed comparison with experiment, and experiments should be designed with regard to avoiding, if possible, and measuring, if necessary, the instabilities qualitatively predicted by theory.

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PLASMA STABILITY

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I. PROBLEMS IN PLASMA STABILITY

In plasma stability analysis, the energy principle of classical magnetohydrodynamics has been extensively used as a very powerful tool for inhomogeneous plasmas in static equilibrium [1]. On the other hand, the microinstabilities of a homogeneous plasma can be found by solving a dispersion relation for a complex frequency ω [2]. But, as soon as one becomes interested in inhomogeneous plasmas which are not in static equilibrium, or if one wants to include such corrections as finite Larmor radius or finite conductivity effects, none of these methods can be used to solve the stability problem.

So far no practical way of handling general stability problems has been made available, and convenient techniques for each particular case must always be sought.

In this paper we will limit ourselves to conservative systems for which the linearized equations of the perturbed motion can be written in the compact form:

$$H(\omega)\xi = 0, \tag{1}$$

in which ξ is a set of perturbed variables and H is an operator depending on the equilibrium quantities and function of ω as a parameter. It has been assumed that ξ has the time dependence

$$\xi \sim e^{i\omega t}, \tag{2}$$

and H has the property

$$H(\omega^*) = H^+(\omega). \tag{3}$$

It is self-adjoint for ω real.

For the following discussion, we shall give a more explicit form to H, namely;

$$H(\omega) \equiv -\omega^2 N + 2i\omega J + U \tag{4}$$

in which N, iJ, and U are hermitian operators and we assume $(\xi, N\xi)$ to be positive definite. This is not a restriction but only an example to illustrate the method, and all the results can be readily extended to more general forms of H. This expression of H is actually the one we would get, for any conservative system, from a Lagrangian formulation [3, 4].

Thus we have to solve the equation

$$-\omega^2 N\xi + 2i\omega J\xi + U\xi = 0 \tag{5}$$

with appropriate boundary conditions and find the corresponding eigenfrequencies ω. If one of these values is complex, the system will be unstable.

If ξ is a solution of the equation,

$$-\omega^2(\xi, N\xi) + 2i\omega(\xi, J\xi) + (\xi, U\xi) = 0 \tag{6}$$

then ω is given by a quadratic equation. We can state the two important conditions [5]:

1. If $(\xi, U\xi) \geq 0$ for any ξ which satisfies the boundary conditions, the system is stable.
2. If $|(\xi, J\xi)|^2 + (\xi, N\xi)\ (\xi, U\xi) \geq 0$ for any ξ which satisfies the boundary conditions, the system is stable.

These two conditions are sufficient for stability. The second one is less stringent than the first one but is more difficult to apply. In the case where $J \equiv 0$, both of these conditions are equivalent and also necessary.

The proof is straightforward and left to the reader [5].

These results are the only ones which are available without going to a detailed investigation of the equation, and that can only be done on specific problems. Fortunately, it frequently occurs that for a particular choice of parameters in the equilibrium and with some approximations, it is possible to solve the problem completely. The question is how to obtain more information on stability for other values of the parameters and in a more refined approximation. A partial answer is LOW's theorem [6].

Low's Theorem

Let H_0 be the operator corresponding to an idealized system and δH the small change in H due to a small variation of the system. ξ_0 is a solution of the unperturbed equation corresponding to the real frequency ω_0.

We shall look for a solution ξ in the form

$$\xi = \xi_0 + \xi_1$$

and

$$\omega = \omega_0 + \omega_1, \tag{7}$$

in which ξ_1 and ω_1 are assumed to be of the same order as δH. Up to this order, Eq. (1) becomes

$$H_0(\omega_0)\xi_1 = 2\,\omega_1 [\,\omega_0 N_0 \xi_0 - i\, J_0 \xi_0] - \delta H(\omega_0)\xi_0 . \tag{8}$$

Taking the scalar product of both sides by ξ_0 and using Eq. (1), we get

$$\omega_1 = \frac{(\xi_0, \delta H(\omega_0)\xi_0)}{2\omega_0(\xi_0, N_0\xi_0) - 2i(\xi_0, J_0\xi_0)} . \tag{9}$$

Since δH, N_0, and iJ_0 are hermitian operators, ω_1 is real and the mode remains stable. The first part of the theorem can now be stated:

Any stable mode of a system remains stable in a small change of the parameters of this system.

However this result breaks down when the coefficient of ω_1 vanishes, i.e.,

$$2\omega_0(\xi_0, N_0\xi_0) - 2i(\xi_0, J_0\xi_0) \equiv \left[\xi_0, \left(\frac{\partial H}{\partial \omega_0}\right)\xi_0\right] = 0. \tag{10}$$

This corresponds to a marginally stable mode in which ω_0 is a double root of the dispersion relation.

In this case, Eq. (8) cannot be solved for ξ_1, unless $(\xi_0, \delta H\xi_0) = 0$. That means that our original assumption regarding the respective orders of magnitude of ξ_1, ω_1, and δH is not consistent. Let us write

$$\omega = \omega_0 + \omega_1 + \omega_2$$

and (11)

$$\xi = \xi_0 + \xi_1 + \xi_2.$$

δH is now assumed to be of second order in this expansion.

The first order terms in Eq. (1) give

$$H_0\xi_1 = 2\omega_1[\omega_0 N_0\xi_0 - iJ_0\xi_0] \tag{12}$$

and this equation has a solution since Eq. (10) holds true.

$$\xi_1 = \omega_1 X_1$$

and (13)

$$H_0 X_1 = \frac{\partial H_0}{\partial \omega_0}\xi_0 .$$

The second order terms give an equation for ξ_2, namely;

$$H_0\xi_2 = \omega_1^2\frac{\partial H_0}{\partial \omega_0} X_1 + \omega_2\left(\frac{\partial H_0}{\partial \omega_0}\right)\xi_0 + \omega_1^2 N_0\xi_0 - \delta H(\omega_0)\xi_0. \tag{14}$$

Taking the scalar product by ξ_0 and using Eqs. (10) and (13) we get

$$\omega_1^2 = \frac{(\xi_0, \delta H\,\xi_0)}{(\xi_0, N_0\xi_0) + (X_1, H_0 X_1)}. \tag{15}$$

The numerator and the denominator of this expression are still real quantities but the mode will be stable or unstable according to whether ω_1^2 is positive or negative.

The second part of the theorem is now achieved [7].

The only modes which can be made unstable by a small change in the parameters of the system are those which are marginally stable, and the perturbed frequency is given by Eq. (15).

Remarks

(1) All the preceding results remain valid for more general operators $H(\omega)$; only N should be replaced by $-\frac{1}{2}\,\partial^2 H/\partial\omega^2$ in Eq. (15).

(2) This method cannot take into account the existence of new modes which can appear if δH changes the structure of the operator. This is the reason for stating that this theorem does not solve completely the stability problem.

Example

As an example, we have taken an equation which has been derived by ROSENBLUTH and SIMON [8] for a cylindrical low-β plasma in a uniform magnetic field including finite Larmor radius effects, macroscopic rotation and a gravity equivalent to the effects of curvature.

$$-H(\omega)\xi \equiv \frac{1}{r}\frac{d}{dr}\left(T\frac{d\xi}{dr}\right) + \left[\frac{1-m^2}{r^3}T + \left(m^2 g + r\omega^2\right)\frac{d\rho}{dr}\right]\xi = 0$$

and (16)

$$T = \left(\omega - \frac{m E_0}{Br}\right)^2 \rho r^3 + m\left(\omega - \frac{m E_0}{Br}\right)\frac{r^2}{\Omega}\frac{dP}{dr}$$

in which ρ is the density, P the plasma pressure m the azimuthal wave number, Ω the ion cyclotron frequency, and E a radial electric field giving a macroscopic rotation.

$$(\xi, H\xi) = \int_0^\infty \xi\, H\xi\, r\,dr$$

$$= \int_0^\infty T\left(\frac{d\xi}{dr}\right)^2 + \left[\frac{m^2-1}{r^2}\,T - (m^2 g + r\omega^2)r\frac{d\rho}{dr}\right]\xi^2 dr \qquad (17)$$

Discussion

i) $m = 1$. There is a marginally stable mode when $g = 0$:

$$\xi = \xi_0 = \text{Const.},\ \omega_0 = 0. \qquad (18)$$

It exists for any density profile, macroscopic velocity and Larmor radius size, all effects which appear only in T. The effect of a small gravity will thus be of very general interest.

Since $(\partial H_0/\partial \omega_0)\xi_0 = 0$, we have $X_1 = 0$ and Eq. (15) gives:

$$\omega_1^2 = \frac{\frac{1}{2}\int_0^\infty g(r)\frac{d\rho}{dr} r\, dr}{\int_0^\infty \rho r\, dr} . \tag{19}$$

(1) If $g > 0$, $d\rho/dr < 0$ and $\omega_1^2 < 0$, then the configuration is unstable.
(2) If $g > 0$ but ρ has a maximum outside the axis then the sign will be determined by a balance between bad and good regions.
(3) For $g = \bar{g}r$ ($\bar{g}$ = const.), we have $\omega_1^2 = -\bar{g}$. An example of a configuration which can be stable is in a double cusp geometry. There cannot be any confinement on the axis where the field is decreasing, but only in a region, as in Fig. 1. g is always negative and the contribution of the region where the density gradient is negative can be made dominant.

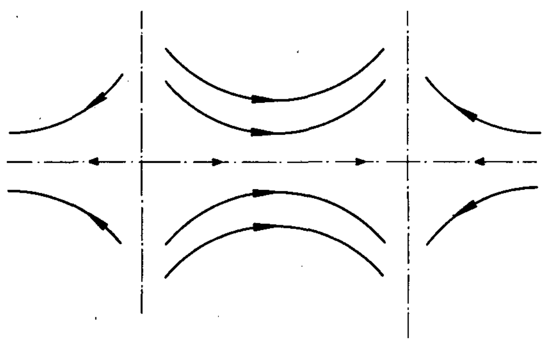

Fig. 1

Example of a field configuration

ii) $m \geq 2$. Equation (16) has been solved completely by making the following assumptions [9]:

$$E_0 = \bar{E}_0 r,\ P \sim \rho \sim e^{-\frac{r^2}{r_0^2}},\ \text{and } g = \bar{g} r.$$

It reduces to a Whittaker's equation and we can find the solutions which vanish at a boundary far enough from the axis.

The result for the lowest radial mode is

$$\xi \sim r^{m-1}$$

$$\omega = (m-1)(\omega_0 + \nu) \pm \left[(m-1)^2(\omega_0+\nu)^2 - m(m-1)\,\omega_0(\omega_0+2\nu) - m\bar{g}\right]^{\frac{1}{2}} \tag{20}$$

where $\omega_0 = \bar{E}_0/B$ and $\nu = \rho i \; V_{th,i}/r_0^2$.

If $\omega_0 = 0$, all the modes $m \geq 2$ will be stable by finite Larmor radius effect if $\nu^2 > 2\bar{g}$

The effect of the electric field depends on the sign of the quantity $\omega_0(\omega_0 + 2\nu)$. A stabilization effect is to be expected only when

$$-2\nu < \omega < 0. \tag{21}$$

We would now be able to study the effects of a small change in density profile, including temperature gradients and electric field gradient, the only condition being to start from a marginally stable mode where,

$$(m-1)^2\nu^2 - m(m-1)\omega_0(\omega_0+2\nu) - m\bar{g} = 0. \tag{22}$$

II. INSTABILITIES OF A PLANE CURRENT SHEET

It is well known from the Magnetohydrodynamics energy principle that a plane current sheet in equilibrium with its own magnetic field is stable. This is because of the constraint of the flux conservation. If we relax this constraint by taking into account additional terms in the Ohm's law, there can be perturbations which modify the magnetic field lines' topology. This can be done with a resistivity term in the Ohm's law.

We shall show that another interesting result can be derived by taking into account the finite electron mass.

In order to keep a symmetry between ions and electrons we shall use a two-fluid model with scalar pressure for simplicity.

For each species, the equations of motion are thus;

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot n\vec{v} = 0, \tag{23}$$

$$nm \frac{d\vec{v}}{dt} = nq\,(\vec{E} + \vec{v} \times \vec{B}) - \vec{\nabla} p, \tag{24}$$

$$\frac{d}{dt}(pn^{-\gamma}) = 0, \tag{25}$$

and the Maxwell's equations are written down in the quasi neutrality condition;

$$n_i = n_e = n, \tag{26}$$

$$\frac{\partial \vec{B}}{\partial t} = - \vec{\nabla} \times \vec{E}, \tag{27}$$

and

$$\vec{\nabla} \times \vec{B} = \mu_0 nq(\vec{v}_i - \vec{v}_e). \tag{28}$$

Equilibrium

It is easy to confirm that the following quantities verify the Eqs.(23)to (28) with $\partial/\partial t \equiv 0$.

$$n = \frac{n_0}{\mathrm{Ch}^2 \alpha x}, \tag{29}$$

$$\vec{B} = B_0 \,\mathrm{th}\, \alpha x \, \vec{e}z, \tag{30}$$

$$\vec{v}_i = - u_i \vec{e}_y, \ \vec{v}_e = u_e \vec{e}_y, \tag{31}$$

and

$$p_{i,e} = n_0 \theta_{i,e} \quad (\theta = kT), \tag{32}$$

where u_i and u_e are positive constants and α, n_0, B_0 are related by:

$$B_0^2 = 2\mu_0 n_0(\theta_i + \theta_e), \tag{33}$$

$$\alpha^2 = \frac{\mu_0 n_0 q^2 (u_i + u_e)^2}{\theta_i + \theta_e}. \tag{34}$$

The electric field is determined by the quasi neutrality condition:

$$E\left(\frac{1}{\theta_i} + \frac{1}{\theta_e}\right) = \left(\frac{u_i}{\theta_i} + \frac{u_e}{\theta_e}\right) B. \tag{35}$$

We can always choose a reference system in which

$$\frac{u_i}{\theta_i} = \frac{u_e}{\theta_e} \tag{36}$$

because the system is invariant under any Galilean transformation (in this non-relativistic approximation).

n and B_z have the form shown in Fig. 2.

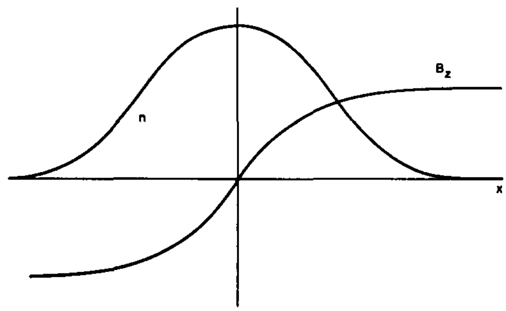

Fig. 2

n and B_z curves versus x

Stability of tearing modes perturbations

We shall study the stability of this equilibrium against perturbations which are invariant along the current ($\partial/\partial_y \equiv 0$).

Let us write the perturbed equations for the ions:

$$-nm_i\omega^2\xi_{ix}^2 = nq\left(E_x + \delta v_y B - u_i \frac{dA_Y}{dx}\right) - qu_i B\delta n_i - \frac{d}{dx}(\delta p_i) \tag{37}$$

$$\delta v_y = -\frac{q}{m_i}(A_y + \xi_{ix} B) \tag{38}$$

$$-nm_i\omega^2\xi_{iz}^2 = nq(E_z - iku_i A_y) - ik\,\delta p_i \tag{39}$$

where we have used ξ_{ix}, ξ_{iz}, A_y as new variables in term of which we have

$$\delta v_{ix} = i\omega\xi_{ix} \tag{40}$$

$$\delta v_{iz} = i\omega\xi_{iz} \tag{41}$$

$$E_y = -i\omega A_y \tag{42}$$

$$B_x = -ik A_y \tag{43}$$

$$B_z = \frac{dA_y}{dx} \tag{44}$$

$$\delta n_i = -\vec{\nabla}\cdot n\vec{\xi}_i \tag{45}$$

$$\delta p_i = -\theta_i\left[\gamma n\vec{\nabla}\cdot\vec{\xi}_i + \xi_{ix}\frac{dn}{dx}\right]. \tag{46}$$

A similar set of equations are derived for the electrons and the remaining conditions from Maxwell's equations give:

$$\frac{1}{\mu_0}\left(k^2 A_y - \frac{d^2A_y}{dx^2}\right) = nq\,(\delta v_{iy} - \delta v_{ey}) - q\,(u_i + m)\,\delta n \tag{47}$$

$$k\left(\frac{dE_z}{dx} - ik\,E_x\right) = i\omega^2\mu_0 nq\,(\xi_{ix} - \xi_{ex}) \tag{48}$$

$$\vec{\nabla}\cdot n\vec{\xi}_i = \vec{\nabla}\cdot n\vec{\xi}_e \tag{49}$$

From this system we can form a variational principle for ω^2, which is a quadratic form in ξ_{ix}, ξ_{iz}, ξ_{ex}, ξ_{ez} and A_y subject to the constraint condition given in Eq. (49). This is easily done by multiplying respectively Eqs. (37) and (39) by ξ_{ix}^* and ξ_{iz}^*, the corresponding equations for the electrons by ξ_{ex}^* and ξ_{ez}^* Eq. (47) by A_y^*, and integrating over x. Using Eqs. (38), (48) and (49) to eliminate δv_y and the other electric field components we finally get after some integrations by parts:

$$\omega^2\int_{-\infty}^{+\infty} n\left\{m_i|\xi_i|^2 + m_e|\xi_e|^2 + \frac{\mu_0 q^2 n}{k^2}|\xi_{ix} - \xi_{ex}|^2\right\}dx$$

$$= \int_{-\infty}^{+\infty} dx\left\{\gamma\theta_i n\left|\vec{\nabla}\cdot\vec{\xi}_i - \frac{qu_i}{\gamma\theta_i}(A_y + \xi_{ix}B)\right|^2 + \frac{nq^2}{m_i}\left[1 + (\gamma-1)\frac{m_i u_i^2}{\gamma\theta_i}\right]|A_y + \xi_{ix}B|^2\right.$$

$$+ \gamma\theta_e n\left|\vec{\nabla}\cdot\vec{\xi}_e - \frac{qn_e}{\gamma\theta_e}(A_y + \xi_{ex}B)\right|^2 + \frac{nq^2}{m_e}\left[1 + (\gamma-1)\frac{m_e n_e^2}{\gamma\theta_i}\right]|A_y + \xi_{ex}B|^2$$

$$\left. + \frac{1}{\mu_0}\left[\left(\frac{dA_y}{dx}\right)^2 + k^2A_y^2 - 2\alpha^2\frac{n}{n_0}A_y^2\right]\right\}. \tag{50}$$

We have put the problem into a variational form which gives at once a necessary and sufficient condition for stability.

The only term which can be made negative in the right-hand side of Eq. (50) is the last one. Indeed we find that with the expression (29) for n we have:

$$I = \int\frac{1}{\mu_0}\left[\left(\frac{dA_y}{dx}\right)^2 + k^2A_y^2 - \frac{2\alpha^2 n}{n_0}A_y^2\right]dx \geq \frac{k^2 - \alpha^2}{\mu_0}\int A_y^2\,dx, \tag{51}$$

the minimum being obtained for

$$A_y \sim \frac{1}{\cosh \alpha x}. \tag{52}$$

Any other function which vanishes at $x = 0$ gives a positive result when inserted in I. Thus all the modes with $k > \alpha$ are stable. If there is a marginally stable mode for $k = \alpha$ we shall be able to find the growth rate of the instability by a perturbation method. That will occur if all the other terms which are positive can be made equal to zero. The only way to formulate it is to take:

$$\vec{\nabla} \cdot \vec{\xi}_i = \vec{\nabla} \cdot \vec{\xi}_e = 0 \tag{53}$$

and

$$A_y + \xi_{ex} B = A_y + \xi_{ix} B = 0.$$

This solution satisfies the constraint condition given in Eq. (49), but unfortunately gives a singular mode, since $A_y \neq 0$ at $x = 0$ when $B = 0$. We cannot calculate the perturbed ω^2 since the integral in the left-hand side of Eq. (50) diverges.

However, let us assume that there exists a regular mode with $\omega^2 < 0$ for $k < \alpha$. In order to get a negative value in the right-hand side, the positive terms must be made as small as possible. That can be achieved by taking $A_y + \xi_{ix} B$ equal to zero everywhere except in a small layer around the point $x = 0$. Let ϵ_i and ϵ_e respectively be the thickness of this layer for ions and electrons.

$\vec{\nabla} \cdot \vec{\xi}_i$ and $\vec{\nabla} \cdot \vec{\xi}_e$ are also zero outside of this region. The dominant term in the left-hand side will come from $\xi_{i,ez} \sim -(1/ik)(d\xi_{i,ex}/dx)$ and will be of the order $1/\epsilon^3$.

We finally get:

$$\left(\frac{m_i}{\epsilon_i^3} + \frac{m_e}{\epsilon_e^3}\right)\frac{n_0}{4\,k^2 B_0^2 \alpha^2}\,\omega^2 = \frac{2\,(k^2 - \alpha^2)}{\alpha\mu_0} + \frac{n_0 q^2}{2}\left(\frac{\epsilon_e}{m_e} + \frac{\epsilon_i}{m_i}\right). \tag{54}$$

By minimizing that expression with respect to ϵ_i and ϵ_e, one finds that the ion contribution is negligible ($\epsilon_i/\sqrt{m_i} = \epsilon_e/\sqrt{m_e}$) and the maximum growth rate is:

$$\gamma \sim \frac{\alpha^3 \rho_i \rho_e^2}{\tau_H}, \tag{55}$$

in which ρ_i and ρ_e are the ion and electron Lamor radius in the external field and τ_H the usual MHD time scale.

The corresponding layer thickness is:

$$\lambda \sim \alpha \rho_e^2. \tag{56}$$

This is much smaller than the electron Larmor radius in the external field. But in the region where B = 0 the particles which carry the current are those which have an almost linear motion along γ-axis, and are very little affected by the magnetic field. A small displacement of these particles along the z-axis is sufficient to create local micro-pinches which modify the magnetic field lines around x = 0.

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QUASI-LINEAR THEORY OF PLASMA TURBULENCE

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I. INTRODUCTION

In this paper we examine the dynamics of a low β, collisionless, unstable plasma, from the time of the onset of the instability to the final quiescent, equilibrium state. The time development of this process can be roughly broken into four phases. During the first phase the "linearized" theory applies and the unstable waves grow exponentially. In the second phase the so-called "quasi-linear" theory applies and the background particle distribution diffuses in such a way as to bring the growth rate of the unstable waves to zero, leaving a quasi-stationary spectrum of waves. In the third phase the waves in the quasi-stationary spectrum interact in such a way as to distort the spectrum but to keep the energy in the spectrum roughly constant. In the fourth phase the waves interact to produce "virtual" waves which in turn lose energy to the bulk of the particles by Landau damping. This "non-linear Landau damping" produces a slow decay of the wave energy (energy drops off roughly as $1/t$) and a slow heating of the plasma, together with a diffusion of the plasma across the magnetic field until all of the wave energy has been given up to the particles and the plasma is in a quiescent, stable state. The first two phases of this process can be fairly well separated in time; however phase III and phase IV occur roughly simultaneously.

We illustrate this process by considering unstable electron plasma oscillations in a low β, collisionless plasma in a large magnetic field.

II. PHASE I – LINEARIZED THEORY

We begin with the Vlasov equation for electrons in a large magnetic field,

$$\frac{\partial f_k}{\partial t} + ik_{\parallel} f_k = -\frac{e}{m} E_k \left(\frac{k_{\parallel}}{k}\right) \frac{\partial F_0}{\partial v} - \frac{e}{m} \sum_q E_{k-q} \frac{(k_{\parallel} - q_{\parallel})}{|k-q|} \frac{\partial f_q}{\partial v}, \tag{1}$$

where the distribution function is given by $F_0(v) + \sum_k e^{ikx} f_k(v, t)$, the other symbols have the usual definitions and since $\omega_p \ll \Omega_e = eB/mc$ the distribution functions depend only on v, the velocity parallel to the magnetic field. The background distribution function, $F_0(v)$, is assumed to have a small bump on the tail, as shown in Fig. 1, and waves with phase velocities, $\omega/k_{\parallel}$, for which $v(\partial F_0/\partial v)_{v=\omega/k} > 0$ will be unstable.

In the "linearized" theory the non-linear sum on the right-hand side of Eq. (1) is neglected and in the usual way we find [1]

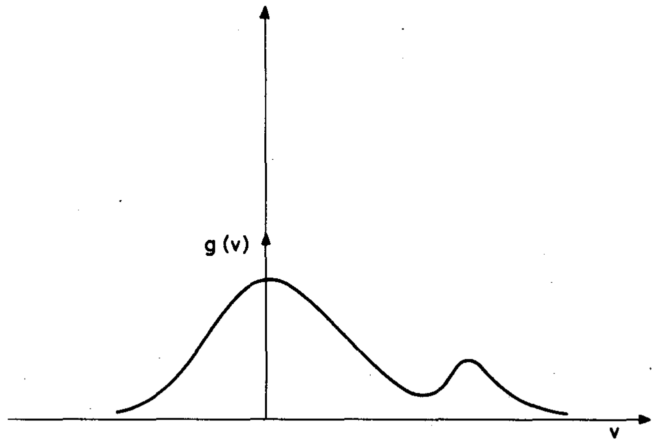

Fig. 1

Background distribution function $F_0(v)$

$$f_0 = -\frac{e}{m}\frac{k_{\|}}{k}\frac{E_k}{(s+ik_{\|}v)}\frac{\partial F_0}{\partial v}, \quad s = -i\omega \tag{2}$$

and

$$\vec{\nabla}\cdot\vec{E}_k = ikE_k = 4\pi\rho_k = -\frac{4\pi e^2}{m}\left(\frac{k_{\|}}{k}\right)E_k\int\frac{\partial F_0/\partial v}{(s+ik_{\|}v)}\,dv. \tag{3}$$

For waves with phase velocity, $\omega/k \gg \bar{v}$, the mean thermal velocity, the velocity integration can be carried out in the usual way and we find

$$\omega_k = \omega_p\frac{k_{\|}}{k} + i\gamma,$$

$$\gamma = \frac{\alpha}{2}\frac{\partial F_0}{\partial v}\bigg|_{v=\omega_k/k_{\|}}, \tag{4}$$

and

$$\frac{\alpha}{2} = 2\pi^2\frac{e^2}{m}\frac{|\omega_k|}{k^2}.$$

We thus see that waves with phase velocities for which $v(\partial F_0/\partial v)_{v=\omega_k/k_{\|}}$ is positive will be unstable and grow exponentially in time, while those for which $v(\partial F_0/\partial v)_{v=\omega_k/k_{\|}} < 0$ will damp.

In the "linearized" theory the unstable waves continue to grow indefinitely and we must proceed to the "quasi-linear" theory to see how the amplitude of these waves is limited.

III. PHASE II – QUASI-LINEAR THEORY

The simplest non-linear correction to the "linearized" theory is obtained by considering only the term with $q = 0$ in the sum on the right hand side of Eq. (1). The remaining terms are the so-called "mode coupling" terms and by neglecting these we obtain the "quasi-linear" theory [2, 3].

We thus have

$$\frac{\partial f_k}{\partial t} + ik_{\parallel} v f_k = -\frac{e}{m}\frac{k_{\parallel}}{k} E_k \frac{\partial g_0}{\partial v}, \tag{5}$$

where $g_0 = F_0(v) + f_0(v, t)$.

For $k = 0$ we include all the terms in the sum on the right-hand side of Eq. (1) and obtain

$$\frac{\partial g_0}{\partial t} = \frac{\partial f_0}{\partial t} = -\frac{e}{m}\sum_q E_{-q}\frac{k_{\parallel}}{k}\frac{\partial f_q}{\partial v}. \tag{6}$$

For $\gamma \ll \omega_p$, $g_0(v, t)$ will vary slowly compared to $E_k(t)$, and we may solve Eq. (5) in the adiabatic approximation, i. e. using time dependences of the form $\exp i \int_0^t \omega(t')dt'$, to obtain

$$f_k = -\frac{e}{m}\frac{E_k}{(s + ik_{\parallel} v)}\frac{\partial g_0}{\partial v} \tag{7}$$

$$\omega(t) \cong \omega_p \frac{k_{\parallel}}{k} + i\gamma(t)$$

$$\gamma(t) = \frac{\alpha}{2}\frac{\partial g_0(v, t)}{\partial v}\bigg|_{v=\omega_k/k_{\parallel}}, \tag{8}$$

which is the same result as in the "linearized" theory except that $F_0(v)$ is replaced by the slowly varying background distribution function $g_0(v, t)$. This simply means that the growth rate at time, t, depends on the slope of the distribution function at time t, rather than the slope of the initial distribution function.

Using Eqs. (6) and (7) we have

$$\frac{\partial g_0}{\partial t} = \left(-\frac{e}{m}\right)^2 \sum_k |E_k|^2 \left(\frac{k_{\parallel}}{k}\right)^2 \frac{\partial}{\partial v}\frac{1}{(s_k + ik_{\parallel} v)}\frac{\partial g_0}{\partial v}. \tag{9}$$

For those velocities, v, for which $\omega_k - k_{\parallel} v$ can be zero, the principal contribution to $\sum_k$ comes from those $\vec{k}$ for which $\omega_k - k_{\parallel} v \cong 0$, or using Eq. (8), from $(k_{\parallel}^2 + k_{\perp}^2)^{\frac{1}{2}} = \omega_p/v$. For these velocities we can write $1/(s_k + ik_{\parallel} v) = (\pi/k_{\parallel})\,\delta(v_0 - v)$ where $v_0 = \omega/k_{\parallel}$. Thus for the resonant velocities we have

$$\frac{\partial g_0}{\partial t} = \frac{\partial}{\partial v} D(v) \frac{\partial g_0}{\partial v}, \text{ where } D(v) = \sum_k \beta \left| E_k \right|^2 \delta(v_0 - v) \tag{10}$$

$$\beta = \left(\frac{e}{m}\right)^2 \pi \frac{k_\parallel^2}{k}.$$

We thus see that $g_0(v, t)$ obeys a diffusion equation in velocity space and the diffusion coefficient for particles with velocity, v, is proportional to the energy in waves which have a phase velocity equal to v. This behaviour of g_0 is simply due to the fact that the waves grow by extracting energy from the resonant particles and the distribution of these particles must change as they lose energy. The energy extracted from these particles is given by

$$\frac{\partial U_r}{\partial t} = \int_{\substack{\text{resonant}\\ \text{particles}}} \frac{1}{2} m v^2 \frac{\partial g_0}{\partial t} dv = \int \frac{1}{2} m v^2 \frac{\partial}{\partial v} D(v) \frac{\partial g_0}{\partial v}$$

$$= -\frac{\pi e^2}{m} \int dv v \sum_k \left| E_k \right|^2 \left(\frac{k_\parallel}{k}\right)^2 \delta \frac{(v_0 - v)}{k_\parallel} \frac{\partial g_0}{\partial v}$$

$$= -\frac{\pi e^2}{m} \sum_k \left| E_k \right|^2 \frac{k_\parallel}{k^2} v \frac{\partial f_0}{\partial v} \bigg|_{v = \omega_k / k_\parallel}$$

which, using Eq. (8), is

$$\frac{dU_r}{dt} = -2 \frac{d}{dt} \sum_k \frac{\left| E_k \right|^2}{8\pi}. \tag{11}$$

We see that the rate at which energy is extracted from the resonant particles is just twice the rate at which the potential energy of the waves increases. In addition to potential energy the waves also have kinetic energy and we shall show that this increases at the same rate as the potential energy, so that the rate at which energy is extracted from the resonant particles is just equal to the rate at which the total energy of the waves increases.

To understand this we note that the bulk of the particles have velocities much slower than the waves and the kinetic energy of the waves is in the organized motion of the bulk of the particles. Thus although there is no resonant diffusion of the bulk of the particles, their distribution will appear to be broadened by the fact that they are "sloshing" back and forth with the kinetic energy of the waves. This "fake" diffusion can be obtained from Eq. (9) by noting that for the bulk of the particles $k_\parallel v \ll \omega_k$ and thus

$$\frac{\partial g_0}{\partial t} \cong \left(\frac{e}{m}\right)^2 \sum_k \left| E_k \right|^2 \left(\frac{k_\parallel}{k}\right)^2 \frac{1}{-i\omega_k} \frac{\partial^2 g_0}{\partial v^2}.$$

Now $\omega_k = \omega_p (k_\parallel / k) + i\gamma$ and since the real part of ω_k is an odd function of k the imaginary part of $\partial g_0 / \partial t$ is zero. The real part is given by

$$\frac{\partial g_0}{\partial v} \cong \left(\frac{e}{m}\right)^2 \sum_k |E_k|^2 \left(\frac{k_\parallel}{k}\right)^2 \frac{\gamma}{\omega_p^2 (k_\parallel/k)^2} \frac{\partial^2 g_0}{\partial v^2}$$

$$= \frac{4\pi}{\omega_p^2} \left(\frac{e}{m}\right)^2 \frac{\partial^2 g_0}{\partial v} \sum_k \frac{1}{8\pi} \frac{\partial |E_k|^2}{\partial t}$$

and

$$\frac{dU_B}{dt} = \int_{\text{bulk}} \frac{1}{2} mv^2 \frac{\partial g_0}{\partial v} = \sum_k \frac{\partial}{\partial t} \frac{|E_k|^2}{8\pi}. \tag{12}$$

Thus the "sloshing" energy of the particles increases at the same rate as the potential energy of the waves and the total rate of change of particle energy is

$$\frac{dU_T}{dt} = \frac{dU_I}{dt} + \frac{dU_B}{dt} = -\sum_k \frac{\partial}{\partial t} \frac{|E_k|^2}{8\pi}.$$

We are now in a position to determine the time development of the unstable waves. Initially we imagine the distribution function $g_0(v, 0) = F_0(v)$ near the bump on the tail to be as shown in Fig. 2a and we assume the initial

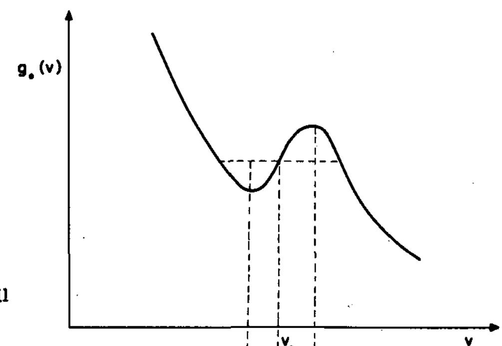

Fig. 2a

Distribution function near the bump of the tail

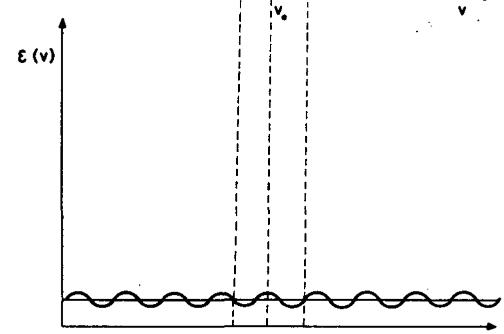

Fig. 2b

Initial spectrum of plasma waves

spectrum of plasma waves $\epsilon(v) = \sum_k |E_k|^2 \delta(\omega/k_\parallel - v)$ to be some smooth function of phase velocity as shown in Fig. 2b. The waves which have phase velocities for which $\partial f/\partial v > 0$ will grow and after a few e-folding times the wave spectrum will be strongly peaked near the fastest growing wave, i.e. near

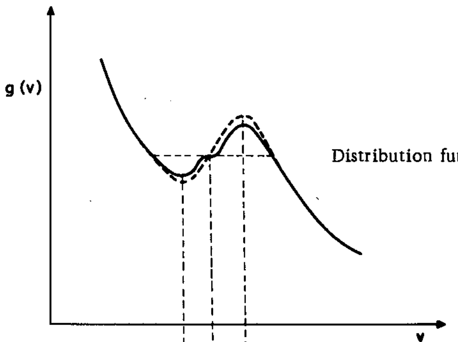

Fig. 3a

Distribution function near the bump of the tail for which $\partial f/\partial v > 0$

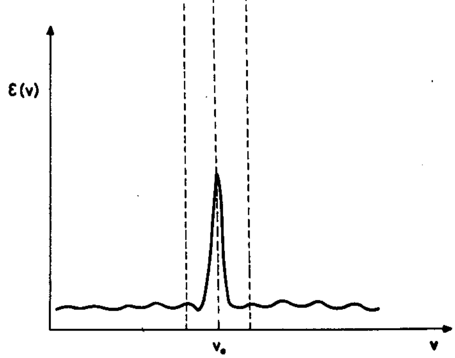

Fig. 3b

Peaked wave spectrum

$v = v_0$, but the distribution function will be virtually unchanged, see Fig. 3a and 3b. At a later time $\epsilon(v)$ will become large enough near $v = v_0$ so that the distribution function will begin to diffuse at particle velocities near v_0. This diffusion tends to flatten the distribution near v_0 and to steepen it on either side of v_0. This is shown in Fig. 4a. This tends to decrease the growth rate of waves near v_0 and to increase the growth rate of waves on either side of v_0, thus widening the spectrum. This is shown in Fig. 4b. This process of flattening the particle distribution function and broadening the wave spectrum continues until the distribution function becomes flat over the region of the wave spectrum and the waves reach an asymptotic spectrum which neither grows nor damps since $\partial g_0/\partial v = 0$ over the entire spectrum. This is shown in Fig. 5a and 5b.

The asymptotic distribution function is uniquely defined by requiring that

$$\int_{u_0}^{u_1} g_\infty dv = \int_{u_0}^{u_1} g(v, 0) dv.$$

The asymptotic wave spectrum can be determined by using Eqs. (8) and (10). From Eq. (8)

$$\frac{\partial |E_k|^2}{\partial t} = \alpha |E_k|^2 \frac{\partial f_0}{\partial v}\bigg|_{v = \omega/k_\parallel}$$

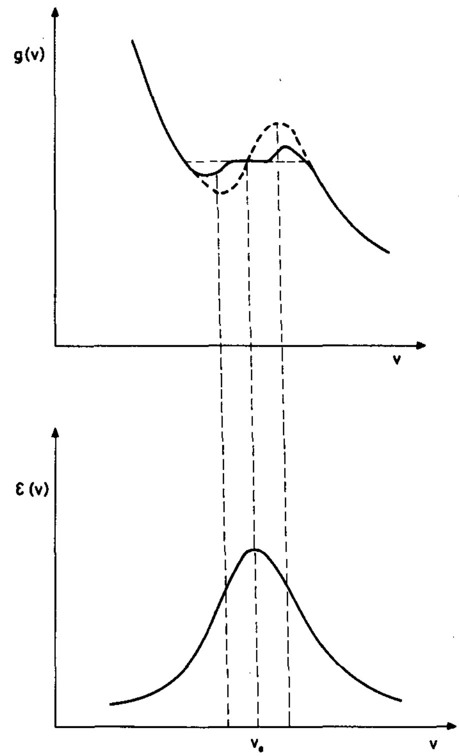

Fig. 4a

Effects of the diffusion on the distribution

Fig. 4b

Widening of the spectrum

Substituting this into Eq. (10) yields

$$\frac{\partial g_0}{\partial t} = \frac{\partial}{\partial v} \sum_k \frac{\beta}{\alpha} \frac{\partial |E_k|^2}{\partial t} \delta\left(\frac{\omega}{k_{\parallel}} - v\right)$$

or

$$\frac{\partial}{\partial t}\left\{g_0 - \frac{\partial}{\partial v} \sum_k \frac{\beta}{\alpha} |E_k|^2 \delta\left(\frac{\omega}{k_{\parallel}} - v\right)\right\} = 0.$$

If the initial level of waves is so small that we can neglect it we obtain

$$\frac{\partial}{\partial v} \sum_k \frac{\beta}{\alpha} |E_k|^2 \delta\left(\frac{\omega}{k_{\parallel}} - v\right) = g_0(v, t) = g_0(v, 0),$$

and as $t \to \infty$

$$\sum_k \frac{\beta}{\alpha} |E_k|^2 \delta\left(\frac{\omega}{k_{\parallel}} - v\right) = \int_{u_0}^{v} [g_\infty - g_0(v', 0)]dv'.$$

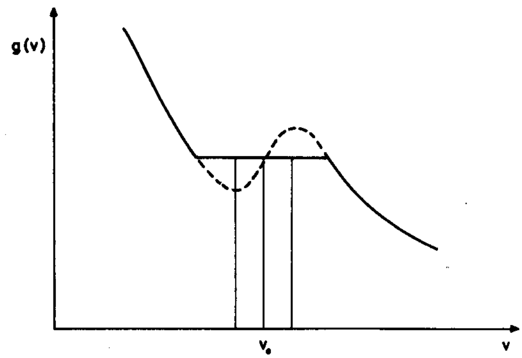

Fig. 5a

Flattening of the distribution function

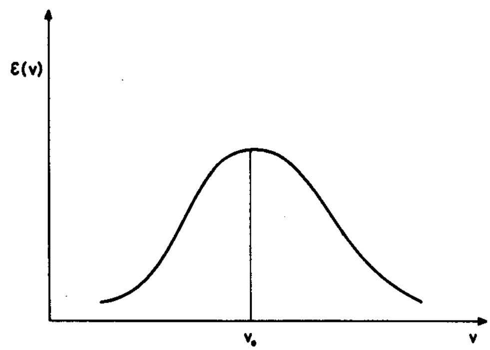

Fig. 5b

Broadening of the wave spectrum

To estimate the size of this we set $g_\infty - g_0 \cong (\Delta g/\Delta v)\,\Delta v \cong (1/\alpha)\gamma\Delta v$ where $\Delta v \approx u_1 - u_0 \simeq$ the width of the bump, and we have

$$\sum_k \beta\,|E_k|^2\,\delta\!\left(\frac{\omega}{k_\parallel} - v\right) \simeq \gamma(\Delta v)^2.$$

Integrating over v we have approximately

$$\sum |E_k|^2 = \frac{1}{\beta}\,\gamma(\Delta v)^3 \simeq \left(\frac{\gamma}{\omega}\right)\left(\frac{\Delta v}{v}\right)^2 4\pi(nmv\Delta v).$$

We thus see that the "quasi-linear" theory leads to the development of a quasi-equilibrium spectrum which persists indefinitely, and that the energy in this spectrum is small in that it is proportional to $\gamma/\omega \ll 1$. This is the essential feature of the quasi-linear theory for it shows that correction terms which are of higher order in $\sum_k |E_k|^2$ will be smaller by a factor of $\gamma/\omega \ll 1$ and thus an expansion in powers of $|E_k|^2$ should lead to reasonable results.

The mode-coupling terms neglected in the "quasi-linear" lead to a slow distortion and damping of the "quasi-linear" equilibrium spectrum and these are considered in the following sections.

IV. PHASE III — RESONANT MODE COUPLING

The "mode coupling" terms neglected in the quasi-linear theory may be evaluated by methods which are similar to the methods of quantum mechanics. After factoring out the quasi-linear term, Eq. (1) can be written as

$$\frac{\partial f_k}{\partial t} + ik_\parallel v f_k = -\frac{e}{m} E_k \frac{k_\parallel}{k} \frac{\partial g_0}{\partial v} - \frac{e}{m} \sum_q{}' E_{k-q} \frac{|k_\parallel - q_\parallel|}{|k-q|} \frac{\partial f_q}{\partial v}. \tag{4.1}$$

where the prime on $\sum_k'$ denotes that the term with $q = 0$ is to be deleted. Assuming the same type of time dependence as in the quasi-linear theory, Eq. (4.1) can be integrated directly to give

$$\begin{aligned} f_k = & -\frac{e}{m}\frac{k_\parallel}{k}\int_0^t dt'\, e^{-ik_\parallel v(t-t')} E_k(t') \frac{\partial g_0(t')}{\partial v} - \\ & -\frac{e}{m}\sum_q \frac{(k_\parallel - q_\parallel)}{|k-q|}\int_0^t dt'\, e^{-ik_\parallel v(t-t')} E_{k-q}(t') \frac{\partial f_q(t')}{\partial v} \end{aligned} \tag{4.2}$$

$$= -\frac{e}{m}\frac{k_\parallel}{k}\frac{E_k(t)\,\partial g_0/\partial v}{S_k(t) + ik_\parallel v} - \frac{e}{m}\sum_q \frac{(k_\parallel - q_\parallel)}{|k-q|}\frac{E_{k-q}(t)}{S_{k-q} + S_q + ik_\parallel v}\frac{\partial f_q(t)}{\partial v}, \tag{4.3}$$

where we have neglected the slow dependence of $g_0(t)$ on t. To lowest order

$$f_q(t) = -\frac{e}{m}\frac{q_\parallel}{q}\frac{E_q(t)}{S_q + iq_\parallel v}\frac{\partial g_0}{\partial v}. \tag{4.4}$$

Inserting this into Eq. (4.2), integrating the result over velocity and using $\vec{\nabla}\cdot E_k = ikE_k = 4\pi e\int d^3v f_k$ we obtain

$$\epsilon(S_k)\, E_k(t) = \sum_q M_{kq}\, E_{k-q}(t)\, E_q(t), \tag{4.5}$$

where

$$\epsilon(S_k) = I + \frac{4\pi e^2 k_\parallel}{ik^2 m}\int d^3v \frac{1}{S_k + ik_\parallel v}\frac{\partial g_0}{\partial v}$$

$$M_{kq} = -\frac{4\pi e^3}{ikm^2}\sum_q \frac{(k_\parallel - q_\parallel)}{|k-q|}\frac{q_\parallel}{q}\int \frac{d^3v}{S_{k-q} + S_q + ik_\parallel v}\frac{\partial}{\partial v}\frac{1}{(S_q + iq_\parallel v)}\frac{\partial g_0}{\partial v}. \tag{4.6}$$

Now the quasi-linear dispersion solution is obtained by equating $\epsilon(S_k^0)$ to zero. This determines the time dependence S_k^0 for no mode coupling. To determine the time dependence with mode coupling we expand $\epsilon(S_k)$ about $S_k = S_k^0$.

$$\epsilon(S_k) = \epsilon(S_k^0) + \frac{\partial\epsilon}{\partial S_k}(S_k - S_k^0)$$

but

$$S_k E_k \equiv \frac{\partial E_k}{\partial t}$$

and thus Eq. (4. 5) becomes

$$\epsilon'\left(\frac{\partial E_k}{\partial t} - S_k^0 E_k\right) = \sum M_{kq} E_{k-q} E_q$$

or

$$\frac{\partial E_k}{\partial t} = S_k^0 E_k + \sum_q M_{kq}^{(1)} E_{k-q} E_q \tag{4. 7}$$

where

$$M_{kq}^{(1)} = M_{kq}/\epsilon'.$$

Letting $E_k(t) = \hat{E}_k e^{-i\omega_k t}$ where the amplitude $\hat{E}_k(t)$ is a slowly varying function of time we can re-write Eq. (4. 7) as

$$\frac{\partial \hat{E}_k}{\partial t} = \sum_q M_{kq}^{(1)} \hat{E}_{k-q} \hat{E}_q e^{-i\Delta\omega_q t} \tag{4. 8}$$

where

$$\Delta\omega_q = \omega_{k-q} + \omega_q - \omega_k.$$

Neglecting (for the moment) the slow time variation of $\hat{E}_{k-q}\hat{E}_q$ Eq. (4. 8) yields

$$\hat{E}_k(t) = \hat{E}_k(0) + \sum_q M_{kq}^{(1)} \hat{E}_{k-q} \hat{E}_q \frac{e^{-i\Delta\omega_q t} - 1}{-i\Delta\omega_q}. \tag{4. 9}$$

Multiplying Eq. (4. 8) by Eq. (4. 9) then gives

$$\frac{1}{2}\frac{\partial}{\partial t} |E_k|^2 = \sum_q \sum_p \overline{M}_{kp}^{(1)} \overline{\hat{E}}_{k-p} \overline{\hat{E}}_p M_{kq}^{(1)} \hat{E}_{k-q} \hat{E}_q \, e^{i\Delta\omega_p t} \left(\frac{e^{-i\Delta\omega_q t} - 1}{-i\Delta\omega_q}\right). \tag{4. 10}$$

We now wish to average over the initial phases of the waves. It can be easily shown that

$$\langle E_q(t)\overline{E}_p(t)\rangle = \delta_{qp} |E_q|^2 + 0(|E|^4),$$

if $\langle E_q(0)E_p(0)\rangle = \delta_{qp}$. Similarly

$$\langle E_{k-p}E_pE_{k-q}E_q\rangle = |E_{k-q}|^2 |E_q|^2 \left[\delta_{pq} + \delta_{p,k-q}\right] + 0(|E|^6)$$

Inserting this into Eq. (4.10) yields

$$\frac{1}{2}\frac{\partial |E_k|^2}{\partial t} = \sum_q \left(|M_{kq}^{(1)}|^2 + M_{kq}\overline{M}_{k,k-q}\right)|E_{k-q}|^2 |E_q|^2 \left(\frac{1 - e^{i\Delta\omega_q t}}{-i\Delta\omega_q}\right) \tag{4.11}$$

For $\omega_k t \to \infty$ the function $(1 - e^{i\Delta\omega_q t})/-i\Delta\omega_q \to \pi\delta(\Delta\omega_q)$ and we obtain

$$\frac{1}{2}\frac{\partial |E_k|^2}{\partial t} = \pi \sum_q \left[M_{kq}(\overline{M}_{kq} + M_{k,k-q})\right]|E_{k-q}|^2 |E_q|^2 \,\delta(\Delta\omega_q), \tag{4.12}$$

which is just the usual form for time proportional transitions. This corresponds to the diagram in Fig.(6) in which the waves E_{k-q} and E_q scatter to form the wave E_k.

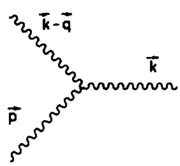

Fig. 6

Scattering of the waves E_{k-q} and E_q to form the wave E_k

If we now take into account the time dependence of $\hat{E}_{k-q}$ and $\hat{E}_q$ we obtain in addition terms which account for the loss of energy by E_k as it scatters with other waves. The final result is thus of the form

$$\begin{aligned}\frac{\partial |E_k|^2}{\partial t} = &\sum_q H_{kq}^{(1)} \; |E_{k-q}|^2 |E_q|^2 \,\delta(\omega_{k-q} + \omega_q - \omega_k) \\ &- \sum_q H_{kq}^{(2)} \; |E_{k-q}|^2 |E_k|^2 \,\delta(\omega_{k-q} + \omega_k - \omega_{-q}) \\ &- \sum_q H_{kq}^{(3)} \; |E_q|^2 |E_k|^2 \,\delta(\omega_k + \omega_{-q} - \omega_{k-q}).\end{aligned} \tag{4.13}$$

Thus resonant mode coupling leads to a distortion of the quasi-linear equilibrium spectrum and the time scale associated with this is of the order of $|E|^2 \sim \gamma_0$ so that this process is on the same time scale as the quasi-linear terms. An essential feature of this resonant mode coupling is that it leaves the total energy in the wave spectrum virtually constant. In particular in the zero temperature limit

$$\sum_k \frac{\partial |E_k|^2}{\partial t} = 0.$$

It is true that the resonant mode coupling does put energy into waves which have phase velocities for which $\partial g_0/\partial v < 0$ and that these waves damp (i. e. waves are produced which are outside the region for which $\partial g_0/\partial v = 0$), but this is a relatively slow process for the case at hand because it is fairly difficult to satisfy $\Delta\omega_q = 0$.

Thus although the resonant mode coupling leads to some loss of wave energy we must look for further mechanisms which tend to thermalize the plasma.

V. PHASE IV – NON-LINEAR LANDAU DAMPING

To obtain resonant mode coupling the function $\epsilon(S_k)$ in Eq. (4. 5) was expanded about $S_k = S_k^0$. This was because for resonant mode coupling we are interested in small changes in the time behaviour of $E_k(t)$. However from Eq. (4. 5) we see that the driving term, $E_{k-q}(t)\,E_q(t)$ has a time dependence of $\exp[-i(\omega_{k-q}+\omega_q)t]$ which may be very different from the natural time dependence of $E_k(t)$ (goes as $\exp^{-i\omega_k t}$), i. e. $\omega_{k-q}+\omega_q$ in general is not close to ω_k. This source thus forces E_k to have terms with the time dependence of the source rather than the natural time dependence of E_k. This is somewhat analogous to driving a harmonic oscillator with a non-resonant source. The result is that E_k must be divided into two parts,

$$E_k = E_k^{(1)}(t) + E_k^{(2)}(t). \tag{5. 1}$$

$E_k^{(1)}(t)$ is the part of E_k which oscillates with its natural frequency and for which the resonant mode coupling terms in the last section apply and $E_k^{(2)}(t)$ is the part of E_k which arises because of the non-resonant parts of the source. Eq. (4. 5) can be solved for $E_k^{(2)}$ to yield

$$E_k^{(2)}(t) = \sum_q \frac{M_{kq}^{(2)}\,E_{k-q}^{(1)}(t)\,E_q^{(1)}(t)}{\epsilon_k(S_{k-q}+S_q)} \tag{5. 2}$$

and $E_k^{(2)}(t)$ does not grow "time proportionally" as does $E_k^{(1)}$ but comes to an equilibrium amplitude given by Eq. (5. 2). An important feature of these waves is that $\omega_{k-q}+\omega_q$ can be almost zero so that the phase velocity of these waves, $(\omega_{k-q}+\omega_q)/k_\parallel$ can be much smaller than the electron thermal velocity. In fact the phase velocity can be even smaller than the ion thermal velocity. This means that the $E_k^{(2)}$ waves can undergo Landau damping, giving their energy to both electrons and ions.

The $E_k^{(2)}$ waves can be viewed as virtual states in the "quantum mechanical" sense. The important difference is that these "virtual" waves can interact with the background particles and thus give up energy to the particles. These waves come to a quasi-equilibrium in which they take energy from the $E_k^{(1)}$ waves and give energy at the same rate to the particles. Note that $E_k^{(2)}$ is of second order in E.

Because these waves can have phase velocities comparable to the ion velocities we must include the ion terms in $\epsilon_k(S_k)$. Thus

$$\epsilon_k(S_k) = 1 + \sum_j \frac{4\pi e^2 k_\parallel}{ik^2 m_j} \int \frac{d^3v}{(S_k + ik_\parallel v)} \frac{\partial g_{0j}}{\partial v}. \tag{5.3}$$

To calculate the rate at which energy is given to the particles by this process we must keep terms of third order in the Vlasov equation. If we iterate Eq. (4.2) and then integrate over velocity we obtain (to third order) an equation of the form

$$E_k = (\quad)E_k + \sum_q (\quad)E_{k-q}E_q + \sum_{q,p} (\quad)E_{k-q}E_{q-p}E_p. \tag{5.4}$$

Each E_k in this equation must now be replaced by $E_k = E_k^{(1)} + E_k^{(2)}$. The equation then divides into two parts

$$\epsilon_k E_k^{(2)} = \sum_q (\quad)E_{k-q}^{(1)}E_q^{(1)}, \tag{5.5}$$

which yields Eq. (5.2), and

$$\begin{aligned}\epsilon_k E_k^{(1)} &= \sum_q (\quad)E_{k-q}^{(1)}E_q^{(1)} + \sum_q (\quad)E_{k-q}^{(1)}E_q^{(2)} \\ &+ \sum_q (\quad)E_{k-q}^{(2)}E_q^{(1)} + \sum_{q,p} (\quad)E_{k-q}^{(1)}E_{q-p}^{(1)}E_p^{(1)}.\end{aligned} \tag{5.6}$$

Expanding $\epsilon_k(S_k)$ in Eq. (5.6) about $S_k = S_k^0$ and multiplying by $\overline{E}_k(t)$ we obtain after averaging over the initial phases

$$\begin{aligned}\frac{\partial |E_k|^2}{\partial t} + i\omega_k |E_k|^2 &= \sum_q (\quad)\langle E_{k-q}^{(1)}E_q^{(1)}E_{-k}^{(1)}\rangle \\ &+ \sum_q (\quad)\langle E_{k-q}^{(1)}E_q^{(2)}E_{-k}^{(1)}\rangle \\ &+ \sum_q (\quad)\langle E_{k-q}^{(2)}E_q^{(1)}E_{-k}^{(1)}\rangle + \sum_{qp} (\quad)\langle E_{k-q}^{(1)}E_{q-p}^{(1)}E_p^{(1)}E_{-k}^{(1)}\rangle.\end{aligned} \tag{5.7}$$

The first term on the right-hand side of Eq. (5.7) is just the resonant mode coupling term discussed in section IV. The second and third terms on the right-hand side are associated with $E^{(2)}$ and lead to diagrams such as that in Fig. 7 where $E_k^{(1)}$ and $E_{-q}^{(1)}$ interact to form the virtual state $E_{k-q}^{(2)}$. $E_{k-q}^{(2)}$ subsequently decays to $E_k^{(1)}$ and $E_{-q}^{(1)}$. This is similar to the "self energy"

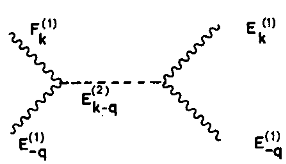

Fig. 7

Interaction of $E_k^{(1)}$ and $E_{-q}^{(1)}$ decaying into $E_k^{(1)}$ and $E_q^{(1)}$ through a virtual state $E_{k-q}^{(2)}$

diagrams of field theory except that because of the Landau damping of $E_{k-q}^{(2)}$ which arises from the imaginary parts of $\epsilon(S_{k-q})$ the interaction is not hermition. Thus not only is $\mathrm{Re}\,\omega_k$ slightly changed but also this leads to an $\mathrm{Im}\,\omega_k$.

The last term in Eq. (5. 7) leads to a 4-wave diagram such as shown in Fig. 8. This also has non-hermition parts and leads to a change in γ_k.

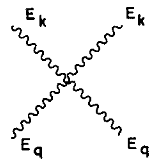

Fig. 8

Four-wave diagram

After averaging over the initial phases the last three terms in Eq. (5. 7) can be combined to give a contribution to $\partial|E_k^{(1)}|^2/\partial t$ of

$$\left.\frac{\partial|E_k^{(1)}|^2}{\partial t}\right|_{4\text{-wave}} = -\sum_q H_{kq}|E_{k-q}|^2|E_q|^2 \tag{5.8}$$

which is of the same order in $|E|^2$ as the resonant scattering and the quasi-linear terms.

As might be expected H_{kq} is quite complicated and will not be given here. The essential point is that these 4-wave processes lead to a non-linear Landau damping which transfers energy to both electrons and ions and this leads to a slow decay of the wave energy.

Further, the energy is put into the slow particles rather than the particles on the tail of the distribution and this leads to a "heating" of the bulk of the plasma.

The decay rate can be estimated by summing Eq. (5. 8) over all k. Letting $\Sigma|E_k|^2 = \mathcal{E}$, we obtain roughly

$$\frac{\partial\mathcal{E}}{\partial t} \cong -\overline{H}\,\mathcal{E}^2$$

which can be integrated to give

$$\mathcal{E} = \mathcal{E}_0/(1+\overline{H}\,\mathcal{E}_0 t),$$

where $\mathcal{E}_0$ is the energy in the "quasi-linear equilibrium spectrum". We note that asymptotically $\mathcal{E}$ does not damp exponentially but as $1/t$ so that this non-linear Landau damping leads to only a rather slow decay of the wave energy.

In closing, we note that the slow $|E^{(2)}|$ waves lead to a diffusion of the plasma (both ions and electrons) across magnetic field lines. The origin of this process is simply that particles drift across field lines with the velocity

$$\vec{v}_\perp = \frac{C\vec{E}\times\vec{B}}{B^2}$$

and the mean square distance they go is

$$\langle r_\perp^2 \rangle = \left(\frac{C}{B}\right)^2 \left\langle \left(\int_0^t E_\perp dt'\right)^2 \right\rangle = \left(\frac{C}{B}\right)^2 \left\langle \left(\sum_q E_q \frac{\exp i(q_\parallel v_\parallel - \omega_q)t - 1}{-i(q_\parallel v_\parallel - \omega)}\right)^2 \right\rangle$$

$$= \left(\frac{C}{B}\right)^2 t \sum_q |E_q|^2 \pi \delta(q_\parallel v_\parallel - \omega_q) \cong Dt$$

where

$$D(v_\parallel) = \left(\frac{C}{B}\right)^2 \sum_q |E_q|^2 \pi \delta(q_\parallel v_\parallel - \omega_q).$$

For $E_q^{(1)}$ waves $\omega_q/q_\parallel >> \bar{v}$ and only the few particles on the tail of the distribution diffuse. However, for $E_q^{(2)}$ waves $\omega_q/q_\parallel \lessapprox \bar{v}_e, \bar{v}_i$ and the slow bulk of particles can diffuse. The dependence of the diffusion coefficient on B is determined in each case by the "quasi-linear equilibrium spectrum" which in general depends on B.

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PLASMA TURBULENCE
General topics

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1. INTRODUCTION

It is known that under experimental conditions plasma often shows chaotic motion. Such motion, when many degrees of freedom are excited to levels considerably above the thermal level, will be called turbulent. The properties of turbulent plasma in many respects differ from the properties of laminar plasma. It can be said that the appearance of various anomalies in plasma behaviour indicates the presence of turbulence in plasma. In order to verify directly the presence of turbulent motion in plasma we must, however, measure the fluctuation of some microscopic parameters in plasma. Let us suppose that we introduce two electric probes in the plasma at $\vec{r}$ and $\vec{r}$. Then we can measure the fluctuation of the electric potential of these probes $\varphi(\vec{r}, t)$ and $[(\vec{r}, t')$. Using special electronic devices we can average $\varphi(\vec{r}, t)\varphi(\vec{r}', t')$ and determine the correlation function of the electric fields in two points at different moments of time. If plasma is stationary and homogeneous in the average, then this relation function depends only upon the differences $(\vec{r} - \vec{r}')$ and $(t - t')$. Instead of this correlation function it is more convenient to use its Fourier transform $I_{k\omega}$, so that

$$\langle\varphi(\vec{r}, t)\varphi(0, 0)\rangle = \int [\exp(-i\omega t + i\vec{k}\cdot\vec{r})] I_{k\omega}\, d\vec{r}\, d\omega.$$

It is this function that should be determined by the theory.

By measuring the correlation function $\langle\varphi(\vec{r}, t)\varphi(\vec{r}', t')\rangle$ one can directly establish whether the turbulence is strong or weak. Weak turbulence represents a group of weak interacting waves. In the case of weak turbulence, only one eigen-frequency corresponds to each wave number so that in the k, ω space, the intensity of oscillation is located in the vicinity of $\omega = \omega_k$, as shown in Fig. 1. In other words,

$$I_{k\omega} = I_k\, \delta(\omega - \omega_k).$$

Respectively, only one k corresponds to each given ω and, therefore, if we measure the correlation function of the potential for a given frequency $\langle\varphi_\omega(\vec{r})\varphi_\omega(\vec{r}+\vec{x})\rangle$ in relation to the distance between two probes we must obtain a nearly periodic function corresponding to one k. If the interaction between different waves increases, the dependence of $I_{\omega x}$ on ω spreads out and the turbulence tends to be strong, see Fig. 2.

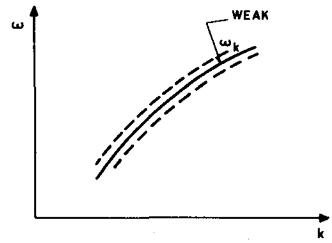

Fig. 1

Weak turbulence.
The intensity of oscillation is located in the vicinity of $\omega = \omega_k$.

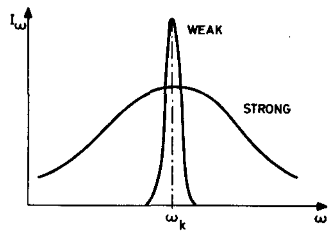

Fig. 2

The interaction between different waves increases, the dependence of $I_{\omega x}$ on ω spreads out.

At present, we have a precise theory only for the weak turbulence of plasma, when perturbation methods can be used. A short survey of this theory will be given and then the strong turbulence will be discussed.

II. THERMAL FLUCTUATIONS IN PLASMA

The turbulent fluctuations develop only in plasma not in thermodynamic equilibrium. This non-equilibrium may be caused either by non-homogeneity of plasma or by distribution-function anisotropy in space velocity. In equilibrium plasma these fluctuations cannot be distinguished from the usual thermal noises. It is therefore obvious that the complete theory should cover both the weak turbulence and thermal fluctuations. Let us first consider the case of thermal fluctuations in stable plasma. It is known that the Vlasov kinetic equation

$$\frac{\partial f_m}{\partial t} + \vec{v} \cdot \vec{\nabla} f_m + \frac{e}{m} \vec{E} \frac{\partial f_m}{\partial \vec{v}} = 0 \tag{1}$$

is an exact equation for the microscopic distribution function

$$f_m = \sum_j \delta(\vec{r} - \vec{r}_j)\, \delta(\vec{v} - \vec{v}_j).$$

The summation is made for the particles of a given species.

Let us split this function into two parts

$$f_m = f_0 + f,$$

where f_0 is a function averaged over small macroscopic volumes and f is the fluctuating part. If we neglect the collisions, then in the absence of external magnetic fields the particles move freely under $\vec{v}$ = const., and $\vec{r} = \vec{r}_0 + \vec{v}t$. Noting by f^μ the part of f, corresponding to non-interacting particles, we shall obtain

$$\langle f^\mu(\vec{r}, \vec{v}, t)\, f^\mu(\vec{r}\,', \vec{v}\,', t')\rangle = \delta(\vec{v} - \vec{v}\,')\, \delta(\vec{r} - \vec{r}\,' - \vec{v}[t - t'])\, f_0(\vec{v}). \tag{2}$$

In Fourier representation this relation has the form

$$\langle f^\mu_{\vec{k}\omega}(\vec{v})\, f^\mu_{\vec{k}'\omega'}(\vec{v}\,')\rangle = \delta(\vec{k} - \vec{k}')\, \delta(\omega - \omega')\, \delta(\vec{v} - \vec{v}\,')\, \delta(\omega - \vec{k}\cdot\vec{v})\, \frac{f_0(\vec{v})}{(2\pi)^3}, \tag{3}$$

where $f^\mu_{\vec{k}\omega}$ is the Fourier component of

$$f^\mu = \int f^\mu_{\vec{k}\omega} \exp(-i\omega t + i\vec{k}\cdot\vec{r})\, d\omega\, d\vec{k}. \tag{4}$$

In addition to f^μ the fluctuations of the distribution function contain that part which corresponds to the electric field fluctuations. If the magnitude of these fluctuations is small enough, we may linearize the kinetic equation (1), so that in the Fourier representation we obtain

$$f_{\vec{k}\omega} = (\vec{g}_{\vec{k}\omega}\, \vec{k})\, f_0\, \varphi_{\vec{k}\omega} + f^\mu_{\vec{k}\omega}, \tag{5}$$

where

$$\vec{g}_{\vec{k}\omega} = \vec{g}(\omega - \vec{k}\cdot\vec{v} + i\nu)^{-1}\, \frac{e}{m}\frac{\partial}{\partial \vec{v}}. \tag{6}$$

Here the small positive quantity ν is introduced to ensure a proper choice of integration contour. Inserting the expression (5) for $f_{\vec{k}\omega}$ in the Poisson equation

$$k^2 \varphi_{\vec{k}\omega} = 4\pi e \int f_{\vec{k}\omega}\, d\vec{k}\, d\omega \tag{7}$$

we obtain

$$\epsilon(\vec{k}\omega)\, \varphi_{\vec{k}\omega} = \frac{4\pi e}{k^2} \int f^\mu_{\vec{k}\omega}\, d\vec{k}\, d\omega, \tag{8}$$

where ϵ is the plasma dielectric constant

$$\epsilon(\vec{k}\omega) = 1 + \frac{4\pi e^2}{m} \int \frac{(\partial f_0/\partial \vec{v})\, d\vec{v}}{\omega - \vec{k}\cdot\vec{v} + i\nu}. \tag{9}$$

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Thus, in the case of small level fluctuations, when the linearization of the Vlasov equation is justified, the intensity of thermal fluctuations in plasma may be found by squaring and averaging the relation (8) using (3), namely

$$I_{\vec{k}\omega} = \frac{2}{\pi}\frac{e^2}{k^4|\epsilon(\vec{k},\omega)|^2}\int f(\vec{v})\,\delta(\omega - \vec{k}\cdot\vec{v})d\vec{v}. \tag{10}$$

The quantity $I_{\vec{k}\omega}$ takes into account both the short-distance fluctuations ($kD > 1$) and the long-distance wave fluctuations ($kD < 1$). In the long wave length region ($kD \ll 1$) the imaginary part $\epsilon'' = \mathrm{Im}\,\epsilon$ is small, so that

$$\frac{1}{|\epsilon(\vec{k},\omega)|^2} = \frac{1}{|\epsilon'|^2 + |\epsilon''|^2} \approx \frac{1}{|\epsilon''|}\,\pi\,\delta(\epsilon'),$$

where $\epsilon' = \mathrm{Re}\,\epsilon$. In this region $I_{\vec{k}\omega} = I_{\vec{k}}\,\delta(\omega - \omega_k)$, where according to (10) the quantity $I_{\vec{k}}$ is given by

$$I_{\vec{k}} = \frac{2e^4}{k^4|\gamma|}\left(\frac{\partial\epsilon'}{\partial\omega}\right)^{-2}\int f(\vec{v})\,\delta(\omega - \vec{k}\cdot\vec{v})\,d\vec{v}; \tag{11}$$

here γ means the rate of plasma waves damping:

$$-\gamma = \epsilon''/\frac{\partial\epsilon'}{\partial\omega}. \tag{12}$$

As we see from (11), when approaching unstable situations ($\gamma \to 0$) the intensity of thermal fluctuations tends to infinity, and therefore weak turbulence must develop in plasma.

III. KINETIC EQUATION FOR WAVES

To describe the weak turbulent fluctuations, it is necessary to retain non-linear terms in the kinetic equations (1). Splitting these equations into two parts by using the averaging operation, for the case of stationary fluctuations we obtain:

$$\frac{\partial f_0}{\partial t} + (\vec{v}\vec{\nabla})f_0 = S_{ef} = -i\frac{e}{m}\frac{\partial}{\partial\vec{v}}\int \vec{k}\langle\varphi^*_{\vec{k}'\omega'}\,f_{\vec{k}\omega}\rangle\,d\vec{k}\,d\vec{k}'\,d\omega\,d\omega', \tag{13}$$

$$f_{\vec{k}\omega} = f^{\mu}_{\vec{k}\omega} + (\vec{g}_{\vec{k}\omega})f_0\,\varphi_{\vec{k}'\omega} + \int(\vec{g}_{\vec{k}\omega}\vec{k}')(\varphi_{\vec{k}'\omega'}f_{\vec{k}''\omega''} - \langle\varphi_{\vec{k}'\omega'}f_{\vec{k}''\omega''}\rangle), \tag{14}$$

where $\vec{k}'' = \vec{k} - \vec{k}'$ and $\omega - \omega' = \omega''$. The term S_{ef} may be considered as a collision term.

If the amplitude of electrical potential oscillations $\varphi_{\vec{k}\omega}$ is small enough we may use the series expansion for $f_{\vec{k}\omega}$. This series may be obtained by iteration of expression (14), namely

$$f_{\vec{k}\omega} = f^{\mu}_{\vec{k}\omega} + (\vec{g}_{\vec{k}\omega}\vec{k})f_0\varphi_{\vec{k}\omega} + \int (\vec{g}_{\vec{k}\omega}\vec{k}')(\vec{g}_{\vec{k}''\omega''})f_0 R\,\varphi_{\vec{k}'\omega'}\varphi_{\vec{k}''\omega''}\,d\vec{k}'\,d\omega$$

$$+ \int (\vec{g}_{\vec{k}\omega}\vec{k}')(\vec{g}_{\vec{k}''\omega''}\vec{k}_1)$$

$$\times(\vec{g}_{\vec{k}''-\vec{k}_1,\omega''-\omega_1}(\vec{k}''-\vec{k}_1)\,f_0 R\,\varphi_{\vec{k}'\omega'}\varphi_{\vec{k}_1\omega_1}\,\varphi_{\vec{k}''-\vec{k}_1,\omega''-\omega_1})d\vec{k}'\,d\omega\,d\vec{k}_1\,d\omega_1\,. \qquad (15)$$

Here R means the subtractions of averaged values, as shown below:

$$R\varphi\psi = \varphi\psi - \langle\varphi\psi\rangle, \quad R\varphi\psi\eta = \varphi\psi\eta - \varphi\langle\psi\eta\rangle - \langle\varphi\psi\eta\rangle, \ldots\ .$$

If we now insert this expression for $f_{\vec{k}\omega}$ in Eq. (7), then in addition to the terms written before we obtain non-linear terms of the $\varphi\varphi$, $\varphi\varphi\varphi$ type, namely

$$\epsilon(\vec{k}\omega)\varphi_{\vec{k}\omega} = \frac{4\pi e}{k^2}\int f^{\mu}_{\vec{k}\omega}\,d\vec{k}\,d\omega + \int V_{\vec{k}\omega,\vec{k}'\omega'}\,R\,\varphi_{\vec{k}'\omega'}\,\varphi_{\vec{k}''\omega''}\,d\vec{k}'\,d\omega'$$

$$+ \int M_{\vec{k}\omega,\vec{k}'\omega'}\,R\,\varphi_{\vec{k}'\omega'}\,\varphi_{\vec{k}_1\omega_1}\,\varphi_{\vec{k}''-\vec{k}_1,\omega''-\omega_1}\,d\vec{k}'\,d\omega'\,d\vec{k}_1\,d\omega_1, \qquad (16)$$

where V and M are matrix elements of the interaction.

If the amplitude of the oscillations is small, we can use the perturbation theory. Namely, in zero approximation we can neglect the wave interaction, and assume that waves are statistically independent. We note the zero approximation amplitude as $\varphi^{(0)}$. If we insert this zero amplitude in the quadratic term, then this non-linear term will act as a driving force. Thus, the amplitudes of forced oscillations $\varphi^{(1)}_{\vec{k}\omega}$ are given by the relation

$$\varphi^{(1)}_{\vec{k}\omega} = \frac{1}{\epsilon(\vec{k}\omega)}\int V_{\vec{k}\omega,\vec{k}'\omega'}\,R\,\varphi^{(0)}_{\vec{k}'\omega'}\,\varphi^{(0)}_{\vec{k}''\omega''}\,dk'\,d\omega'.$$

Now let us multiply Eq. (16) by $\varphi^{*}_{\vec{k}\omega}$ and average it over phases of free oscillations $\varphi^{(0)}$. In the case of many excited degrees of freedom these phases may be considered as random phases. In the approximation which we need, we can substitute $\varphi^{(0)}$ for φ in the third right-hand term of Eq. (16). The second term, proportional to $V_{\vec{k}\omega,\vec{k}'\omega'}$, disappears in the zero approximation so that it is necessary to take into account the corrections $\varphi^{(1)}$ in order to obtain the input of the same order of magnitude. Retaining only the linear and quadratic term in I and neglecting the difference between $\langle\varphi\varphi\rangle$ and $\langle\varphi^0\varphi^0\rangle$ in non-linear terms we obtain the following equation for waves

$$\epsilon_{\vec{k}\omega} I_{\vec{k}\omega} = \frac{e^4}{\epsilon^*(\vec{k},\omega+i\nu)}\int f(\vec{v})\,\delta(\omega-\vec{k}\cdot\vec{v})d\vec{v} + I_{\vec{k}\omega}\int R_{\vec{k}\omega,\vec{k}'\omega'}\,I_{\vec{k}'\omega'}\,d\vec{k}'\,d\omega'$$

$$+ I_{\vec{k}\omega}\int\frac{V_{\vec{k}\omega,\vec{k}''\omega''}\,V_{\vec{k}''\omega'',\vec{k}\omega}}{\epsilon(u'',\omega''+i\nu)}\,I_{\vec{k}'\omega'}\,d\vec{k}'\,d\omega'$$

$$+ \frac{1}{2\epsilon^*(\vec{k},\omega+i\nu)}\int\left|V_{\vec{k}\omega,\vec{k}'\omega'}\right|^2 I_{\vec{k}'\omega'}\,I_{\vec{k}''\omega''}\,d\vec{k}'\,d\omega', \qquad (17)$$

where

$$R_{\vec{k}\omega,\vec{k}'\omega'} = \frac{4\pi e}{k^2}\int(\vec{k}'\vec{g}_{\vec{k}\omega})\{(\vec{k}\vec{g}_{\vec{k}''\omega''}\vec{g}_{\vec{k}'-\omega'})(\vec{k}\vec{g}_{\vec{k}\omega})+(\vec{k}'\vec{g}_{\vec{k}''\omega''})(\vec{k}\vec{g}_{\vec{k}\omega})\}f\,d\vec{v} \tag{18}$$

$$v_{\vec{k}\omega,\vec{k}''\omega''} = \frac{4\pi e}{k^2}\int\{(\vec{g}_{\vec{k}\omega}\vec{k}')(\vec{g}_{\vec{k}''\omega''}\vec{k}'')+(\vec{g}_{\vec{k}\omega}\vec{k}'')(\vec{g}_{\vec{k}'\omega'}\vec{k}')\}f\,d\vec{v}. \tag{19}$$

Integrating by parts we can show that $k^2 v_{\vec{k}\omega,\vec{k}''\omega''} = \vec{k}''^2 v_{\vec{k}''\omega'',\vec{k}\omega}$. Using this relation and introducing the "number of waves" (number of quasi-particles):

$$N_{\vec{k}} = \frac{k^2}{8\pi}\left|\frac{\partial\epsilon}{\partial\omega}\right| I_{\vec{k}}, \tag{20}$$

for the case of weak turbulent state ($\gamma \ll \omega$), Eq.(17) can be rewritten as

$$\frac{1}{2}\frac{\partial N_{\vec{k}}}{\partial t} = \gamma_{\vec{k}} N_{\vec{k}} + \int W_{\vec{k}\vec{k}'} N_{\vec{k}} N_{\vec{k}'}\,d\vec{k}'$$
$$+\frac{1}{2}\int U_{\vec{k}\vec{k}'} N_{\vec{k}'} N_{\vec{k}''}\,\delta(\omega_{\vec{k}} - \omega_{\vec{k}'} - \omega_{\vec{k}''})\,d\vec{k}', \tag{21}$$

where

$$W_{\vec{k}\vec{k}'} = \frac{8\pi}{k'^2}\frac{\operatorname{Im} R_{\vec{k}\omega,\vec{k}'\omega'}}{\left|\frac{\partial\epsilon'}{\partial\omega'}\cdot\frac{\partial\epsilon}{\partial\omega}\right|} + \frac{8\pi}{k'^2\left|\frac{\partial\epsilon'}{\partial\omega'}\cdot\frac{\partial\epsilon}{\partial\omega}\right|}\operatorname{Im}\left\{\frac{v^2_{\vec{k}\omega,\vec{k}''\omega''}}{\epsilon''}\right\},$$

$$U_{\vec{k}\vec{k}'} = \frac{8\pi^2\left|v_{\vec{k}\omega,\vec{k}'\omega'}\right|^2 k^2}{k'^2 k''^2\left|\frac{\partial\epsilon}{\partial\omega}\frac{\partial\epsilon'}{\partial\omega'},\frac{\partial\epsilon''}{\partial\omega''}\right|};\quad \epsilon' = \epsilon(\vec{k}',\omega'),\quad \epsilon'' = \epsilon(\vec{k}'',\omega''). \tag{22}$$

In Eq. (21) the last term and the contribution from the residue of the third term at $\epsilon'' = 0$ describe the wave decay processes whereas the other non-linear terms describe the scattering of waves by particles.

In the expressions for $R_{\vec{k}\omega,\vec{k}'\omega'}$ and $v_{\vec{k}\omega,\vec{k}'\omega'}$ we can neglect the exponentially small contribution from residues at $\omega = \vec{k}\cdot\vec{v}$ and $\omega' = \vec{k}'\cdot\vec{v}$. In this approximation $v_{\vec{k}''\omega'',\vec{k}\omega} = v^*_{-\vec{k}'',v\omega'',\vec{k}'\omega'}$. Using this relation we can show that the wave scattering by particles does not change the net number of the quasi-particles $\int N_{\vec{k}}\,d\vec{k}$. As to the wave-wave scattering, on the other hand, it conserves the net energy for oscillations.

Eq. (21) is valid only for electrons. In order to include the ion motion, it is sufficent to take into account the ion contribution to R, v and ϵ. It is not difficult to generalize this equation for the case when the external magnetic field is present. For this purpose, it is sufficient to change the operator $\vec{g}_{\vec{k}\omega}$ in a proper way.

The second term in the right-hand side of Eq. (21) describes the thermal fluctuations. Thus, schematically, Eq. (21) can be written as

$$\frac{1}{2}\frac{\partial N}{\partial t} = \gamma N + q - \alpha N^2, \tag{23}$$

where the first term describes the oscillation growth with the increment γ, q is the source of thermal noise, and where the non-linear terms describe the interaction of waves. One sees, therefore, that in the turbulent steady-state, when the increment is large, it is possible to neglect q and, consequently, $I = \gamma/\alpha$. On the other hand, when γ is negative and not very large in its absolute value, one can, in Eq. (23), neglect the non-linear term, thus $I = q/|\gamma|$. In this case, the plasma has thermal noise only. When the intensity of the thermal fluctuations tends to infinity, the non-linear term should be retained to define $I_{\vec{k}}$. The real picture of weak turbulent steady-state is, of course, more complicated because the non-linear terms are a form of diffusion nature in k-space.

IV. INTERACTION OF PARTICLES WITH WAVES

Let us consider now the equation for the averaged function (13). The expression on the right-hand side of this equation is denoted by S_{ef}. As can be seen from Eq. (13) the term describing the collisions of particles and waves may be written

$$S_{ef} = \mathrm{Im}\,\frac{e}{m}\frac{\partial}{\partial \vec{v}}\int \vec{k}\,P_{\vec{k}\omega}(\vec{v})\,d\vec{k}\,d\omega, \tag{24}$$

where the correlation function $P_{\vec{k}\omega}$ is defined by the relation

$$P_{\vec{k}\omega}(\vec{v})\,\delta(\omega-\omega')\,\delta(\vec{k}-\vec{k}') = \langle \varphi^{*}_{\vec{k}'\omega'},\ f_{\vec{k}\omega}\rangle. \tag{25}$$

Using the expression for $f_{\vec{k}\omega}$ we obtain in the quasi-linear approximation

$$\dot{P}_{\vec{k}\omega}(\vec{v}) = \frac{4\pi e f\,\delta(\omega - \vec{k}\cdot\vec{v})}{(2\pi)^3 k^2 \epsilon^*(\vec{k}, \omega + i\nu)} + I_{\vec{k}\omega}(\vec{k}\,g_{\vec{k}\omega})f. \tag{26}$$

In this expression the first term describes the slowing down of particles due to the polarization of the medium and the Cerenkov radiation of the longitudinal waves.

Let us first consider the simplest case of a stationary stable plasma where we can neglect the quadratic terms in I. In this case the intensity of fluctuations $I_{\vec{k}\omega}$ is given by (10). Substituting this value in the second term of Eq. (26) we find

$$S_{ef} = \frac{2e^4}{m^2}\int \frac{\delta(\vec{k}\cdot\vec{v} - \vec{k}'\cdot\vec{v}')}{k^4|\epsilon(\vec{x}, \vec{k}\cdot\vec{v})|^2}\left\{f(\vec{v})\vec{k}\,\frac{\partial f(\vec{v}')}{\partial \vec{v}'} - f(\vec{v}')\vec{k}\,\frac{\partial f}{\partial \vec{v}}\right\} d\vec{k}\,d\vec{v}'. \tag{27}$$

When $kD>1$ (where D is the Debye radius), the dielectric constant ϵ may be set equal to unity and the integral in Eq. (27), as can be easily shown, converts into the Landau collision term. (One should moreover cut off the integral in the upper limit at $k \simeq 1/\rho_0$, where ρ_0 is the minimum distance between particles at binary collisions.)

When approaching instability the amplitude of thermal noises increases, in the expression (26) the second term prevails. By neglecting the first term we obtain the quasi-linear approximation treatment. In this case, and in the case of slightly unstable plasma, it is more convenient to consider a non-stationary problem with given initial conditions. In the wave kinetic equation we should, therefore, take into account a term with a time derivative, whereas in the expression for S_{ef} we should consider $I_{\vec{k}}$ to be the time function.

V. NON-LINEAR INTERACTION OF LANGMUIR AND ION-SOUND WAVES

Let us now consider the simplest example of Langmuir oscillations. The Langmuir wave spectrum is non-decaying; hence, the scattering of waves by particles will be the main non-linear process. When $kD>\sqrt{m/M}$ we may neglect the wave scattering by ions, and the kinetic equation (21) becomes:

$$\frac{\partial N_{\vec{k}}}{\partial t} = \frac{4\pi}{\omega_0 mn^2} \int \frac{(\vec{k}\cdot\vec{k}')^2}{k^2k'^2} (\vec{k}\cdot\vec{v})^2 \delta(\omega'' - \vec{k}''\cdot\vec{v}) \vec{k}'' \frac{\partial f_0}{\partial \vec{v}} \, d\vec{v} \, N_{\vec{k}} N_{\vec{k}'} \, d\vec{k}'. \tag{28}$$

From this relation, one sees that the total number of waves $\int N_{\vec{k}} d\vec{k}$ really remains constant. According to Eq. (28) the wave scattering process by electrons results in the diffusion of the wave packet in the $\vec{k}$-space towards small k. The characteristic damping rate of this diffusion is of the order of magnitude

$$\nu \simeq \omega_0 (kD)^3 \frac{\epsilon}{nT}, \tag{29}$$

where ϵ is the energy of the Langmuir waves.

When $kD<\sqrt{m/M}$ the ion scattering prevails, and instead of Eq. (28) we obtain

$$\frac{\partial N_{\vec{k}}}{\partial t} = \frac{\pi\omega_0 T_i^2}{(T_e+T_i)^2 Mn^2} \int \frac{(\vec{k}\cdot\vec{k}')^2}{k^2k'^2} \delta(\omega'' - \vec{k}''\cdot\vec{v}) \vec{k}'' \frac{\partial f}{\partial \vec{v}} N_{\vec{k}} N_{\vec{k}'} \, d\vec{k}'. \tag{30}$$

As we can see, the ion scattering also leads to the decrease of k. The oscillation energy ϵ tends to a finite limit, and, consequently the process of scattering cannot by itself result in the complete relaxation of waves in homogeneous medium.

Let us now consider ion-sound waves. Such waves can be excited in a plasma with $T_i \ll T_e$ and in the presence of electric current, when the mean (drift) velocity of electrons is higher than the sound velocity $c_s = \sqrt{T_e/M}$.

In this case the scattering of waves by ions also results in a flux in the k-space towards small k. As can be shown, the wave kinetic equation becomes:

$$\frac{\partial N_{\vec{k}}}{\partial t} = 2\,\gamma_k N_k + 2\,A\,N_k\left(k^6 \frac{\partial N_{\vec{k}}}{\partial k} + 4\,k^5 N_k\right), \tag{31}$$

where $A = \Omega_0^2 T_i\,\theta^2/2gT_e^2 n$, $\Omega_0^2 = 4\pi e^2 n/M$, and θ is the mean square root value of the angle between k and the z-axis, which was supposed to be small.

When ion-sound instability occurs $\gamma_k = \alpha k$, where α is a constant, and from Eq. (31) we obtain

$$N_k \cong \frac{\alpha}{2Ak^4} \ln \frac{1}{kD}. \tag{32}$$

We see that non-linear interaction leads to the spectrum sharply decreasing with k.

Using the wave kinetic equation we can investigate various weak turbulent states. Unfortunately, this kinetic equation cannot be applied to the cases when the interaction between waves is not small. For such strong turbulent states we have no suitable strict mathematical methods. We have, therefore, to use some approaches similar to the weak coupling approximation or semi-empirical theory based on the mixing length concept.

VI. WEAK COUPLING APPROXIMATION

Up to now, the wave interaction has been considered to be very small. Let us see what happens when the matrix element increases. Let us consider the model equation

$$(\omega - \omega_{\vec{k}})C_{\vec{k}\omega} = \int V_{\vec{k}\omega,\,\vec{k}'\omega'}\, C_{\vec{k}'\omega'}\, C_{\vec{k}''\omega''}\, d\vec{k}'\, d\omega', \tag{33}$$

where the eigen-frequency $\omega_{\vec{k}}$ is a complex number. We assume that the matrix element of the interaction V increases approaching of the order of unity. It is clear that the interaction between waves results in the broadening of I_ω as a function of ω. Hence, in the case of strong turbulence, the dependence of $I_{\vec{k}\omega}$ upon the frequency cannot be approximated by $\delta(\omega - \omega_{\vec{k}})$. We cannot therefore use the kinetic equation in form (21). If the matrix element still remains lower than unity then, even in the case of stronger turbulence, one can use the weak coupling approximation.

Note that according to Eq. (33), a single wave k, ω interacts only with two quite different waves $\vec{k}', \omega'$ and $\vec{k}'', \omega''$. As we have seen above, one of the main effects of non-linear interaction is the damping of a single wave which is defined by the right-hand side of Eq. (33). Let us rewrite this equation in the form

$$(\omega - \omega_{\vec{k}} + \eta_{\vec{k}\omega}) C_{\vec{k}\omega} = \eta_{\vec{k}\omega} C_{\vec{k}\omega} + \int V_{\vec{k}\omega,\vec{k}'\omega'} C_{\vec{k}'\omega'} C_{\vec{k}''\omega''} \, d\vec{k}' \, d\omega'. \tag{34}$$

The term $\eta_{\vec{k}\omega} C_{\vec{k}\omega}$ in the left-hand side of this equation takes into account the part of non-linear interaction proportional to $C_{\vec{k}\omega}$. In the right-hand side of Eq. (34) from which we have already picked up the "damping" of every single wave, there remains only the effect of the driving force. We shall consider this force to be small, which is in fact so if $V \ll 1$. According to this, we shall write $C = C^{(0)} + C^{(1)}$, where $C^{(1)} \ll C^{(0)}$. For the amplitude of the forced oscillations $C^{(1)}$ in the right-hand side of Eq. (34), it will be sufficient to retain the non-linear term only, thus we shall have

$$C^{(1)}_{\vec{k}\omega} = (\omega - \omega_{\vec{k}} + \eta_{\vec{k}\omega})^{-1} \int V_{\vec{k}\omega,\vec{k}'\omega'} C^{(0)}_{\vec{k}'\omega'} C^{(0)}_{\vec{k}''\omega''} \, d\vec{k}' \, d\omega'. \tag{35}$$

Let us multiply Eq. (34) by $C^{*}_{\vec{k}\omega}$ and average the result over the random phase of oscillations. In the non-linear term we substitute $C = C^{(0)} + C^{(1)}$, assuming that $C^{(0)}$ should be statistically independent. The averaging of the non-linear term leads to three terms, two of which are proportional to $I_{\vec{k}\omega}$, and the third one contains $I_{\vec{k}'\omega'} \, I_{\vec{k}''\omega''}$ as the integrands. Defining the quantity $\eta_{\vec{k}\omega}$ so as to cancel the terms in the right-hand side of the mentioned equation, which are proportional to $I_{\vec{k}\omega}$, we obtain the following equations:

$$\left|\omega - \omega_{\vec{k}} + \eta_{\vec{k}\omega}\right|^2 I_{\vec{k}\omega} = \frac{1}{2} \int \left| v_{\vec{k}\omega,\vec{k}'\omega'} \right|^2 I_{\vec{k}'\omega'} \, I_{\vec{k}''\omega''} \, dk' \, d\omega', \tag{36}$$

$$\eta_{\vec{k}\omega} = - \int \frac{v_{\vec{k}\omega,\vec{k}''\omega''} \, v_{\vec{k}''\omega'',\vec{k}\omega}}{\omega'' - \omega_{\vec{k}''} + \eta_{\vec{k}''\omega''}} \, I_{\vec{k}'\omega'} \, d\vec{k}' \, d\omega', \tag{37}$$

where

$$v_{\vec{k}'\omega,\vec{k}'\omega'} = V_{\vec{k}\omega,\vec{k}'\omega'} + V_{\vec{k}\omega,\vec{k}''\omega''}.$$

Defining a new function $S_{\vec{k}\omega}$ by the relation $S_{\vec{k}\omega} = (\omega - \omega_{\vec{k}} + \eta_{\vec{k}\omega})^{-1}$, we rewrite these equations in the form

$$I_{\vec{k}\omega} = \frac{1}{2} \left| S_{\vec{k}\omega} \right|^2 \int \left| v_{\vec{k}\omega,\vec{k}'\omega'} \right|^2 I_{\vec{k}'\omega'} \, I_{\vec{k}''\omega''} \, d\vec{k}' \, d\omega', \tag{38}$$

$$S_{\vec{k}\omega} = S^{(0)}_{\vec{k}\omega} - S^{(0)}_{\vec{k}\omega} \int S_{\vec{k}''\omega''} \, v_{\vec{k}\omega,\vec{k}''\omega''} \, v_{\vec{k}''\omega'',\vec{k}\omega} \, I_{\vec{k}'\omega'} \, d\vec{k}' \, d\omega', \tag{39}$$

where

$$S^{(0)}_{\vec{k}\omega} = (\omega - \omega_{\vec{k}})^{-1}.$$

The functions $S_{\vec{k}\omega}$ and $S^{(0)}_{\vec{k}\omega}$ have a simple physical meaning which can be easily understood if a small additional external force $f_{\vec{k}\omega}$ is added to the right-hand side of Eq. (8). Repeating the calculations it will be easy to verify that $S_{\vec{k}\omega}$ is the Green function describing the response of the turbulent medium to the "small force" $f_{\vec{k}\omega}$. $S^{(0)}_{\vec{k}\omega}$ is the Green function in the linear approximation.

The equation of this form for ordinary fluid was suggested by Kraichnan. Wild has shown that such equations may be derived using the selective summation of the perturbation theory series. Similar equations may be used to treat strong turbulence phenomena in plasma.

VII. SEMI-EMPIRICAL APPROACHES

Even if the weak coupling approximation were to be applied to strong plasma turbulence it would require a large number of numerical calculations. That is why it is desirable to use simpler methods of treatment of strong turbulent motions in plasma. In many specific cases, the semi-empirical methods similar to those used in ordinary hydrodynamics may be developed for plasma turbulence. We shall consider here only one specific case, namely, the turbulent convection in mirror traps. Such turbulent convection was observed experimentally by Ioffe. In his experiment, the collisionless plasma at low pressure was produced by acceleration of ions in a radial electrical field. This plasma, with an ion temperature of the order of 1 keV and density $\simeq 10^9 \mathrm{cm}^{-3}$, escapes from the trap at an anomalous time during $t \simeq 10^{-4}$ s.

Experiments show that decay of plasma is induced by flute instability. This instability leads to convection of plasma in traps. In the Ioffe device the electrons are cold. Their temperature is of the order 10 eV. Hence, the instability arises due to force $F \simeq nT_i/R$ which acts on the unit volume of a single plasma tube in radial direction. R is the mean radius of curvature of magnetic lines. As can be shown easily, this force leads to the acceleration of a single tube of the order of $g_0 = (T_i/Rm)[\Omega_0^2/(\Omega_c^2 + \Omega_0^2)]$, where $\Omega_c = eH/mc$, $\Omega_0^2 = 4\pi e^2 n/m$. If this tube is surrounded by plasma, then the magnitude of the acceleration is $g \simeq (n'/n)g_0$, where n' is the excess of plasma density compared to the background density.

When the turbulent convection develops, the amplitude of pulsation n' at a distance x from the wall is of the order of $n' \simeq x\, dn/dx$. The amplitude of velocity pulsation is of the order $v' \simeq (gx)^{\frac{1}{2}} \simeq x(g_0 d\ln n/dx)^{\frac{1}{2}}$, so that the coefficient of turbulent diffusion $D \simeq v'x \simeq x^2 (g_0 d\ln n/dx)^{\frac{1}{2}}$. Thus, near the wall, the flux q is equal to $-Ax^2(dn/dx)^{\frac{3}{2}}(g_0/n)^{\frac{1}{2}}$, where A is a numerical factor of the order of unity. To obtain q and therefore the plasma lifetime τ we must estimate the minimal scale of pulsations x_0. To do this, we must take into account that plasma motion is produced by fluctuations of the electrical fields. Namely, the transverse component of these fields $E_\perp = -\nabla_\perp \varphi$ leads to transverse velocity $\vec{v}_\perp = c(\vec{E}\times\vec{H})/H^2$. Since cold electrons are not confined by magnetic mirrors, the plasma potential must be positive. But in the vicinity of the wall, namely at the distance $x \simeq \rho_i = (T_i/m\Omega_c^2)^{\frac{1}{2}}$, the plasma tubes lose their ions, so that in this region the potential cannot be positive. It means that the boundary of plasma is equipotential and the radial component of velocity $v_r \simeq c\,E_\varphi/H$ must be zero at a distance of the order of about ρ_i near the wall. From these arguments it follows that the minimal scale of pulsation x_0 must be of the order of the ion Larmor-radius ρ_i. Estimating q as $q \simeq A\,n\sqrt{g_0 x_0} \simeq A\,n\sqrt{g_0\rho_i}$, we obtain for the plasma lifetime in a magnetic trap of radius a the following relation

$$\tau = \frac{\pi a^2 n}{2\pi a q} = Ca\left(\frac{T_i\,\rho_i}{mR}\,\frac{\Omega_0^2}{\Omega_c^2+\Omega_0^2}\right)^{\frac{1}{2}}.$$

At low density, when $\Omega_0^2 < \Omega_c^2$, this relation leads to $\tau \simeq \sqrt{n}$. This dependence was verified experimentally. Moreover, the experimental data on electrical field fluctuations fit fairly well into the qualitative theoretical picture of turbulent convection.

This fact demonstrates the validity of using semi-empirical approaches to plasma turbulence. Such an approach cannot, of course, give quantitative results, but it is adequate for a qualitative consideration of the turbulent phenomena. One may express the hope that together with the weak coupling approximation, this approach can give quantitative as well as qualitative results.

LANDAU DAMPING AND FINITE RESISTIVITY INSTABILITY IN PLASMAS

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I. LANDAU DAMPING AND FINITE AMPLITUDE WAVES

The main topics of my lectures will be anomalous damping in collision-free plasmas of finite amplitude disturbances and, in particular, of shock waves.

First, let me consider a laminar mechanism of collisionless damping, the well-known Landau damping. As one can see from the linear theory described in the paper by SIMON [1], this damping is related to resonant particles having a velocity close to the wave phase velocity. Now we shall take into account a non-linear phenomenon which plays an important role. As is known, Landau damping is determined by the value of the derivative of the distribution function

$$\frac{dF}{dv}\left(v=\frac{\omega}{k}\right). \tag{1}$$

One of the most important non-linear phenomena is the distortion of the distribution function, especially in the resonant region $v \simeq \omega/k$. In order to understand what will happen, we start from the quasi-linear approximation (QLA) [2], where these distortions are described by a diffusion-type term

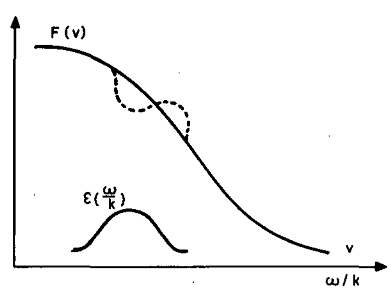

Fig. 1

The distortion of the distribution function in the quasi-linear theory.

in the kinetic equation for the "background" function $\langle F \rangle$, Fig. 1. It is quite clear now that the larger the oscillation amplitude the stronger their relaxing action on the form of the distribution function F(v) in the neighbourhood of $v \simeq \omega/k$ will be. Hence, one may expect that the damping decrement value $(1/\mathscr{E})\,d\mathscr{E}/dt$ ($\mathscr{E}$ - wave energy) which is proportional to dF/dv (i.e. $v = \omega/k$) decreases with the increase of $\mathscr{E}$. The quasi-stationary slope dF/dv will be found from the equation

$$\frac{d}{dv} D \frac{dF}{dv} = St(F) \tag{2}$$

where, for the resonant particles $v \simeq \omega/k$, the expression for D(quasi-linear coefficient of diffusion) can be reduced to

$$D \simeq \frac{e^2 \bar{E}^2}{m^2 \omega}. \tag{3)*$$

In the expression for F_{coll} we shall keep the term of the highest order containing the second derivative $(\nu T/m) d^2(f_M - F)/dv^2$, where f_M is Maxwell's function of distribution. Such a simplified form of the collision integral takes into account the reduction of the local equilibrium**. By integrating Eq. (2) once we shall get

$$\frac{dF}{dv} \simeq \frac{df_M}{dv} \frac{1}{1 + e^2 \bar{E}^2 / m\omega\nu T}. \tag{4}$$

The damping decrement of small amplitude waves $(e^2\bar{E}^2/m\omega\nu T < 1)$ tends to

$$\nu \rightarrow \nu_m = \frac{\pi \omega_0}{2} \left(\frac{\omega}{k}\right)^2 \frac{dF_M}{dv}, \quad \left(v = \frac{\omega}{c}\right),$$

i.e. Landau damping decrement. At $e^2\bar{E}^2/m\omega T\nu > 1$ amplitudes, the linear theory is no longer applicable. The damping decrement for such waves, as follows from Eq. (4), should decrease with the increase of amplitude as E^{-2}.

The above quasi-linear consideration reflects a general feature of damping decrease for finite amplitude waves. However, in the quasi-linear approximation we could not see the microscopic mechanisms of entropy production, since QLA means random phase approximation from the beginning.

Let us now make a qualitative consideration, where we can see the mechanism of phase mixing. What would happen to the Maxwellian distribution function if a monochromatic wave with amplitude φ_0 of electric potential were given in a plasma for t = 0 (t = time)? Our main interest lies in a time evolution of resonant particles. These particles will be trapped by moving potential wells if $(m/2)(v - \omega/k)^2 < e\varphi_0$, and will oscillate inside the wells (Fig. 2). The periods of such oscillations are different for the various particles having different energies. The distribution function must keep along the characteristics of the collisionless kinetic equation. These characteristics are the particle orbits. Now, on the basis of period differences, we can conclude that the distribution function at any fixed x will

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* We shall consider an example of Langmuir electronic oscillations.

** A precise consideration of Landau's form of collision integrals gives similar results.

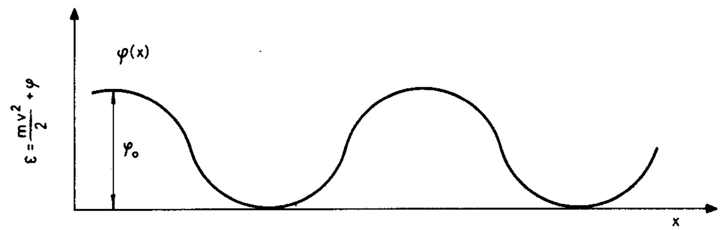

Fig. 2

Trapping by moving potential wells

be modulated in space velocity, and this modulation will grow with time (see Fig. 3).

This is the microscopic mechanism of plateau formation. Because of the fast growing of modulation frequencies in velocity space (i. e. second

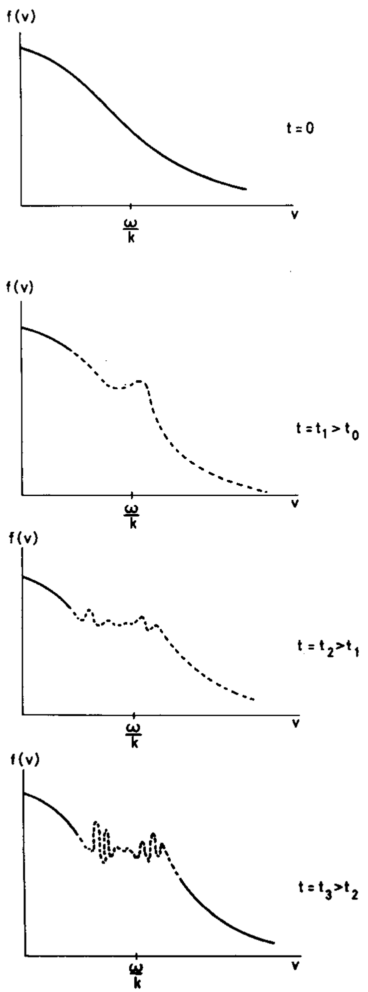

Fig. 3

Modulation of the distribution function

derivatives $\partial^2 f/\partial v^2$), conventional collisions will very soon need to be taken into account and we can obtain, at last, a real plateau. It is interesting to know the final result for the wave damping rate in this case.

Moreover, it is a more complicated task to find the Landau damping for finite amplitude monochromatic waves when the quasi-linear method is inapplicable. As is well-known, in this case we have non-damping oscillations in the limit when interparticle collisions may be neglected – these being the stationary waves of Bernstein-Green-Kruskal.

Let us now calculate the damping decrement. As an example, we take longitudinal plasma waves of finite (but not small) amplitude assuming the collision frequency to be sufficiently low but not neglecting it at all. The method is the following: the electron distribution function f(v, x) is derived from a kinetic equation (we assume that the amplitude change of the wave connected with its damping may be neglected so that it can be considered stationary in the system where the wave is at rest). The wave damping is now given by the formula:

$$\frac{1}{\mathscr{E}}\frac{d\mathscr{E}}{dt} = -\frac{\overline{\vec{E}\cdot\vec{j}}}{\mathscr{E}}, \quad j = \int_{-\infty}^{+\infty} vf dv, \tag{5}$$

where E is the wave field amplitude, and j the current density induced by the wave (in the system where the plasma is at rest and $\mathscr{E}$ is the energy density of the wave). The dash means the average over the period of oscillations. Thus, everything is reduced to the search for the distribution function.

Ions, for simplicity, are assumed to be infinitely heavy (at rest) and uniformly distributed in space.

The kinetic equation for this function in the system of the resting wave may be written down in view of collisions as*

$$u\frac{\partial f}{\partial y} - \varphi'(y)\frac{\partial f}{\partial u} = \frac{\partial f}{\partial u}\left[\frac{\partial f}{\partial u} + (\alpha + u)f\right], \tag{6}$$

where all values are reduced to the dimensionless form:

$$\varphi(y) = \frac{e\varphi}{T}, \qquad y = kx, \qquad \varphi(x) = \varphi_0 \cos^2\frac{kx}{2},$$
$$\alpha = \frac{v_{ph}}{(T/m)^{\frac{1}{2}}}, \qquad u = \frac{3}{2kv_T\tau_D}, \qquad \tau_D \simeq \frac{m^2 v_{ph}}{8\pi e^4 nL}, \tag{7}$$

where $\varphi(x)$ is the wave potential in the system at rest, k and v_{ph} are its wave number and phase velocity respectively, τ_D is the effective time of electron collisions whose velocity coincides with the wave phase velocity, L are Coulomb logarithms and $\varphi(y)$, y, α and u are the dimensionless potential energy, co-ordinate, phase velocity and collision frequency, respectively.

---

* We shall now use more real forms of collision integrals.

Let us introduce one restriction which makes the problem soluble (in practice):

$$u \ll \varphi_0 \ll 1: \tag{8}$$

this corresponds to a wave of finite but small amplitude and low frequency of collisions.

Now it will be more convenient to deal with new independent variables for the distribution functions

$$u_D y \rightarrow \epsilon = \frac{u^2}{2} + \varphi(y), \quad y,$$

where ϵ is the dimensionless total particle energy in the wave field. Then Eq. (6) takes the form

$$\frac{\partial f}{\partial y} = \nu \frac{\partial}{\partial \epsilon} \left\{ \pm [\epsilon - \varphi(y)]^{\frac{1}{2}} \left(f + \frac{\partial f}{\partial \epsilon} \right) + cf \right\}, \tag{9}$$

$$c = \frac{\alpha}{\sqrt{2}} = \frac{v_{ph}}{(2T/m)^{\frac{1}{2}}}, \qquad \nu = \sqrt{2}u = \frac{3}{\sqrt{2}kv_T \tau_D},$$

where the ± signs before the root correspond to the different directions of electron velocities (plus sign is taken for particles overtaking the wave).

In solving Eq. (9) we must consider two cases: $\epsilon > \varphi_0$ (external region) and $\epsilon < \varphi_0$ (internal region, see Fig. 2). We find the solution of Eq. (9) as an expansion in series:

$$f(y, \epsilon) = f_0(\epsilon) + \nu f_1(\epsilon, y) + \dots . \tag{10}$$

By substituting Eq. (10) into Eq. (6) we get

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial \epsilon} \left\{ \pm [\epsilon - \varphi(y)]^{\frac{1}{2}} \left(f_0 + \frac{\partial f_0}{\partial \epsilon} \right) + cf_0 \right\}, \tag{11}$$

where $f_0(\epsilon)$ (zero approximation) is defined (in the external region) by the condition $f_1(y)$ is periodic, and by the boundary condition that at $\epsilon \gg \varphi_0$ the value $f_0(\epsilon)$ should asymptotically approach the Maxwell distribution for the plasma moving at the velocity $-c$ relative to the wave. Thus the equation for $f_0(\epsilon)$ at $\epsilon > \varphi_0$ will be

$$\frac{d}{d\epsilon} \left\{ J(\epsilon) \left[f_0 + \frac{df_0}{d\epsilon} \right] + cf \right\} = 0, \tag{12}$$

where $J(\epsilon) = \pm \frac{1}{2\pi} \int_{-\pi}^{\pi} (\epsilon - \varphi_0 \sin^2 \frac{y}{2})^{\frac{1}{2}} dy$ are some elliptic integrals. Now we can find $f_0(\epsilon)$ and substitute it in Eq. (11). Finally we must have f_1^{ext} and $W_{ext} = \int jE dx$, which, as one can see, will be expressed in terms of some elliptic integrals.

Let me write this, omitting a calculation and using some expansions of elliptic integrals for the case $\varphi_0 \ll 1$,

$$\dot{W}_{ext} \simeq 0.1\, \nu \varphi^{\frac{1}{2}}\, c\, e^{-c^2} \omega T n. \tag{13}$$

The same procedure must be used also in the internal region ($\epsilon < \varphi_0$). However, the distribution function here is quite symmetrical $f_+(\epsilon) = f_-(\epsilon)$. This means that its contribution to wave damping $\dot{W}$ is quite small, and we can therefore neglect it.

The major problems arise when we want to find distribution functions inside the singular domain lying between external and internal regions. Fortunately, as shown by ZACHAROV and KARPMAN [3], we do not need to know in detail the behaviour of this function to calculate W_{sing}: it is enough to know its value at both sides of the singular region. In fact,

$$\dot{W}_{sing} = \frac{\omega T v_T}{\sqrt{8\pi}} \int_{-\pi}^{+\pi} dy \int_{\varphi_0-\delta}^{\varphi_0+\delta} \frac{d\epsilon}{[\epsilon - \varphi(y)]^{\frac{1}{2}}} \varphi'(y)(f^+ + f^-)$$

$$= \frac{-\omega T v_T}{\sqrt{8\pi}} \int_{-\pi}^{+\pi} dy \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\epsilon \frac{d[\epsilon - \varphi(y)]^{\frac{1}{2}}}{dy} (f^+ + f^-),$$

where δ is the width of the singular region.

Now, by using integration by parts and taking into account only the lowest derivatives of the distribution function inside a singular region, we obtain

$$\int_{-\pi}^{+\pi} dy \left\{ \int_{\varphi_0-\delta}^{\varphi_0+\delta} d\epsilon \frac{d[\epsilon - \varphi(y)]^{\frac{1}{2}}}{dy} [f^+(y, \epsilon) + f^-(y, \epsilon)] \right\}$$

$$\simeq \varphi_0 \cos^2 \frac{y}{2} \left(\frac{\partial f_0^{\pm}}{\partial \epsilon} \Bigg|_{\epsilon = \varphi_0 + 0} - \frac{\partial f_0^{\pm}}{\partial \epsilon} \Bigg|_{\epsilon = \varphi_0 - 0} \right).$$

If we substitute here the values of $\partial f_0 / \partial \epsilon$, which we find from external ($\epsilon = \varphi_0 + 0$) and internal ($\epsilon = \varphi_0 - 0$) regions, we finally have

$$\dot{W}_{sing} \simeq 3 \nu \varphi_0^{\frac{1}{2}} e^{-c^2} c \omega T n. \tag{14}$$

In order to explain the physical meaning of the results for monochromatic waves, we shall give a semi-quantitative estimate of the dependence of increment on amplitude. The QLA formula (4) may be interpreted as follows: let us represent it as $\nu = \nu_m/(1 + \tau_1/\tau_2)$. Here ν_m is the decrement obtained in the linear approximation ("Landau damping"), τ_1 is the characteristic time for the establishment of Maxwell's local distribution, τ_2 is the characteristic time of distortion of the distribution function under the influence of wave packet field. If $\tau_1 \ll \tau_2$, i.e. the collisions succeed in "Maxwellizing" the distribution function, we get the conventional Landau damping. With the increase of the wave amplitude the distortion introduced by it turns out to be so large that the collisions lag behind "Maxwellizing" and the decrement of damping decreases.

With the aid of a similar interpretation we can estimate the absorption in the case of "monochromatic waves", and on condition that τ_1 and τ_2 are chosen correctly. Let φ be the amplitude of potential in the wave. Then the particles having velocities relative to the waves of the order $\omega/k - (e\varphi/m)^{\frac{1}{2}} < v < \omega/k + (e\varphi/m)^{\frac{1}{2}}$ will be responsible for the absorption. This means that the distribution function is distorted most strongly in the Δv region with the width of the order $(e\varphi/m)^{\frac{1}{2}}$. Because of the Coulomb collisions (small-angle scattering) the local equilibrium in this region will evidently be reconstructed within the time $\tau_1 \simeq e\varphi/\nu T$. The time of the non-linear distortion under the influence of the wave field is of the order $\tau_2 \simeq \lambda/(e\varphi/M)^{\frac{1}{2}}$, where λ is the wave length. We shall finally get

$$\nu \simeq \frac{\nu_m}{1 + (e\varphi)^{3/2}(T\lambda\nu\sqrt{m})^{-1}}. \tag{15}$$

This means that for "monochromatic" waves the damping decrement falls off as $E^{3/2}$ with the increase of amplitude. A rigorous consideration as we have seen above confirms this dependence.

Hence, it follows that non-linear wave damping takes place in the distribution of resonant particles responsible for damping. Nevertheless, this does not guarantee us that non-linear stable waves, once they have appeared, will exist for a long period of time. It is also necessary to see whether they are stable with respect to the different, random distortions. If they turn out to be unstable it would mean that their energy has passed to some other forms of motion of plasma, possibly to the irregular turbulent motion. Then we speak about the effective damping.

II. THE CONNECTION BETWEEN FINITE RESISTIVITY INSTABILITY (PRECISE THEORY) AND BOHM DIFFUSION (SPECULATIONS)

Let us now consider the finite-resistivity instability of plasma confinement. The great importance of finite resistivity has been shown in the paper by FURTH [4]. I shall talk about a different aspect of this problem, in particular, about the finite resistivity instability in a plasma column situated in a strong magnetic field with straight lines of force (Fig. 4). I exclude effective gravity, longitudinal currents (in undisturbed equilibrium) and will try to look for instabilities induced only by the non-uniformity of plasma density. In order to make clear the physical meaning of the instability looked for, I shall use some simplifications which, as far as I know, do not change the final results.

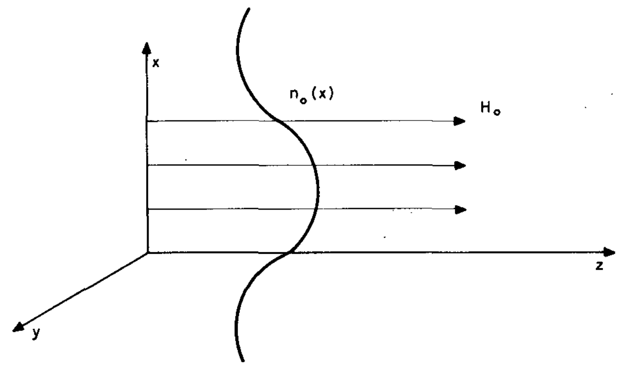

Fig. 4

Plasma column in a strong magnetic field with straight lines of force.

$$
\begin{aligned}
&p_0 \ll \frac{H_0^2}{8\pi} \\
&\vec{E}_1 = -\vec{\nabla}\varphi_1 \\
&T_i \ll T_e \equiv T_0 \\
&\omega \ll \Omega_{Hi}
\end{aligned}
\tag{16}
$$

Let us consider the behaviour of disturbances in the form

$$\varphi_1 \simeq \exp(i\omega t + ik_z z + ik_y y).$$

Then the equations for perturbations will be

$$i\omega n + c\frac{E_y}{H_0} n_0' = 0 \qquad \left(n_0' = \frac{dn_0}{dx}\right). \tag{17}$$

(These are the continuity equations for ions, where we neglected the ion motion along z. It means that we consider a perturbation with $k_y \gg k_z$.) However, we take into account the electron motion along $\vec{H}_0$, neglecting inertia of the electrons.

$$-ik_z n_1 T_0 - en_0 E_z - mn_0 v_z \nu = 0. \tag{18}$$

Finite resistivity means a friction term in Eq. (18). The above form of Eq. (18) is, in fact, a generalized Ohm's Law (with the pressure gradient). It is there that we find the only difference from Furth's problems.

If we examine the electron continuity equation

$$i\omega n + c\frac{E_y}{H} n_0' + ik_z v_z n_0 = 0,$$

we shall see from the quasi-neutrality condition, that the last term here $+ ik_z v_z n_0$ must be compensated by some additional ion currents, δv_i, which is equal to

$$M c \left[c \frac{d}{dt} \frac{[\vec{E} \times \vec{H}]}{H^2} \times \vec{H} \right] \Big/ \; eH^2,$$

the inertial drift of ions. So we have

$$ik_z n_0 v_z = -ik_y n_0 M \frac{c^2 i\omega}{eH_0^2} E_y . \tag{19}$$

Consequently, we have three equations, (17)-(18), for three variables: n, E, v_z. According to the standard procedure we find the following dispersion relation

$$\omega = \omega_e - i \frac{\omega^2}{\omega_s} , \tag{20}$$

where

$$\omega_s = \left(\frac{k_z^2}{k_y^2} \right) \Omega_{He} \Omega_{Hi} \frac{1}{\nu_e} ,$$

$$\omega_e = k_y \frac{cT_0}{eH_0} \frac{n_0'}{n_0} .$$

Now we see the simple physical meaning of the disturbances which are obtained. In fact we have drift waves. If we put $\nu_e \to 0$, i. e. $\omega_s \to \infty$, we find

$$\omega = \omega_e. \tag{21}$$

It is easy to see that the imaginary part, arising from finite conductivity, corresponds to the instability.

If we cannot put k_z^2 arbitrarily, for instance, because of the finite length of the plasma column, we shall always have high ω_s values. This means that the growth rate of instability will be small in this case;

$$\text{Im } \omega \approx \frac{\omega_e^2}{\omega_s} \ll \omega_e. \tag{22}$$

Further, if the choice of k_z is not limited, we can have arbitrary ω_s. The imaginary part of ω has a maximum,

$$\text{Im } \omega \simeq \omega_e, \tag{23}$$

when $\omega \simeq \omega_s$. This is the most unstable case.

These calculations ignored the precise eigenvalue problem and are only a rough estimate of the growth rate of instability.

Now we shall discuss this instability, considering the strong eigenvalue problem. In fact, we must construct perturbations in the precise form

$$[\exp(i\omega t + ik_z z + ik_y y)]\,\varphi(x). \tag{24}$$

Instead of deriving the eigenfunction equation, repeating the above calculations for the new form of perturbations Eq. (24), we note that

$$k_{y1}^2 \to k_y^2 + k_x^2 \to k_y^2 - \frac{d^2}{dx^2}. \tag{25}$$

Using it in Eq. (20), we find

$$\frac{d^2\varphi}{dx^2} - \left[1 - i\frac{\omega_s}{\omega}\left(1 - \frac{\omega_e}{\omega}\right)\right] k_y^2\varphi = 0, \tag{26}$$

where $\omega_s = \omega_s(x)$, $\omega_e = \omega_e(x)$.

I shall now say a few words about the philosophy of finite resistivity instability.

There is one classical example in the field of finite dissipation instability, that is, the well-known Poiseuille flow instability. The situation is very similar in these two cases. There are small parameters before the highest derivatives of the eigenfunction equations. The chosen example is easier: Eq. (26) is an equation of the second order.

A rigorous solution of the type II differential equation requires an accurate knowledge of the density profile. We shall restrict ourselves to the case when density varies so slowly that ω_s can be considered as a constant. Close to the point where n_0'/n_0 is maximum, ω_e may be written in the form

$$\omega_e \simeq \omega_{e0} - \beta x^2. \tag{27}$$

We then obtain

$$\frac{d^2\varphi}{dx^2} - \left\langle \left\{1 - \frac{\omega_s}{\omega}\left[1 - \frac{\omega_{e0}}{\omega}\right]\right\} - i\frac{\omega_s}{\omega^2}\beta^2 x^2 \right\rangle k_y^2\varphi = 0. \tag{28}$$

The solution of Eq. (28) is completely similar to the solution of Schroedinger equation for a linear harmonic oscillator. As a result, for the eigenvalues and the eigenfunctions we obtain:

$$E = -\left\{1 - i\frac{\omega_s}{\omega}\left[1 - \frac{\omega_{e0}}{\omega}\right]\right\} k_y^2, \tag{29}$$

$$k = -\frac{i\omega_s\beta}{\omega^2} k_y^2, \tag{30}$$

with

$$\frac{E}{\gamma^2} = 2n + 1 \tag{31}$$

$$\varphi(x) \simeq H_n(\gamma x) \exp(-\gamma^2 x^2). \tag{32}$$

Here $H_n(\gamma x)$ are Hermitian polynomials and

$$\gamma^2 = k_y \frac{\sqrt{\omega_s \beta}}{\omega_1^2 + \omega_2^2} \frac{1}{\sqrt{2}} [(\omega_1 + |\omega_2|) - i(\omega_1 - |\omega_2|)], \tag{33}$$

where

$$\omega_1 = \text{Re}\ \omega, \qquad \omega_2 = \text{Im}\ \omega.$$

These solutions, as may easily be seen, decrease in both directions from $x = 0$, a point where n_0'/n_0 is maximum.

If we examine the eigenvalues thus obtained carefully, we can see that the rough estimates of the type (20), (23) and (25) give a correct idea of the order of magnitude of the instability increment.

In a weakly ionized plasma, it would be necessary to take into account the neutral gas effect. We shall give the expressions for the frequency and increment of developing instability, without any derivations, taking into account ion-neutral collisions

$$\omega_1 \simeq \frac{\omega_e}{1 + \nu_{0i}/\omega_s}, \qquad \omega_2 \simeq \frac{\omega_1}{\omega_s(1 + \nu_{0i}/\omega_s)^3},$$

(if $\omega_s + \nu_{0i} \gg (\omega_e\omega_s)^{\frac{1}{2}}$), where ν_{0i} is the ion collision frequency.

Thus, in addition to the already known stabilizing factor which is due to the short plasma-column length, we obtain yet another factor – ion friction with the neutral gas.

The development of instability must give rise to a "turbulent" regime in the plasma and to turbulent diffusion. Let us make dimensional estimates of the expected diffusion, as is generally done in turbulence theory.

The diffusion coefficient can be written in the form

$$D \simeq \vec{v}^2 \tau.$$

Here $\vec{v}$ is the rate of plasma pulsations and τ the time characteristic of the disappearance of correlations. In this case, $\tau \simeq 1/\omega_2$, since there is no other time factor which would determine the irreversibility of the "turbulent" regime. The pulsation amplitude will be defined according to the following considerations. On the one hand, instability induces the growth of the pulsation amplitude $\partial v/\partial t \simeq \omega_2 v^*$; on the other hand, the non-linear terms of the type $(v\vec{\nabla})v$ and $\vec{\nabla}\cdot n\vec{v}$ induce the transfer of energy into the short wave section of the spectrum where the fluctuations fade. The stationary value of the pulsation amplitude is defined from the conditions of equilibrium existing between these two processes:

$$\omega_2 n^* \simeq \frac{v^*}{\lambda_\perp} n^*,$$

where $\lambda_\perp$ is the dimension characteristic of the turbulent pulsations taking the direction perpendicular to H.

We now obtain for D:

$$D \simeq \omega_2 \lambda_\perp^2 \simeq \omega_2 \lambda_\perp^2.$$

It is obvious to take for $\lambda_\perp$ the instability wave length

$$\lambda_\perp \simeq \frac{2\pi}{k_\perp}.$$

We are interested in obtaining the minimum diffusion coefficient, that is why we shall take the minimum permissible $k_\perp \simeq 2\pi/r$, where r is the characteristic transverse dimension of the system.

After this, we finally obtain

$$D_{\perp\,max} \simeq \frac{cT_0}{2\pi\, eH_0}.$$

This coincides with the diffusion coefficient adopted in the well-known Bohm hypothesis.

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SHOCK WAVES IN COLLISION FREE PLASMAS

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In ordinary gases, the density rise across a shock wave occurs in a distance of the order of a few collision mean free paths for strong shocks and becomes larger for weaker shocks. For MHD shocks in a plasma with a mean free path short compared to the gyro radius the shock thickness will also be larger than the mean free path. In both of these cases the shock thickness can be derived in a relatively straightforward manner from the steady one-dimensional flow equation taking into account the standard kinetic theory dissipation coefficients of the medium viscosity, electrical conductivity, etc. [1, 2].

In a collision free plasma, or more precisely a plasma with a mean free path large compared to the gyro radius, shock waves have thicknesses less than a mean free path. In this case, the mechanisms by which the directed flow energy ahead of the shock is converted to random energy behind the shock are more complex. The purpose of this lecture is to discuss some of the mechanisms by which this dissipation can occur.

Among the reasons for interest in collision free shock waves is the possibility of using shock waves as a heating mechanism for plasmas. If a plasma sample smaller than the particle collision mean free path is to be heated by a shock wave we must understand the dissipative mechanism in such a shock. A more fundamental reason is that since shock waves require some dissipation the study of collision free shock waves provides an opportunity for the study of turbulent dissipation mechanisms.

In the collision dominated case, the isentropic theory of characteristics (discussed in an earlier lecture) leads to the conclusion that a pressure pulse steepens. Eventually, the gradients become so steep that the dissipative effects become important, limit the steepening process, and result in a steady shock structure in which the density and temperature rise monotonically. In a collision free plasma the steepening process still exists in some cases. For example, a zero temperature collision free plasma is completely described by the MHD equations in the limit of ion gyro period small compared to the time scale of variation of the flow parameters. The theory of characteristics therefore again leads to the steepening of a compression pulse and we conclude that a shock wave can be formed on a scale small compared to a mean free path.

In the collision free case, however, as the gradients increase the steepening process is not inhibited directly by dissipative effects but rather by non-dissipative changes in the dispersion relation of small amplitude waves. When gradients are reached such that the pulse has significant components at wavelengths of the order of the ion gyroradius these components will propagate at different speeds than the longer wavelengths components. Thus the basic assumption in the characteristic theory (propagation speed independent of wavelength) which led to steepening is violated and we might expect small scale structure of the order of the ion gyroradius to appear. The problem of the collision free shock structure is to determine the nature of these fine scale variations and how they lead to dissipation.

HYDRAULIC ANALOGY

There exists an analogy between the collision free plasma shocks and hydraulic jumps in water which is helpful in demonstrating some of the possible mechanisms which may occur. Waves in shallow water with wavelengths long compared to the water depth, h, propagate at a speed $\sqrt{gh}$ which is independent of wavelength, (g is the acceleration due to gravity). Thus the theory of characteristics applies and it can easily be shown that a finite amplitude pulse steepens. When wavelengths comparable to either the water depth or a length $(zT/\rho g)^{\frac{1}{2}} \approx 0.5$ cm which is determined by the surface tension, T, are reached the propagation speeds of small amplitude waves change. (T is the surface tension and ρ the density). At these wavelengths the damping due to viscosity is, however, still small. Thus in the water case as in the plasma case non-dissipative changes in the dispersion relation limit the steepening process. In both cases a small scale structure will appear which is not in itself dissipative but results in dissipation. Thus examination of shock structures in shallow water can be helpful in illustrating some of the effects which may occur in plasmas. The analogy should, however, not be extrapolated to the point of quantitative comparison.

The water experiments to be described as well as much of the discussion of the analogy is based on the work of WITTING [3] although the existence of the analogy had been previously suggested by several authors for example [4]. The water experiment is sketched in Fig. 1. A long tank is separated by a divider which does not quite reach the bottom. The pressure in the chamber in the left is reduced slightly giving rise to a difference in water level between the two parts of the tank. When the cap is removed manually the pressure rises and water flows out producing a shock wave propagating to the right.

Three distinct types of shock waves are observed in water. Two of these are regular laminar patterns in which a wave train which is stationary in shock co-ordinates appears and is gradually damped by viscosity. In one of these the wave train is behind the shock and in the other it is ahead of the shock. The third type occurs in strong shock waves and is an irregular turbulent pattern.

The top picture in Fig. 2 is a side view of a weak shock propagating into relatively deep water (surface tension unimportant). The horizontal line is the initial water level while the wavy line is the water surface at an instant in time. The wave train decreases in amplitude towards the left. Thus we may regard the wave train as having been produced at the shock front and damping as it moves behind the shock. The picture shown would be entirely steady in a co-ordinate system moving with the shock, however, since the medium is dispersive the phase and group velocities are not equal and the energy associated with the wave train can be moving relative to the shock.

The lower picture in Fig. 2 is a top view of a weak wave travelling into shallower water (surface tension significant). A regular wave train is again

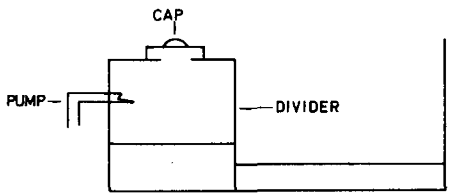

Fig. 1

Apparatus for water experiment

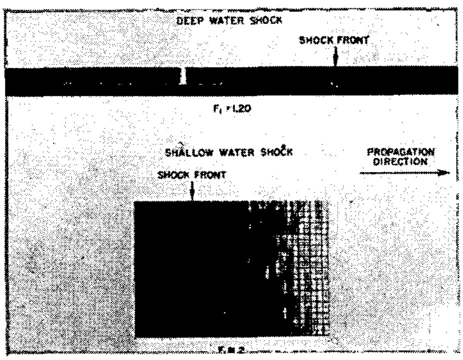

Fig. 2

Weak hydraulic shocks

observed. However, in this case it appears to damp as one goes to the right, i.e., ahead of the shock.

Both top and side views of stronger shocks in both deep and shallow water are illustrated in Fig. 3. In these cases the shock is seen to be highly turbulent. The water surface, particularly in the deep water case, appears to fluctuate in a highly random fashion. The side view indicates a

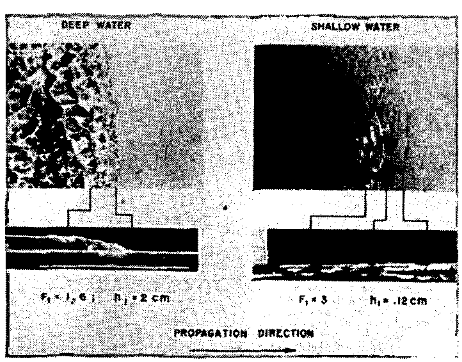

Fig. 3

Strong hydraulic shocks

gradual rise in water level to the level behind the shock with no remnant of the wave train behind. The shallow water top view may indicate some more pronounced regular features superimposed on the turbulent background. Thus, there might be several classes of turbulent shocks. However, for both shallow and deep water the shock waves become turbulent for sufficiently strong shock waves.

Let us first discuss the wave trains for weak shocks in terms of small amplitude waves. The right hand side of Fig. 4 shows the dispersion re-

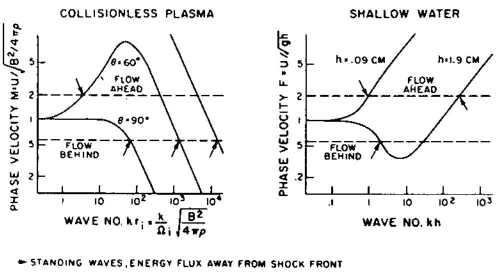

Fig. 4

Dispersion relations for infinitesimal waves

lation for water for two different water levels. In order to obtain a wave train standing in shock co-ordinates the phase velocity must equal the flow velocity. Possible flow velocities ahead and behind are also indicated. The intersection of these lines with the dispersion curves are possible cases in which a wave could stand in shock co-ordinates. However, we must also require that if the wave train is produced by the shock the energy flux be away from the shock. Thus a wave train ahead of the shock must have a group velocity greater than the phase velocity and a wave train behind must have a group velocity less than the phase velocity. The points which satisfy this condition are indicated by arrows. Note that, for shallow water, the only possible case is a wave train ahead of the shock. For deep water, waves either behind or ahead would be possible at considerably different wavelengths. The experiments indicate that the longer wavelength is the dominant one as one might expect if one viewed the shock as having been formed by a steepening process.

PLASMA WAVE TRAINS

The dispersion relations for a zero temperature plasma are shown in the left part of Fig. 4 for a shock propagating perpendicular to the magnetic field and at 60°. (The curve on the far right corresponds to electron plasma oscillations and probably has too short a wavelength to be of interest). For shocks propagating at an angle to the field we see that the allowed case with the longest wave length corresponds to a wave train ahead of the shock with a wavelength of the order of the ion gyro-radius. For propagation perpendicular to the magnetic field the wave train would exist behind and have a wavelength about 40 times smaller. Examining the dispersion relation at various angles shows that the wave train behind would only exist for a small range of angles ($\sim\sqrt{me/mi}$) in the neighbourhood of 90°. Thus the more common case would have wavelengths comparable to the ion gyroradius and the wave train ahead.

Thus far we have discussed the wave trains entirely as small amplitude waves. The water pictures show a finite amplitude which retains its shape as the shock propagator. Thus we must still ask whether such time independent finite amplitude solutions exist. It has been shown, both for water [5, 6] and for plasmas [7, 8], that such non-linear solutions exist. In the complete absence of dissipation these solutions give only a solitary pulse. However, with any small dissipation a non-linear wave train develops which then gradually damps. The damping in the plasma case might be due to the remaining collisional effects or Landau damping.

TURBULENT PLASMA SHOCKS

It is clear that for sufficiently weak shock waves the laminar solutions discussed above will apply. However at some shock strength, which is at present not clearly determined, we would expect instabilities to become important and lead to a turbulent shock. Several specific instabilities have been suggested recently [9, 10]. We will discuss a model which has been suggested for a fully turbulent shock [11] propagating perpendicular to the magnetic field into a zero temperature plasma. We will first discuss the formal equations which ought to be solved [12] and then indicate the extent to which a possible solution can be suggested.

The general picture consists of a gradual change in density, magnetic field and flow velocity upon which is superimposed a random field of waves. For moderate shock strengths these wave amplitudes will be small enough so that quasi-linear theory may be applied. The wave spectrum will be determined by the wave interactions with one another and with the overall flow. The flow in turn will be coupled to the wave spectrum by the momentum and energy fluxes associated with the waves.

For zero temperature there will be no resonant particle effects, thus the wave kinetic equation is of the form

$$\frac{Dn_k}{Dt} = \frac{\partial n_k}{\partial t} + \frac{dx}{dt}\frac{\partial n_k}{\partial x} + \frac{dk_x}{dt}\frac{\partial n_k}{\partial k_x} = \left(\frac{\partial n_k}{\partial t}\right)_{\text{wave-wave}} \tag{1}$$

where the non-linear interaction term is due entirely to wave-wave scattering. n_k is the number of quasi-particles or the energy at wave number k divided by $\hbar w$. (See paper by Kadomtsev*). The terms dx/dt and dk_x/dt are to be evaluated following a wave packet. Thus

$$\frac{dx}{dt} = u + v_g, \tag{2}$$

where u is the flow velocity and v_g is the x-component of group velocity re-

* These proceedings.

lative to the fluid. From the condition that the frequency in shock fixed coordinates $\omega + k_x u$ is a constant in a steady state shock we obtain

$$\frac{dk_x}{dt} = -kx\,\frac{du}{dx} \tag{3}$$

where we have also assumed that the dispersion relation contains no explicit dependence on x. This assumption makes use of two results to be described below. We have assumed that the relevant wave spectrum consists of the whistler mode at frequencies well above the ion cyclotron frequency and well below the electron cyclotron frequency. In this frequency range ω is a function of B/ρ and $\vec{k}$ only. Thus, from Eq.(6) below there is no explicit dependence on x. Rewriting Eq.(1) for a steady state shock structure making use of Eqs.(2) and (3) we have

$$(u + v_g)\,\frac{\partial n_k}{\partial x} - k_x\,\frac{du}{dx}\,\frac{\partial n_k}{\partial k_x} = \left(\frac{\partial n_k}{\partial t}\right)_{\text{wave-wave}}. \tag{4}$$

This expression relates the wave spectrum to the mean flow through u and du/dx. In order to obtain the effect of the waves on the flow we must write the conservation equations. These take the form

$$\rho u = \rho_1 u_1 \tag{5}$$

$$\frac{B}{\rho} = \frac{B_1}{\rho_1} \tag{6}$$

$$\rho u^2 + \frac{B^2}{8\pi} + p_{xx} = \rho_1 u_1^2 + \frac{B_1^2}{8\pi} \tag{7}$$

$$(\epsilon + p_{xx})u + qx + \tfrac{1}{2}\rho u^3 + \frac{B^2 u}{4\pi} = \tfrac{1}{2}\rho_1 u_1^3 + \frac{B_1^2 u_1}{4\pi} \tag{8}$$

where

$$p_{xx} = \int d^3k\, n_k \hbar k_x v_g \tag{9}$$

$$\epsilon = \int d^3k\, n_k \hbar\omega \tag{10}$$

$$qx = \int d^3k\, n_k \hbar\omega v_g \tag{11}$$

where B is to be interpreted as the average field. The contribution to the magnetic stress tensor due to the fluctuating fields of the waves is contained in p_{xx}. p_{xx} can be evaluated by summing the contributions to the magnetic stress from each of the waves. It is, however, obtained more simply in the form given in Eq.(9) by observing that pressure is a momentum flux and is therefore the product of the momentum of a quasi-particle (wave) $\hbar k$ and the velocity at which it transports energy or momentum, v_g, summed over the total number of quasi-particles. Similarly ϵ and qx are the wave energy density and energy flux relative to the fluid.

In principle we should now try to find a solution of Eqs.(4) to (8). In practice this is extremely difficult since a non-linear partial differential equation is involved. Formally the equations do not differ very much from the coupling of the Boltzmann Equation with the fluid equations which must be solved for ordinary shock waves. There is, however, one great simplification which can be used for ordinary gases, which is absent in the present case. Namely, there is an equilibrium particle distribution (Maxwellian) which can be used as a base distribution function. One can then make either a perturbation expansion about the Maxwellian distribution function, as is done in the Chapman-Enshog procedure, or make use of Maxwellians in a parametric description of the distribution function, as was done by MOTT-SMITH [13]. In the present case however the wave spectrum at thermodynamic equilibrium is much too small to produce significant effects and we have no a priori knowledge of an expected turbulent wave spectrum. Some suggestions for general features of the spectrum were made in Ref. [12]; however they did not lead to a completed argument.

Let us attempt to estimate the shock thickness for a possible solution of this set of equations by satisfying three minimum criteria [11]. First, there must be an energy source for the turbulent wave energy. This is contained in the equations, as may be seen by multiplying Eq.(4) by $\hbar\omega$ and integrating over wave number space. The wave collision term does not contribute since wave energy is conserved in wave-wave scattering. Making use of some integrations by parts, this energy moment becomes

$$\frac{d(\epsilon u + q_x)}{dx} + p_{xx}\frac{du}{dx} = 0. \tag{12}$$

The first term is the divergence of the wave energy flux and therefore the second term must be a wave energy source. Whether it is a positive or negative depends on the sign of du/dx. For a compressive flow, as in the shock, du/dx is negative corresponding to a source. This energy source may be regarded physically as simply the work done against the wave pressure in an adiabatic compression associated with the compressive flow.

Secondly, we must require that the waves have a sufficient group velocity so that they can propagate upstream against the flow. If they did not they would simply be blown downstream by the flow and leave the shock transition region. If we assume that, as in the case of weak waves, the longest wavelength which can do this will be important, we must ask for the smallest value of k which has a group velocity equal to the flow velocity ahead of the shock, in order to define a typical wave number in the wave spectrum. For waves somewhat above the ion cyclotron frequency the dispersion relation may be approximated by

$$\omega = \frac{V_A^2}{\Omega_i} k k_z \tag{13}$$

where V_A is the Alfvén speed, Ω_i the ion cyclotron frequency and k_z is the component of k parallel to the magnetic field. Setting the x-component of group velocity equal to the flow speed ahead, i.e. the shock Mach number, M, times the Alfvén speed, we obtain

$$\frac{V_A}{\Omega_i}\frac{k_x k_z}{k} = M \tag{14}$$

or since the minimum magnitude of k is achieved for $k_x = k_z$

$$k = 2M\frac{\Omega_i}{V_A}. \tag{15}$$

The third criterion which must be satisfied is that there be some randomization process to provide the entropy change required by the conservation conditions. The only randomizing process present in the equations is the wave-wave scattering term. The shock thickness must therefore be sufficient to allow for wave-wave scattering. On the other hand, the shock cannot be too many scattering lengths wide, since the wave energy must increase from an extremely small value ahead of the shock to a finite value behind. This would correspond to a very strong shock in an ordinary gas in which the temperature ratio across the shock is very large. Since this requires a large entropy charge the gradients must be very steep, i.e. of the order of the scattering length. The mean time for scattering of a wave by other waves may be evaluated from the explicit form of the wave-wave scattering term and, assuming a fairly broad wave spectrum, turns out to be of the order of

$$\sigma \approx \frac{B^2}{8\pi\epsilon}\frac{1}{\omega} \tag{16}$$

the corresponding distance a wave travels before being scattered, which we shall take as an estimate of the shock thickness, is then the time given above multiplied by the group velocity or roughly

$$L \approx \frac{B^2}{8\pi\epsilon}\frac{1}{K}. \tag{17}$$

Eq.(17) defines the shock thickness in terms of the mean wave number k which is defined in Eq.(15) and the wave energy ϵ which is determined by the conservation equations (5) to (8) $8\pi\epsilon/B^2$ is a rapidly increasing function of Mach number thus the shock thickness decreases rapidly with increasing Mach number. At a Mach number of two we find that the shock thickness is roughly

$$L \approx \frac{10\,V_A}{\Omega_i}. \tag{18}$$

EXPERIMENTS

Direct experimental evidence for the structure of collision free shock waves is at present very meagre. At the time that Refs. [11] and [12] were written we thought that they had been produced in the laboratory and had a thickness in agreement with the above estimate. There is, however, an alternate explanation of the observed structure in terms of ionization processes [14] which casts considerable doubt on interpreting the experiments as collision free shock waves.

Recent satellite experiments have, however, shown clear evidence for the existence of a collision free shock wave standing ahead of the magnetosphere. The mean free path for particle collisions in the solar wind plasma is greater than the distance to the sun, thus, if a shock wave is to exist ahead of the magnetosphere it must be collision free. Magnetometer data

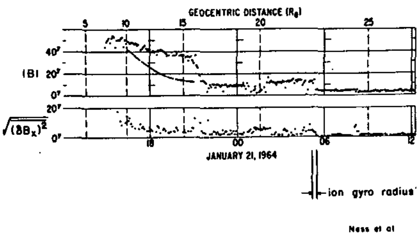

Fig. 5

IMP magnetometer data

taken by NESS [15] in the IMP satellite is shown in Fig. 5. The upper curve is the magnitude of the average magnetic field and the lower curve is a measure of the rms variations in field strength. At large distances from the earth the field has a relatively steady value. At about 23 earth radii there is a sudden jump in magnitude as well as an increase in fluctuation level. This shock transition appears almost discontinuous in magnitude of B. However the fluctuation level seems to increase over a distance of the order of the gyroradius. In data from other passes, the magnitude of B also increases in a finite distance. Behind the shock the fluctuation level appears to remain high. At about 16 earth radii the field again increases at the magnetosphere boundary.

The data from IMP has, at present, not been analysed in sufficient detail in the neighbourhood of the shock, to make clear quantitative statements about the shock thickness or dissipation mechanism. It is, however, clear that shock thicknesses, of the order of the ion gyroradius or less, occur and that significant turbulence is produced in a strong shock.

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ADVANCED KINETIC THEORY

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1. OUTLINE OF THE GENERAL THEORY

In this paper we always consider a simple model of a plasma consisting of N particles of charge e, mass m, enclosed in a cubic box of volume Ω, the average density being $C = N/\Omega$. The total charge is neutralized by a continuous background of opposite charge, which otherwise plays no role in the dynamics. The Hamiltonian of this system is therefore

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2m} + e^2 \sum_{j<n}\sum V_{jn}\left(|\vec{x}_j - \vec{x}_n|\right), \tag{1.1}$$

where

$$V_{jn} = \frac{1}{|\vec{x}_j - \vec{x}_n|} = \frac{8\pi^3}{\Omega} \sum_{\vec{l}} V_l\, e^{i\vec{l}\cdot(\vec{x}_j - \vec{x}_n)} , \tag{1.2}$$

the Fourier transform V_l being

$$V_l = \frac{1}{2\pi^2} \frac{1}{l^2} . \tag{1.3}$$

The system is described by a N-particle distribution function $f_N(\vec{x}_1 \ldots \vec{x}_N, \vec{v}_1 \ldots \vec{v}_N; t) \equiv f_N(x, v; t)$, which is normalized to one:

$$\int (d\vec{x})^N (d\vec{v})^N f_N = 1. \tag{1.4}$$

From this function, one can define reduced distribution functions $f_s(\vec{x}_1 \ldots \vec{x}_s, \vec{v}_1 \ldots \vec{v}_s; t)$ by integrating over N-s particles:

$$f_s(\vec{x}_1 \ldots \vec{x}_s, \vec{v}_1 \ldots \vec{v}_s; t) = \frac{N!}{(N-s)!} \int (d\vec{x})^{N-s} (d\vec{v})^{N-s} f_N. \tag{1.5}$$

The factor in front of the integral takes care of the fact that we are inquiring about the probability of having one particle in $\vec{x}_1$, $\vec{v}_1$, one particle in $\vec{x}_2$, $\vec{v}_2$, ... regardless of their identity. The reduced functions are the only quantities of real interest, because all relevant physical quantities (density, local velocity, two-point correlations, etc.) are expressed in terms of f_1 and f_2. Another important quantity is the one-particle velocity distribution $\varphi(\vec{v};t)$:

$$\varphi(\vec{v};t) = \frac{1}{N}\int d\vec{v}\, f_1(\vec{x},\vec{v};t). \tag{1.6}$$

Note that

$$\int d\vec{v}\varphi(\vec{v};t) = 1. \tag{1.7}$$

The N-body distribution obeys the Liouville equation

$$\mathcal{L} f_N \equiv \left[\partial_t + \sum_j \vec{v}_j \cdot \vec{\nabla}_j - \frac{e^2}{m}\sum_{j<n} (\vec{\nabla}_j V_{jn}) \cdot \vec{\partial}_{jn}\right] f_N = 0, \tag{1.8}$$

where

$$\partial_t = \partial/\partial t$$

$$\vec{\nabla}_j = \partial/\partial\vec{x}_j$$

$$\vec{\partial}_j = \partial/\partial\vec{v}_j$$

$$\vec{\partial}_{jn} = \vec{\partial}_j - \vec{\partial}_n$$

Our general method is based on a formal solution of the Liouville equation (1.8) which is a linear equation. The solution is expanded as a power series in e^2 and the various terms are analysed by means of a diagram technique. The latter enables one to choose among all terms those which are relevant for a given problem (in general, an infinite subseries of the complete perturbation series). The latter are then summed explicitly, or they are shown to satisfy a new equation, different from the original Liouville equation, which is the kinetic equation of the problem at hand. We shall outline the method here; readers interested in details are referred to the author's book [1].

The solution of Eq. (1.8) (which is linear) can be expressed in terms of the Green's function, which obeys the equation

$$\mathcal{L}\mathcal{G}(xvt\,|\,x'v't') = \delta(x-x')\,\delta(v-v')\,\delta(t-t') \tag{1.9}$$

with the causal condition

$$\mathcal{G}(xvt\,|\,x'v't') = 0 \qquad \text{for} \qquad t'>t. \tag{1.10}$$

The variables $x'v't'$ are mere parameters in Eq. (1.9). However, one can regard $\mathcal{G}$ as an operator whose matrix elements are

$$\langle xv\,|\mathcal{G}(t,t')|\,x'v'\rangle \equiv \mathcal{G}(x,v,t\,|\,x'v't'). \tag{1.11}$$

Its operation on a function is defined by

$$\mathscr{G}f \equiv \int dx\, dv\, \mathscr{G}(xvt\,|\,x'v't')f(x'v't'). \tag{1.12}$$

With this definition it is easily shown that the solution of the initial value problem for Eq. (1.8) is given by

$$f_N(t) = \mathscr{G}(t, 0)f_N(0). \tag{1.13}$$

Hence $\mathscr{G}(t, 0)$ is the operator which, acting on the initial value, transforms it into the solution at time t. It is shown that $\mathscr{G}(t, t')$ actually depends only on the difference $t - t'$ if the interaction V does not depend on time. One can then introduce the Laplace transform of the Green's function, called the resolvent operator $\mathscr{R}(z)$:

$$\mathscr{R}(z) = \int_0^\infty d\tau\, \mathscr{G}(\tau)e^{iz\tau} \tag{1.14}$$

and, feeding this back into Eq. (1.13):

$$f_N(t) = \frac{1}{2\pi}\int_C dz\, e^{-izt}\mathscr{R}(z)f_N(0), \tag{1.15}$$

C being a contour parallel to the real axis, lying above all singularities of the integrand. Eq. (1.15) is our fundamental starting point.

We now do perturbation theory. We therefore make the following decomposition:

$$\mathscr{L} = \mathscr{L}_0 + e^2\mathscr{L}'$$

$$\mathscr{L}_0 = \partial_t + \sum \vec{v}_j \cdot \vec{\nabla}_j \tag{1.16}$$

$$\mathscr{L}' \equiv \sum_{j<n} \mathscr{L}'_{jn} = \sum_{j<n} (-m^{-1})(\vec{\nabla}_j \cdot V_{jn}) \cdot \vec{\partial}_{jn}.$$

Let us also call $\mathscr{G}^0(\tau)$ the Green's function of the unperturbed Liouville equation, and $\mathscr{R}^0(z)$ the corresponding resolvent. It is then demonstrated [1] that the complete Green's function obeys the following integral equation:

$$\mathscr{G}(t) = \mathscr{G}^0(t) - e^2\int_0^t dt'\, \mathscr{G}^0(t - t')\mathscr{L}'\mathscr{G}(t') \tag{1.17}$$

and correspondingly

$$\mathscr{R}(z) = \mathscr{R}^0(z) - e^2\mathscr{R}^0(z)\mathscr{L}'\mathscr{R}(z). \tag{1.18}$$

Hence, if $\mathscr{R}^0(z)$ is known, Eq. (1.18) can be solved by successive iterations, and the result substituted into Eq. (1.15) gives the series

$$f_N(t) = \frac{1}{2\pi}\int_C dz\, e^{-izt} \sum_{n=0}^{\infty} (-e^2)^n \mathscr{R}^0(z)\,[\mathscr{L}'\mathscr{R}^0(z)]^n f_N(0). \tag{1.19}$$

This is the general solution of the Liouville equation in terms of an infinite perturbation series.

In order to make calculations with Eq. (1.19) it is convenient to work in a representation in which the operator $\mathscr{R}^0(z)$ is diagonal. It is easily seen that this is the case in Fourier representation (indeed, $\mathscr{L}^0$, Eq. (1.16), is then a purely algebraic operator). We therefore expand the distribution function as

$$f_N(x, v; t) = \sum_k e^{ikx} \rho_k(v; t) \tag{1.20}$$

(indeed, k stands for the set $\vec{k}_1 \ldots \vec{k}_N$, and $\vec{k}x$ means $\sum_j \vec{k}_j \cdot \vec{x}_j$). The matrix elements of any operator are defined as

$$\langle k|A|k'\rangle = \int dx\, e^{-ikx} A e^{ik'x}. \tag{1.21}$$

Eq. (1.19) then becomes an equation for the Fourier components:

$$\rho_k(v; t) = \frac{1}{2\pi}\sum_{n=0}^{\infty} \int dz e^{-izt}(-e^2)^n \sum_{k'} \langle k|\mathscr{R}^0(z)[\mathscr{L}'\mathscr{R}^0(z)]^n|k'\rangle \rho_{k'}(v; 0). \tag{1.22}$$

From Eq. (1.16) one obtains:

$$\langle k|\mathscr{R}^0(z)|k'\rangle = \frac{1}{i(kv - z)}\delta(k - k') \tag{1.23}$$

and, using Eqs. (1.2) and (1.16):

$$\langle k|\mathscr{L}'_{jn}|k'\rangle = \frac{8\pi^3}{\Omega}\frac{1}{m} V_{|\vec{k}'_j - \vec{k}_j|}\; i(\vec{k}'_j - \vec{k}_j)$$

$$\cdot \vec{\partial}_{jn}\; \delta(\vec{k}_j + \vec{k}_n - \vec{k}'_j - k'_n) \prod_{r \neq j, n} \delta(\vec{k}_r - \vec{k}'_r). \tag{1.24}$$

There appears a selection rule stating that at each elementary interaction, the sum of wave-vectors is conserved. We must stress that $\langle k|\mathscr{L}'|k'\rangle$ is an operator acting on functions of the velocities. The Liouville equation (1.8), transformed to Fourier space reads as:

$$\partial_t \rho_k + i\sum_j \vec{k}_j \cdot \vec{v}_j \rho_k = e^2 \sum_{k'} \langle k|\mathscr{L}'|k'\rangle \rho_{k'} . \tag{1.25}$$

The formula (1.22), when written out explicitly, contains of course an enormous number of terms, corresponding to all possible combinations of wave-vectors in the intermediate states. In order to classify these terms we introduce a graphical representation of each of the terms in this series, by using the following rules:

(a) With each matrix element of $\mathscr{R}^0(z)$: $\langle k|\mathscr{R}^0(z)|k'\rangle$ we associate a set of superposed lines; the number of lines equals the number of non-vanishing wave-vectors in the set k.

(b) Each line is labelled with an index, representing the particle associated with the corresponding wave vector.

(c) With each matrix element of $\mathscr{L}'_{jn}$: $\langle k'|\mathscr{L}'_{jn}|k\rangle$ we associate a vertex which is the concourse of the lines labelled j and n in the set k (if any) and of the lines labelled j and n in the set k' (if any).

For instance the diagram shown in Fig. 1 corresponds to the contribution to $\rho_{\vec{k}}(\vec{v}_\alpha|\ldots;t)$ from $\rho_{\vec{k}',\vec{k}'}(\vec{v}_\alpha,\vec{v}_n|\ldots;0)$ described by the following term of Eq. (1.22):

$$\rho_k(\vec{v}_\alpha|\ldots;t) = \frac{1}{2\pi}\int dz e^{-izt}(-e^2)^2 \frac{1}{i(\vec{k}\cdot\vec{v}_\alpha - z)}\langle\vec{k}|\mathscr{L}'_{\alpha j}|\vec{k}',\vec{k}-\vec{k}'\rangle$$

$$\times\frac{1}{i(\vec{k}'\cdot\vec{v}_\alpha+(\vec{k}-\vec{k}')\vec{v}_j - z)}\langle\vec{k}',\vec{k}-\vec{k}'|\mathscr{L}'_{jn}|\vec{k}',\vec{k}-\vec{k}'\rangle\frac{1}{i(\vec{k}'\vec{v}_\alpha+(\vec{k}-\vec{k}')\vec{v}_n - z)}$$

$$\times\rho_{\vec{k}',\vec{k}-\vec{k}'}(\vec{v}_\alpha\vec{v}_n)\ldots;0).$$

In this, more explicit notation, $\rho_{\vec{k}}(\vec{v}_\alpha|\ldots;t)$ means the Fourier component with one non-vanishing wave vector $\vec{k}$ corresponding to the particle α.

The diagrams are more than a pictorial representation of the various terms. One can prove a number of theorems relating the shape of the diagram (number and type of vertices, connectedness, etc.) to the order of magnitude of the corresponding term as a function of e^2, c, t, ... Hence, the practical procedure of choice in solving a given problem is as follows. One first decides, on physical grounds, the order of magnitude of the terms one wants to retain. One then draws all possible diagrams which are of that order. One finally sums the corresponding terms. We have no time to go into the details of the method here: many examples have been worked

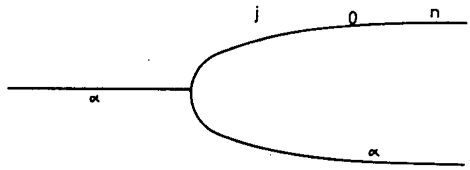

Fig. 1

Diagram corresponding to the contribution to

$\rho_{\vec{k}}(\vec{v}_\alpha|\ldots;t)$ from $\rho_{\vec{k}',\vec{k}''}(\vec{v}_\alpha,\vec{v}_n|\ldots;0)$

out in Ref. [1]. We shall treat here a few problems which will show how the method works.

2. NON-MARKOFFIAN KINETIC EQUATION FOR PLASMAS

We want an equation describing the evolution in time of the reduced one-body distribution of a plasma. Its Fourier decomposition is given by

$$f_1(\vec{x}_\alpha,\vec{v}_\alpha;t) = c[\varphi(\vec{v}_\alpha;t) + \int d\vec{k}\,\rho_{\vec{k}}(\vec{v}_\alpha;t)e^{i\vec{k}\cdot\vec{x}_\alpha}], \tag{2.1}$$

where $\rho_{\vec{k}}(\vec{v}_\alpha;t)$ is the integral of $\rho_{\vec{k}}(\vec{v}_\alpha|\ldots;t)$ over all velocities but $\vec{v}_\alpha$. If the system were homogeneous, the only term left in Eq. (2.1) would be $c\varphi(\vec{v}_\alpha;t)$, the velocity distribution. We consider first this term. We need to write down all the diagrams ending with no line at left. The most general type of diagram is shown in Fig. 2.

We represent by $\mathcal{D}$ the sum of all possible diagrams with no external line and in which no intermediate state is the "vacuum" state. Such diagrams are called diagonal fragments. $\mathcal{C}$ is the sum of all diagrams having no external line at left, but having a number of them at right, and again no intermediate vacuum state: these are the destruction fragments.

Taking advantage of this structure, we may write Eq. (1.22) for $\mathscr{R} = 0$ as follows

$$\rho_0(v;t) = \frac{1}{2\pi}\int_C dz\, e^{-izt}\frac{1}{-iz}[\rho_0(v;0) + A_0(z)\tilde{\rho}_0(v;z) + \sum_k \mathscr{D}_{0k}(z)\rho_k(v;0)]$$

$$\equiv \frac{1}{2\pi}\int_C dz\, e^{-izt}\tilde{\rho}_0(v;z). \tag{2.2}$$

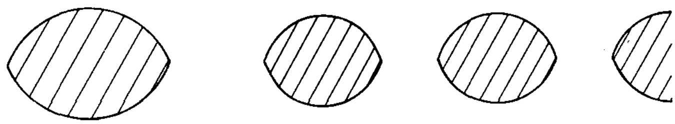

Fig. 2

Diagram of the most general type

The last equality defines the Laplace transform of $\rho_0(v; t)$. The decomposition comes out as follows:

(A) The first term is the zero'th order term.

(B) The second term is the sum of all terms beginning at left with a diagonal fragment; we define

$$A_0(z) = \sum_{\substack{n=1 \\ \text{(all diagonal} \\ \text{fragments)}}}^{\infty} \langle 0 | \mathscr{L}' [\mathscr{R}_0(z) \mathscr{L}']^n | 0 \rangle. \tag{2.3}$$

This operator acts on the Laplace transform of ρ_0 itself, as can be seen from Fig. 2.

(C) The third term represents the sum of all terms beginning at left with a destruction fragment:

$$\mathscr{D}_{0k}(z) = \sum_{\substack{n=0 \\ \text{(all destruction} \\ \text{fragments)}}}^{\infty} \langle 0 | \mathscr{L}' [\mathscr{R}_0(z) \mathscr{L}']^n | k \rangle. \tag{2.4}$$

This operator acts on the Fourier components (correlation functions) at time zero.

To get the kinetic equation, we take the time derivative of both sides of Eq. (2.2), and integrate them over all velocities but $\vec{v}_\alpha$. The result is

$$\partial_t \varphi(\alpha; t) = \int_{(\alpha)} dv \frac{1}{2\pi} \int_C dz\, e^{-izt} [A_0(z) \tilde{\rho}_0(v; z) + \sum_k \mathscr{D}_{0k}(z) \rho_k(v; 0)]. \tag{2.5}$$

This form is not yet satisfactory because of the occurrence of the Laplace transform. We eliminate it by inverting the second Eq. (2.2):

$$\partial_t \varphi(\alpha; t) = \frac{1}{2\pi} \int_{(\alpha)} dv \int_C dz\, e^{-izt} [A_0(z) \int_0^{\infty} d\tau\, e^{iz\tau} \rho_0(v; \tau)$$

$$+ \sum_k \mathscr{D}_{0k}(z) \rho_k(v; 0)] \tag{2.6}$$

or, after some algebra

$$\partial_t \varphi(\alpha; t) = \int_0^t d\tau \frac{1}{2\pi} \int_C dz\, e^{-iz\tau} \int_{(\alpha)} dv\, A_0(z) \rho_0 (v; t-\tau)$$

$$+ \frac{1}{2\pi} \int_C dz\, e^{-izt} \int_{(\alpha)} dv \sum_k \mathscr{D}_{0k}(z) \rho_k (v; 0). \qquad (2.7)$$

This is the basic equation we shall now discuss.

(a) First we note that in all problems involving many bodies, one must assume that the correlations between particles vanish for infinite separation. It is shown that this implies

$$\rho_0 (v; t) = \prod_j \varphi(\vec{v}_j ; t). \qquad (2.8)$$

Actually, if this is assumed at time zero it will be maintained at all later times. This condition closes the equation: (2.7) is therefore a highly non-linear equation.

(b) Eq. (2.7) is a typical non-Markoffian equation. In other words, the evolution of φ at time t is related to the values of φ at all earlier times, as is seen from the first term. This is in contrast with usual kinetic equations (Boltzmann, Vlasov, etc).

(c) The evolution of $\varphi(\alpha; t)$ is a functional of the correlations at the initial time. This is another aspect of the long range memory of the system.

We now note that Eq. (2.7) is still an exact equation: the only assumption there is the large size of the system. In order to specialize it for the case of a plasma to dominant order, we must make a choice among the diagrams. We have no time to go into the detailed arguments justifying this choice. Let us state that we retain all diagrams of order $e^2(e^2c)^P$, which is shown in Fig. 3.

Before showing how the diagrams can be summed let us indicate how the general equation simplifies in special situations. It must first be noted

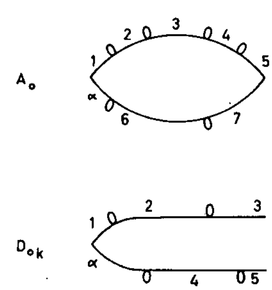

Fig. 3

Diagram of order $e^2(e^2c)^P$

that in simple cases, and in particular in the plasma limit, there appear two widely separated time scales, i. e. a short time scale, of the order of ω_p^{-1}, the inverse plasma frequency, and a long time scale of the order of the relaxation time, or the order of the time rate of change of φ. It turns out that the short time contributions come out of the singularities of $A_0(z)$, which in normal cases are located at distances of order ω_p off the real axis. We may then neglect the retardation in Eq. (2. 7) and write the Markoffian approximation to the kinetic equation as

$$\partial_t \varphi(\alpha; t) = \frac{1}{2\pi} \int_C dz \frac{e^{-izt}}{-iz} \int_{(\alpha)} dv\, A_0(z) \rho_0(v; t)$$

$$+ \frac{1}{2\pi} \int_C dz\, e^{-izt} \int dv \sum_k \mathscr{D}_{0k}(z) \rho_k(v; 0). \qquad (2.9)$$

We now can go a step further. If, as said above, all singularities of $A_0(z)$ and of $\mathscr{D}_{0k}(z)$ are located far down in the complex plane, the corresponding residues give rapidly damped terms, and can be neglected. There remains a single term to be considered, i. e. the residue of the first term in the r. h. s.* at the pole $z = 0$. Hence if we are interested in the long-time behaviour of a normal (in particular, a stable) plasma, Eq. (2. 9) simplifies to

$$\partial_t \varphi(\alpha; t) = \int_{(\alpha)} dv\, A_0(0) \rho_0(v; t). \qquad (2.10)$$

This is the usual kinetic equation of plasmas, valid for long times. It has now reduced to a Markoffian, non-linear equation in which the memory of the initial correlation has disappeared.

Let us now outline schematically how the ring diagrams can be summed. We stress the fact that ring diagrams are among the very few examples in physics in which a perturbation series can be summed in closed form.

We first note that all ring diagrams begin at left with the same vertex: $<$. We therefore re-write the first term in the r. h. s. of Eq. (2. 7) in the following form (the destruction term can be handled in a similar way and will not be considered here).

$$\partial_t \varphi(\alpha) = \int d\vec{l}\, d_\alpha F_{\vec{l}}(\vec{v}_\alpha; t), \qquad (2.11)$$

where $d_\alpha = -8\pi^3 e^2 c\, m^{-1} V_l\, i\vec{l} \cdot \vec{\partial}_\alpha$ is essentially the matrix element corresponding to $<$, and $F_{\vec{l}}(\vec{v}_\alpha; t)$ is the remainder of the series. Note that all velocity integrations have been included in $F_{\vec{l}}$, which is therefore a function of $\vec{v}_\alpha$,

* Right hand side.

and t. The function $F_{\vec{l}}(\vec{v}_\alpha; t)$ is written in the form

$$F_{\vec{l}}(\vec{v}_\alpha; t) = \frac{1}{2\pi} \int_0^t d\tau \int_C dw\, e^{-ilw\tau} F_{\vec{l}}(\vec{v}_\alpha; w) \tag{2.12}$$

(we took the new Laplace variable $w = \frac{z}{l}$ for convenience). $F_{\vec{l}}(\vec{v}; w)$ is a function of $\vec{v}$, w and $t - \tau$, through the distribution function $\varphi(\vec{v}; t - \tau)$. The latter parameter $t - \tau$ will not be written explicitly. An examination of the series of diagrams shows that the sum $F_{\vec{l}}(\vec{v}; w)$ obeys a recurrence relation which is actually an integral equation:

$$\epsilon_{\vec{l}}^-(\nu - w) F_{\vec{l}}(\vec{v}; w) = d_{\vec{l}}(\vec{v}) \int d\vec{v}_1 \frac{F_{\vec{l}}(\vec{v}_1; w)}{l(\nu - w) - \vec{l}\cdot\vec{v}_1} + q_{\vec{l}}(\vec{v}; w), \tag{2.13}$$

where $\nu = \frac{\vec{l}\cdot\vec{v}}{l}$. $\epsilon_{\vec{l}}^-(\omega)$ is the dielectric constant of the plasma which is regular in the lower half-plane of w. $d_{\vec{l}}(\vec{v})$ and $q_{\vec{l}}(\vec{v}; w)$ are some functionals of $\varphi(\vec{v}; t - \tau)$ which we shall not specify. Noting now that the kernel depends only on the component parallel to l of the velocities, we can multiply both sides of the equation by $\delta(\nu - \frac{\vec{l}\cdot\vec{v}}{l})$ and integrate over $\vec{v}$. Calling

$$\bar{f}_{\vec{l}}(\nu) = \int d\vec{v}\, \delta(\nu - \frac{\vec{l}\cdot\vec{v}}{v})\, f_{\vec{l}}(\vec{v})$$

we obtain an equation for the barred functions:

$$\epsilon_{\vec{l}}^-(\nu - w) \bar{F}_{\vec{l}}(\nu; w) = \frac{\bar{d}_{\vec{l}}(\nu)}{l} \int_{-\infty}^{\infty} d\nu_1 \frac{\bar{F}_{\vec{l}}(\nu_1; w)}{\nu - w - \nu_1} + \bar{q}_{\vec{l}}(\nu; w). \tag{2.14}$$

This equation is written for complex w. Remember w must be on a parallel to the real axis above all singularities of $F_{\vec{l}}(\nu; w)$. As it stands, this equation cannot be solved in closed form by any standard method. However, if we are allowed to move w down to the real axis, the kernel $(\nu - w - \nu_1)^{-1}$ becomes

$$\pi i\, \delta_-(\nu - w - \nu_i) \equiv P \frac{1}{\nu - w - \nu_i} + \pi i\, \delta(\nu - w - \nu_i) \tag{2.15}$$

and the equation becomes of a type known as a Cauchy singular integral equation. There exists an extensive literature (see for instance Ref. [2]) treating these equations which can be solved in closed form.

It turns out that, if the proviso is satisfied, the solution of Eq. (2.13) is of the form:

$$E_{\vec{l}}(\vec{v};w) = \frac{q_{\vec{l}}(\vec{v};w)}{\epsilon^{-}_{\vec{l}}(\nu - w)} + \frac{\pi i}{l} d_{\vec{l}}(\vec{v}) \int_{-\infty}^{+\infty} d\nu_1\, \delta_{-}(\nu - \nu_1) \frac{\bar{Q}_1(\nu_1; w)}{\epsilon^{+}(\nu_1)\epsilon^{-}(\nu_1 - w)}, \qquad (2.16)$$

$\bar{Q}_1$ being a functional of $\varphi(\vec{v}; t - \tau)$. We must now discuss whether our initial assumption (about the possibility to move w down to the real axis) is satisfied. This depends on the behaviour of the integral in Eq. (2. 14). The whole point can be seen from the first term of the solution in Eq. (2. 16). The corresponding term in the integrand of Eq. (2. 14) is

$$\frac{\bar{q}_{-\vec{l}}(-\nu_1; w)}{(\nu - w - \nu_1)\,\epsilon^{-}_{-\vec{l}}(-\nu - w)} = \frac{\bar{q}_{-\vec{l}}(-\nu_1; w)}{\epsilon^{+}_{\vec{l}}(\nu + w)(\nu - w - \nu_1)},$$

where $\epsilon^{+}_{\vec{l}}(\omega)$ is the "ordinary" dielectric constant, regular in the upper half-plane. This function has a certain number of zeros; let ξ_+ be the zero closest to the real axis; we can therefore write $\epsilon^{+}(\omega) = \sigma^{+}(\omega)(\omega - \xi_+)$. The integral is therefore of the form

$$\int d\nu_1 \frac{f(\nu_1; w)}{(\nu - w - \nu_1)(\nu + w + \xi_+)}.$$

This function has two cuts: one on the real axis, Im w = 0, and one on the line, Im w = Im ξ_+. Two cases arise:

(i) The plasma is stable. Then all the zeros of the dielectric constant lie in the lower half-plane: Im $\xi_+ < 0$. In this case, in letting w go to the real axis, we cross no cut, the singular integral equation is the proper analytic continuation of the original equation, and hence Eq. (2. 16) is a solution of Eq. (2. 13). Substituting Eq. (2. 16) into Eqs. (2. 12) and (2. 11) (and adding the destruction fragment) we obtain the general non-Markoffian "ring equation" which describes the evolution of a homogeneous plasma for arbitrarily short times. This equation reduces in the Markoffian long-time limit to the well-known ring equation:

$$\partial_t \varphi(\vec{v}_\alpha; t) = \frac{8\pi^4 e^4 c}{m^2} \int d\vec{l} \int d\vec{v}_1 \vec{l}\cdot\vec{\partial}_\alpha V_l^2 \frac{1}{|\epsilon^{+}_{\vec{l}}(v_\alpha)|^2}$$

$$\cdot\, \delta(\vec{l}\cdot\vec{v}_\alpha - \vec{l}\cdot\vec{v}_1)\vec{l}\cdot\vec{\partial}_{\alpha 1}\varphi(\vec{v}_\alpha; t)\varphi(\vec{v}_1; t) \equiv \mathscr{C}\{\varphi\}. \qquad (2.17)$$

(ii) The plasma is unstable. In this case there is one zero (at least) in the upper half-plane: Im $\xi_+ > 0$. Hence, in letting w go to the real axis we have crossed a cut, and therefore Eq. (2. 16) is no longer a solution of the original equation (2. 13).

However, we can still obtain the solution in this case by performing the proper analytical continuations. Starting from the stable solution (Eq. (2. 16)), we let ξ_+ move into the upper half-plane: this involves an adequate deformation of the contour of integration in the second term on the r.h.s of that equation. In this way we obtain a non-Markoffian kinetic equation valid for unstable plasmas.

If we assume that the instability is "weak", i.e. ξ_+ is sufficiently close to the real axis, we may again make a Markoffian approximation and obtain a long-time kinetic equation. The only difference with the general treatment outlined above is that in transforming Eq. (2.9) we must include the residue at the unstable pole $w = \xi_+$ as well as the residue in $w = 0$. The result is a kinetic equation of the general form

$$\partial_t \varphi(\vec{v}; t) = \mathscr{C}\{\varphi\} + \mathscr{I}\{\varphi\}, \tag{2.18}$$

where $\mathscr{C}\{\varphi\}$ is the normal collision term defined in Eq. (2.17) and $\mathscr{I}\{\varphi\}$ is another functional of φ, of the general form:

$$\mathscr{I}\{\varphi\} \approx \int d\vec{l}\, V_l^2 \vec{l}\cdot\vec{\partial}_\alpha D(t) \vec{l}\cdot\vec{\partial}_\alpha \varphi(\vec{v}). \tag{2.19}$$

Hence $\mathscr{I}\{\varphi\}$ is of the general form of a Fokker-Planck equation with a time-dependent diffusion coefficient. Without giving the detailed form of the latter, we retain the property

$$D(t) \sim e^{2i\gamma_0 t}, \qquad \gamma_0 = \operatorname{Im} \xi_+ .$$

The effect of $\mathscr{I}\{\varphi\}$ is a stabilization of the plasma. Starting with a two-humped distribution, the stabilization proceeds through friction – which brings the two maxima closer together – and diffusion – which broadens the humps. The stabilization mechanism is very efficient, as can be seen from the exponential time dependence of $D(t)$: the more unstable the plasma, the more important is the friction and the diffusion.

Before finishing, let us just mention that the analysis of the "inhomogeneity factor" $\rho_{\vec{k}}(\vec{v}; t)$ in Eq. (2.1) can be done along the same lines, leading to a kinetic equation for this function. The calculations have been done by BALESCU and KUSZELL [3]. Here too, for small amplitude inhomogeneities, an exact kinetic equation can be derived, as well for stable as for unstable plasmas.

3. THE DIELECTRIC CONSTANT AND THE THEORY OF BROWNIAN MOTION

Much stress has been laid in recent years on the so-called "dielectric formulation" of the many-body problem. The start of this development has been given by NOZIÈRES and PINES [4] who showed that the ground state energy of a quantum mechanical electron-gas can be expressed in terms of the dielectric constant of the system. This result has later been generalized and it has been shown that the free energy of a system in equilibrium can be expressed in terms of the dielectric constant (for a review of these results see Ref. [5]). It is tempting to investigate the role of the dielectric constant for systems out of equilibrium. It was shown by NOZIÈRES and PINES [4] that the stopping power of an electron gas can be expressed in terms of the dielectric constant in the Born approximation.

In a recent work BALESCU and SOULET [6] have shown that the concept of a dielectric constant plays a central role in the theory of the Brownian motion. Let us first describe the problem. The general Markoffian equation for long times, Eq. (2.10), reduces to a particularly simple form if we consider the system to consist of a test particle moving through a medium which is in equilibrium. In that case

$$\rho_0(v;t) = \psi(\vec{v}_T;t) \prod_{s=1}^{N} \varphi^0(v_s), \tag{3.1}$$

where $\psi(\vec{v}_T;t)$ is the velocity distribution of the test particle, and $\varphi^0(v_s)$ is the equilibrium distribution of the "field" particles. If moreover it is assumed that the coupling between test particle and system is small, then the leading diagonal fragments will contain only two vertices involving the test particle (there cannot be less than two!). Hence the equation resulting from Eq.(2.10) under these circumstances is a second-order differential equation in $\psi(\vec{v}_T;t)$ which can be written in the form:

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial\vec{v}_T}\left\{-\frac{\langle\Delta\vec{v}\rangle}{\Delta t} + \frac{1}{2}\frac{\partial}{\partial\vec{v}_T}\frac{\langle\Delta\vec{v}\,\Delta\vec{v}\rangle}{\Delta t}\right\}\psi(\vec{v}_T;t). \tag{3.2}$$

This is the classical Fokker-Planck equation governing the theory of Brownian motion. It should be stressed that no assumption has been made on the density or interactions of the field particles. Our purpose is the calculation of the two coefficients $\langle\Delta\vec{v}\rangle/\Delta t$ and $\langle\Delta\vec{v}\Delta\vec{v}\rangle/\Delta t$ which characterize the Brownian motion*.

Independently of the previous discussion, we want to define the dielectric constant of a plasma from statistical mechanics. Suppose first that, in a phenomenological theory, we consider an external charge density of the form

$$e_T\rho_{ex}(\vec{x},t) \sim e_T r_{\vec{k}} e^{i\vec{k}\cdot\vec{x} - i\omega t}. \tag{3.3}$$

We want to calculate the fields $\vec{E}(\vec{x};t)$, $\vec{D}(\vec{x};t)$ inside the plasma. The Poisson equation, after Fourier-Laplace transformation, can be written in either of the two forms

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* We mention at this point that there is another circumstance in which Eq.(2.10) reduces to Eq. (3.2) (of course, with different coefficients): it is the case of a test particle, arbitrarily strongly coupled to the medium, but having a mass much larger than the mass of the field particles. All the results discussed below can be generalized to cover this case as well (see Ref. [6]).

$$i\vec{k}\cdot\vec{D}_{\vec{k}}(\omega) = 4\pi\, e_T r_{\vec{k}} e^{-i\omega t}$$
$$i\vec{k}\cdot\vec{E}_{\vec{k}}(\omega) = 4\pi\, [e_T r_{\vec{k}} e^{-i\omega t} + e\, h_{\vec{k}}^{ind}(\omega)], \tag{3.4}$$

where $e h_{\vec{k}}^{ind}(\omega)$ is the induced charge density. We now assume:

$$\vec{D}_{\vec{k}}(\omega) = \epsilon_{\vec{k}}(\omega)\vec{E}_{\vec{k}}(\omega) \tag{3.5}$$

and deduce the value of the dielectric constant $\epsilon_{\vec{k}}(\omega)$ as:

$$\frac{1}{\epsilon_{\vec{k}}(\omega)} - 1 = \frac{e h_{\vec{k}}^{ind}(\omega)}{e_T r_{\vec{k}} e^{-i\omega t}}. \tag{3.6}$$

If it turns out that $e h_{\vec{k}}^{ind}(\omega) \sim e_T r_{\vec{k}} e^{-i\omega t}$, the resulting dielectric constant is independent of the test particle and represents an intrinsic property of the medium.

Let us now calculate the quantity $h_{\vec{k}}^{ind}$ from a microscopic theory. Our system is a plasma of N electrons plus an additional test particle, of charge e_T, assumed to be small: $e_T/e \ll 1$. The Hamiltonian is therefore

$$H = H_0 + e\, e_T H_1$$
$$H_0 = \sum_{s=1}^{N} \frac{1}{2} m v_s^2 + \frac{e^2}{2} \sum_{s \neq s'} V_{ss'} + \frac{1}{2} m_T v_T^2 \tag{3.7}$$
$$H' = \sum_{s=1}^{N} V_{Ts}.$$

Note that H_0 contains now all the interactions among the field particles. The total charge density $e h_{\vec{k}}$ is defined as e times the integral over all velocities of the sum of the inhomogeneity factors corresponding to all particles:

$$e h_{\vec{k}} = \int (dv_s)^N d\vec{v}_T [e \sum_{\sigma=1}^{N} \rho_{\vec{k}}(v_\sigma | \ldots ; t) + e_T \rho_{\vec{k}}(\vec{v}_T | \ldots ; t)]. \tag{3.8}$$

We want to calculate this quantity as a function of t, starting at time zero with an appropriate initital condition. The latter is chosen in such a way that the test particle is not correlated to the medium, and that the test particle distribution is compatible with the assumed form (3.3)

$$\rho_{k_s \vec{k}_T}(v_s, \vec{v}_T; 0) = \rho_{k_s}(v_s; 0) r_{\vec{k}}\, \delta(\vec{k} - \vec{k}_T)\delta(\vec{k}\cdot\vec{v}_T - \omega)\, \chi(\vec{v}_{T\perp}). \tag{3.9}$$

$\chi(\vec{v}_{T\perp})$ is an arbitrary function of the components of $\vec{v}_T$ perpendicular to $\vec{k}_1$, normalized to unity; $\rho_{k_s}(v_s; 0)$ are the initial Fourier components of the system; it is assumed that the latter is homogeneous: $\Sigma\, k_s = 0$.

Applying the previous formulae we obtain

$$\rho_{\vec{k}}(j|\ldots;t) = \sum_{l_s} \langle \vec{k}_{(j)} |\mathscr{G}(t)| l_s; \vec{l}_T = \vec{k}\rangle \rho_{l_s}(v_s;0) r_{\vec{k}} \delta(\vec{k}\cdot\vec{v}_T - \omega) X, \tag{3.10}$$

$\langle \vec{k}_{(j)}|$ being the state with a single non-vanishing wave vector, equal to $\vec{k}$, for particle j.

As the test particle is assumed to be weakly coupled to the system, we want the response only to first order in the perturbation. Hence

$$\mathscr{G}(t) = \mathscr{G}^0(t) - e\, e_T \int_0^t dt_1 \mathscr{G}^0(t - t_1) \mathscr{L}^1 \mathscr{G}^0(t_1). \tag{3.11}$$

We now note that in the unperturbed system, the test particle is not coupled to the plasma; one can then show that

$$\langle l_s; \vec{l}_T |\mathscr{G}^0(t)| l_s'; \vec{l}_T' \rangle = \langle l_s |\mathscr{G}^s(t')| l_s' \rangle e^{-i\vec{l}_T \cdot \vec{v}_T t} \delta_{\vec{l}_T' - \vec{l}_T} \delta(\vec{v}_T' - \vec{v}_T), \tag{3.12}$$

where $\mathscr{G}^s$ is the Green's function of the medium (in absence of the test particle), i. e.

$$\rho_{k_s}(v_s;t) = \sum_{k's} \langle k_s |\mathscr{G}^s(t)| k_s' \rangle \rho_{k's}(v_s;0). \tag{3.13}$$

Substituting into Eq. (3. 8) we obtain, after some algebra:

$$e h_{\vec{k}}(t) = e_T r_{\vec{k}} e^{-i\omega t} - e^2 e_T r_{\vec{k}} \int_0^{t_0} dt_1 \int dv_s$$

$$\times \sum_{i,j=1}^{N} \sum_{l_s} \langle \vec{k}_{(i)} |\mathscr{G}^s(t - t_1)| \{l_s\}', \vec{l}_j + \vec{k} \rangle X_j e^{-i\omega t} \rho_{l_s}(v_s;t), \tag{3.14}$$

where

$$X_j = \frac{8\pi^3}{\Omega} V_k \frac{1}{m} (-i\vec{k}\cdot\vec{\partial}_j) \tag{3.15}$$

is the matrix element $\mathcal{L}_{jT}^{\prime}$ (actually there is an operator $\vec{\partial}_j - \vec{\partial}_T$ in Eq. (3.15), but the second term gives a vanishing contribution after the integration over $\vec{v}_T$). Expressing now the Green's function in terms of the resolvent yields:

$$eh_{\vec{k}}(t) = e_T r_{\vec{k}} e^{-i\omega t} - e^2 e_T r_{\vec{k}} e^{-i\omega t} \int_0^t dt_1 \frac{1}{2\pi} \int_C dz\, e^{-iz(t-t_1)}$$

$$\times \int dv_s \sum_{i,j} \sum_{l_s} \langle \vec{k}_{(i)} | \mathscr{R}^s(z) | \{l_s\}', \vec{l}_j + \vec{k} \rangle X_j e^{i\omega(t-t_1)} \rho_{l_s}(v_s; t_1). \qquad (3.16)$$

We now note that the induced charge density can be identified as

$$eh_{\vec{k}}^{ind} = eh_{\vec{k}} - e_T r_{\vec{k}} e^{-i\omega t}$$

and that this quantity is indeed proportional to $e_T r_k e^{-i\omega t}$, as expected. Hence, the dielectric constant is

$$\frac{1}{\epsilon_{\vec{k}}(\omega)} - 1 = -e^2 \frac{1}{2\pi} \int_0^t d\tau \int_C dz\, e^{-i(z-\omega)\tau}$$

$$\times \int dv_s \sum_{i,j} \sum_{l_s} \langle \vec{k}_{(i)} | \mathscr{R}^s(z) | \{l_s\}', \vec{l}_j + \vec{k} \rangle X_j \rho_{l_s}(v_s; t-\tau). \qquad (3.17)$$

This is the most general formula for the dielectric constant of a plasma. We note that no assumption has been made on the density, intensity of the interactions, etc. Moreover the system has not been assumed to be in equilibrium; therefore the dielectric constant is still a function of time. We note again here the non-Markoffian character of the expression: the dielectric constant depends on the whole history of the system.

If we now assume that the field particles are in equilibrium, $\rho_{l_s} = \rho_{l_s}^0(v_s)$ is independent of time and

$$\frac{1}{\epsilon_k^0(\omega)} - 1 = -\frac{e^2}{2\pi} \int_C dz \frac{e^{-i(z-\omega)t}}{-i(z-\omega)}$$

$$\times \int dv_s \sum_{i,j} \sum_{l_s} \langle \vec{k}_{(i)} | \mathscr{R}^s(z) | \{l_s\}', \vec{l}_j + \vec{k} \rangle X_j \rho_{l_s}^0(v_s).$$

If, moreover, we wait a sufficient time for the system to reach a steady state, i.e. for the transient contributions to die out, we obtain

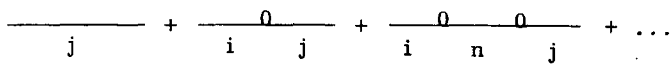
$$\frac{1}{\epsilon_{\vec{k}}^0(\omega)} - 1 = -e^2 \int dv_s \sum_{i,j} \sum_{l_s} \langle \vec{k}_{(i)} | \mathscr{R}^s(\omega) | \{l_s\}_2', \vec{l}_j + \vec{k} \rangle X_j \rho_{l_s}^0(v_s). \qquad (3.18)$$

This is the dielectric constant involved in usual situations in electrodynamics. To make contact with well-known formulae, we note that a simple case contained in this formula obtains if we take

$$\rho^0_{1_s}(v_s) = 0 \qquad \text{for } 1_s \neq 0$$

$$\rho^0_0(v_s) \equiv \pi\,\varphi^0(\vec{v}_s),$$

(i. e. we neglect the equilibrium correlations). Moreover, we retain only the following diagrams for the resolvent:



The summation of these diagrams is very simple (it is just a geometrical progression), and the result is the Vlassov dielectric constant:

$$\epsilon^V_{\vec{k}}(\omega) = 1 - \frac{4\pi e^2 c}{mk^2}\pi i \int d\vec{v}\,\vec{k}\cdot\vec{\partial}\,\varphi^0(v)\,\delta_-(\vec{k}\cdot\vec{v} - \omega).$$

Coming back now to the Brownian motion theory, we shall only quote the results without proof. These results are obtained by comparing the diagrams for the dielectric constant and those for the kinetic equation. The proofs which are straightforward but somewhat lengthy can be found in Ref. [6].

The diffusion tensor in the Fokker-Planck equation is given by

$$\frac{\langle \Delta\vec{v}\Delta\vec{v}\rangle}{\Delta t} = -\frac{2e\,e_T}{\beta m_T^2}\int d\vec{k}\,\frac{\vec{k}\vec{k}}{\vec{k}\cdot\vec{v}_T}\,V_k\,\mathrm{Im}\,\frac{1}{\epsilon^0_{\vec{k}}(\vec{k}\cdot\vec{v}_T)}, \tag{3.19}$$

and the friction coefficient:

$$\frac{\langle \Delta\vec{v}\rangle}{\Delta t} = \frac{e\,e_T}{m_T}\int d\vec{k}\,V_k\,\vec{k}\,\mathrm{Im}\frac{1}{\epsilon^0_{\vec{k}}(\vec{k}\cdot\vec{v}_T)} + \frac{1}{2}\frac{\partial}{\partial\vec{v}_T}\cdot\frac{\langle\Delta\vec{v}\Delta\vec{v}\rangle}{\Delta t}. \tag{3.20}$$

Let us stress once more that, although similar formulae have been known earlier, the present ones are valid quite generally, whatever the nature of the medium. Such formulae are useful because they lend themselves to approximation methods which could afford new approaches to the study of Brownian motion and of transport coefficients.

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TURBULENCE IN HYDRODYNAMICS AND PLASMA PHYSICS

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1. INTRODUCTION

The problem of turbulence is interesting in its own right since turbulence is a widespread and important physical state. But it is also interesting in that, complicated as it is, it represents perhaps the simplest non-linear field problem whose coupling constant, the Reynolds number, is readily varied in a laboratory, right up to infinite coupling. The other non-linear field problems, the quantum many-body problem, liquids, quantum field theory and plasma physics, are all systems of greater complexity than straight homogeneous isotropic incompressible turbulence. Perhaps plasma physics, in which the short-range forces are of little physical interest and consequence, is the next in order of difficulty even though it is so rich in physical phenomena. One can thus expect that a reliable theory of turbulence will prove of value in understanding the more awkward and extreme situations in plasma physics, and these in their turn will help towards a reliable theory of the thermodynamics and transport properties of liquids in general. In the first part of this paper I shall consider turbulence, extending the discussion to m.h.d. turbulence with and without external fields, and finally discuss how to set up general transport equations for plasmas which have validity outside the range considered to date.

2. TURBULENCE IN INCOMPRESSIBLE HYDRODYNAMICS

An incompressible fluid of unit density with viscosity ν satisfies the Navier Stokes equations

$$\frac{\partial \vec{U}}{\partial t} = \nu \nabla^2 \vec{U} - (\vec{U} \cdot \vec{\nabla}) \vec{U} - \vec{\nabla}_p + \vec{\mathscr{F}} \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{U} = 0, \tag{2.2}$$

where an external force $\vec{\mathscr{F}}$ has been added to create motion in the fluid. Introduce the three- and four-dimensional Fourier transform

$$\vec{U}_{\vec{k}}(t) = \int \vec{U}(\vec{r}, t) \exp i \vec{k} \cdot \vec{r} \, d^3 r \tag{2.3}$$

and

$$\vec{U}_k = \int \vec{U}(\vec{r}, t) \exp i(\vec{k}\cdot\vec{r} + k_0 t) d^3 r dt, \tag{2.4}$$

so that eliminating the pressure

$$\frac{\partial U^{\alpha}_{\vec{k}}}{\partial t} = -\nu \vec{k}^2 U^{\alpha}_{\vec{k}} + \int M^{\alpha\beta\gamma}_{\vec{k}\vec{j}\vec{l}} U^{\beta}_{j} U^{\gamma}_{l} d^3_j\, d^3 l + \mathscr{F}^{\beta}_{\vec{k}} \mathscr{D}^{\alpha\beta}_{k}, \tag{2.5}$$

where

$$M^{\alpha\beta\gamma}_{-\vec{k}\vec{j}\vec{l}} = \frac{1}{(2\pi)^3 i} \delta(\vec{k} + \vec{j} + \vec{l}) (k^{\beta} \mathscr{D}^{\alpha\gamma}_{\vec{k}} + k^{\gamma} \mathscr{D}^{\alpha\beta}_{k}), \tag{2.6}$$

and

$$-(ik_0 + \nu\vec{k}^2) U^{\alpha}_{k} + \int M^{\alpha\beta\gamma}_{-kjl} U^{\beta}_{j} U^{\gamma}_{l} d^4 j d^4 l + \mathscr{F}^{\beta}_{k} \mathscr{D}^{\alpha\beta}_{\vec{k}} = 0, \tag{2.7}$$

where

$$M^{\alpha\beta\gamma}_{-kjl} = \frac{1}{i(2\pi)^3} \delta(k + j + l) (k^{\beta} \mathscr{D}^{\alpha\gamma}_{\vec{k}} + k^{\gamma} \mathscr{D}^{\alpha\beta}_{\vec{k}}) \tag{2.8}$$

and

$$\mathscr{D}^{\alpha\beta}_{\vec{k}} = (\delta^{\alpha\beta} - k^{\alpha} k^{\beta} / k^2). \tag{2.9}$$

(Throughout $\vec{k}$ will mean three vector, k four vector).

Now suppose the external force fluctuates in space and time, representing say a shaking grid, and the probability distribution of $\vec{\mathscr{F}}$ is known. It will put energy into the system at a certain rate and preferentially into certain wave numbers. It will then diffuse into other wave numbers, eventually being destroyed by the viscous term when it reaches a large $\vec{k}$ region. A steady state can thus be set up, and energy will enter and leave the system at a constant rate E say, while the energy content will be fixed at E say. Now

$$E = \frac{1}{2} \int U^2(r) d^3 r \tag{2.10}$$

$$= \frac{1}{2} \frac{1}{(2\pi)^3} \int U^{\alpha}_{\vec{k}} U^{\alpha}_{-\vec{k}} d^3 k, \tag{2.11}$$

so that if

$$\langle U^{\alpha}_{\vec{k}} U^{\beta}_{-\vec{k}} \rangle = q_{\vec{k}} \mathscr{D}^{\alpha\beta}_{\vec{k}}, \tag{2.12}$$

then

$$E = \frac{1}{(2\pi)^3} \int q_{\vec{k}} d^3 k. \tag{2.13}$$

From (2.5) by multiplying by $\vec{U}_{-\vec{k}}$ and averaging over $\vec{\mathscr{F}}$ one finds

$$\frac{1}{2}\frac{\partial q_{\vec{k}}}{\partial t} + \nu |\vec{k}|^2 q_k + \langle \int M^{\alpha\beta\gamma}_{-kjl} U^{\beta}_j U^{\gamma}_l U^{\alpha}_{-k} \rangle$$

$$- \langle \mathscr{F}^{\beta}_{\vec{k}} \mathscr{D}^{\alpha\beta}_{\vec{k}} U^{\alpha}_{-\vec{k}} \rangle = 0. \tag{2.14}$$

If $\overleftarrow{\mathscr{F}}$ fluctuates rapidly so that it has a gaussian distribution with

$$\langle \mathscr{F}^{\alpha}_{\vec{k}}(t) \mathscr{F}^{\beta}_{-\vec{k}}(t') \rangle = 2 h_{\vec{k}}\, \delta(t-t')\, \delta^{\alpha\beta} \tag{2.15}$$

it is easily shown that

$$\frac{1}{2}\frac{\partial q_k}{\partial t} + \nu |\vec{k}|^2 q_k + \langle MUUU \rangle = h_{\vec{k}}. \tag{2.16}$$

Clearly the equation says: (Change of energy) + (loss of viscosity) + (transfer in and out) = (gain from outside). (2.17)
Later a rigorous development will be given, but for the moment consider the analogy with the probability f of finding a particle of velocity v in the Boltzmann equation, when one adds a sink of particles proportional to f and a source. The normalization integral is like (2.13)

$$N = \int f(\vec{v})\, d^3 v \tag{2.18}$$

and the Boltzmann equation

$$\frac{\partial f}{\partial t} + \nu f + \int \sigma(\vec{v}\vec{v}'\vec{v}_1\vec{v}_1')W\,(f(\vec{v})f(\vec{v}_1) - f(\vec{v}')f(\vec{v}_1'))\, d(\text{all}) = h, \tag{2.19}$$

i.e.

(change in number) + (loss to sink) + (transfer in and out of v') = gain from outside. (2.20)

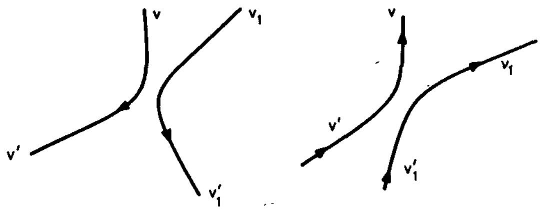

Fig. 1

Processes where $v + v_1 = v' + v_1'$

That the collision term has the form it has is clear from the equations of motion. In (2.5) one sees that "$\vec{k}$" interacts with "$\vec{j}$" and "$\vec{l}$" with $\vec{j}+\vec{l}=\vec{k}$ and the two processes shown in Fig. 1, where

$$\vec{v}+\vec{v}_1=\vec{v}'+\vec{v}_1' \tag{2.21}$$

will have analogues (Fig. 2) so one can guess that the transfer term will look like

$$\int \Lambda_{\vec{k}\vec{j}\vec{l}}\,(q_{\vec{k}}q_{\vec{l}}-q_{\vec{j}}q_{\vec{l}})\,d^3\vec{j}\,d^3\vec{l}, \tag{2.22}$$

Λ containing $\delta(\vec{j}+\vec{l}-\vec{k})$ just as σ contains $\delta(\vec{v}+\vec{v}_1-\vec{v}'-\vec{v}_1')$. To complete the analogy one can look at the quantum form of the cross section σ which, if the interparticle potential is ϕ, is

$$|\phi(\vec{v}+\vec{v}_1-\vec{v}'-\vec{v}_1')|^2\,\delta(E+E_1-E'-E_1')$$

$$=\mathrm{Im}\left[\frac{|\phi(\vec{v}+\vec{v}_1-\vec{v}'-\vec{v}_1')|^2}{\pi(E+E_1-E'-E_1'+i\epsilon)}\right]\quad (E'=\tfrac{1}{2}m\vec{v}'^2 \text{ etc}). \tag{2.23}$$

Readers familiar with second quantization will see in Fig. 2 the "vertex" of a field with equation of motion (2.5), and will recognize M for ϕ and $i\nu k^2$ for E etc. So if the "cross section" is calculated in perturbation theory it can be expected to give

$$\tfrac{1}{2}\frac{\partial q_k}{\partial t}+\nu|\vec{k}|^2 q_k+\int\frac{L_{\vec{k}\vec{l}\vec{j}}\,q_{\vec{l}}(q_{\vec{j}}-q_{\vec{k}})}{(\nu|\vec{k}|^2+\nu|\vec{j}|^2+\nu|\vec{l}|^2)}\,d^3j\,d^3l=h_{\vec{k}} \tag{2.24}$$

where

$$L_{\vec{k}\vec{j}\vec{l}}=\tfrac{1}{2}\,\delta(\vec{k}+\vec{j}+\vec{l})\,M^{\alpha\beta\gamma}_{-kjl}M^{\alpha\beta'\gamma'}_{k-j-l}\,(\mathscr{D}^{\beta\beta'}_{j}\mathscr{D}^{\gamma\gamma'}_{l}+\mathscr{D}^{\beta\gamma'}_{j}\mathscr{D}^{\gamma\beta'}_{l}) \tag{2.25}$$

$$=\delta(\vec{k}+\vec{j}+\vec{l})\,\frac{(k^4+2k^3 j\cos\theta-kj^3\cos\theta)\sin^2\theta}{(k^2+2kj\cos\theta+j^2)} \tag{2.26}$$

$$(\cos\theta=\vec{k}\cdot\vec{j}/|\vec{k}|\,|\vec{j}|)$$

$$=L_{\vec{j}\vec{l}\vec{k}}=L_{\vec{k}\vec{l}\vec{j}}\,. \tag{2.27}$$

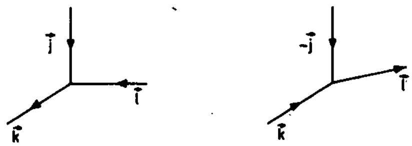

Fig. 2

$Q_k(t)$ as a function of t

(Note that since the viscosity is irreversible in time +only signs appear in the denominator.) To get away from perturbation theory one can argue that some more adequate time scale than the perturbation value $(\nu|\vec{k}|^2+\nu|\vec{j}|^2+\nu|\vec{l}|^2)^{-1}$ should be used; for example some $\omega^{-1}=(\omega_{\vec{k}}+\omega_{\vec{j}}+\omega_{\vec{l}})^{-1}$ say. Finally then the transport equation is

$$\frac{1}{2}\frac{\partial q_k}{\partial t}+\nu|\vec{k}|^2\, q_{\vec{k}}+\int \Lambda_{\vec{k}\vec{j}\vec{l}}\, q_{\vec{l}}(q_{\vec{k}}-q_{\vec{j}})d^3j\, d^3l=h_{\vec{k}} \tag{2.28}$$

having an equilibrium solution, the solution of (2.28) with $\partial q/\partial t$ zero. To use this equation one must find a way of deriving $\Lambda_{\vec{k}\vec{j}\vec{l}}$. The author first attempted to do this by deriving Liouville's equation and from it a transport equation which yielded an approximation to

$$\mathscr{L}^{\alpha\beta}_{\vec{k}}(t)=\langle U^{\alpha}_{\vec{k}}(t)\,U^{\beta}_{-\vec{k}}(0)\rangle \text{ i.e. } \langle U_k U_{-k}\rangle \tag{2.29}$$

(EDWARDS [1]) referred to as [I] in future. Though this will work in some problems it appears impossible from Liouville's equation to get an adequate formalism. The trouble is that Liouville's equation separates time from all the other variables. Now if the time scales of the problem really are completely separate, as for example the time taken over a collision and the time between collisions are, in the usual Boltzmann equation (2.19), then Liouville's equation is a good starting point. But in turbulence there is a continuum of time scales and since "$\vec{k}$" interacts with "$\vec{j}$" and "$\vec{l}$", if one associates a time scale with $\vec{k}$ of say $\omega_{\vec{k}}$, then the "collision" will always make time scales both greater and smaller than $\omega_{\vec{k}}$. It appears then that consideration of the probability that $U_{\vec{k}}(t)=u_{\vec{k}}$ at t, for all $\vec{k}$ i.e. $F(t;\ldots\vec{u}_k\ldots)$ is not a good starting point. One needs rather the probability that $U_{\vec{k}}(t)=u_{\vec{k}}(t)$ and $P(\ldots[u_{\vec{k}}(t)]\ldots)$ i.e. $P(\ldots u_k\ldots)$. Here the comparison is between Hamiltonian mechanics and Lagrangian mechanics, and in the next section I shall develop a new approach to transport theory which I shall call Lagrangian statistical mechanics. An alternative attack which does contain the idea of input and output is due to KRAICHNAN [2]; several applications have been published by KRAICHNAN [3], and a comparison with [1].

3. LAGRANGIAN STATISTICAL MECHANICS

Consider a definite fluid whose velocity field is $\vec{U}(\vec{r},t)$. The probability for its velocity field being $\vec{u}(\vec{r},t)$ is

$$P([\vec{u}]) = \prod_{r,t} \delta(\vec{U}(\vec{r},t) - \vec{u}(\vec{r},t)) \tag{3.1}$$

and

$$\int P\,\delta u = 1, \tag{3.2}$$

where δu means the integral over all functions u. (Those unfamiliar with the vital concept of functional integration can consult GEL'FAND and YAGLOM [4]; or consider space discrete and take the limit in the final formulae.)

If the equation of motion (2.1) is called

$$X^{\alpha}(\vec{r},t) = 0, \tag{3.3}$$

then the equation for P is

$$XP = 0, \tag{3.4}$$

for this implies $P = 0$ unless $X = 0$, and $X = 0$ for $U = u$; and the normalization (3.2) implies (3.1) can be a solution. But (3.4) does not contain the boundary conditions fixing the particular U and so can be applied to an ensemble. Fermi's treatment of the gauge condition (FERMI [5]) has this form. One is now in the position of having an equation which will deal with particular situations or ensembles. Since I wish to discuss ensembles only I should need only one equation, rather than the infinite set (3.4) for all $\vec{r}$, t. The required equation must be

$$\int \frac{\delta}{\delta U^{\alpha}(r,t)} X^{\alpha} P\, d^3 r dt = 0, \tag{3.5}$$

for from this the complete set of symmetrical moments of the Navier-Stokes equations can be recovered and hence all statistical information concerning an ensemble. Equation (3.5) will play the role in Lagrangian statistical mechanics that Liouville's equation does in Hamiltonian mechanics. To see that (3.5) is a practical proposition I will use it to solve a simple but highly suggestive model. Consider one U with viscosity J and external force $\mathscr{F}$

$$\frac{\partial U}{\partial t} = -JU + \mathscr{F}. \tag{3.6}$$

If the functional probability distribution of $\mathscr{F}$ is

$$N \exp\left(-\frac{1}{h}\int \mathscr{F}^2(t)\,dt\right) = P([\mathscr{F}]), \tag{3.7}$$

N being the normalization given

$$\int P(\mathscr{F})\,\delta\mathscr{F} = 1, \tag{3.8}$$

then clearly

$$\langle P([u])\rangle = N \exp\left[-\frac{1}{h}\int (\dot{u} - J u)^2 dt\right] \tag{3.9}$$

so that if

$$U_\omega = \int e^{i\omega t}\, U(t) dt \tag{3.10}$$

$$P([u]) = N \exp[-\frac{1}{n}\int U_\omega U_{-\omega}(\omega^2 + J^2)\, d\omega]. \tag{3.11}$$

(The symbol N will always mean normalization, whatever value the normalization has.)

Hence

$$\langle U_\omega U_{-\omega}\rangle = \frac{2h}{\omega^2 + J^2} \tag{3.12}$$

i.e.

$$\langle U(t)\, U(0)\rangle = \frac{2h}{2\pi}\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega^2 + J^2}\, d\omega \tag{3.13}$$

$$= \frac{h}{J}\, e^{-J|t|} \tag{3.14}$$

and

$$\langle U^2\rangle = \frac{h}{J}\ . \tag{3.15}$$

These results mean that if P, satisfying

$$\int \frac{\delta}{\delta U(t)}\left(\frac{\partial U}{\partial t} - JU + \mathscr{F}\right) dt\, P = 0 \tag{3.16}$$

i.e.

$$\int \frac{\delta}{\delta U_{-\omega}}((i\omega - J)\, U_\omega + \mathscr{F}_\omega)\, P_{d\omega} = 0, \tag{3.17}$$

is averaged over $\mathscr{F}$ using (3.7), then the $(\delta/\delta U_\omega)\mathscr{F}_\omega$ becomes replaced by

$$\frac{\delta}{\delta U_\omega}\left(\frac{\delta}{\delta U_{-\omega}}\, \frac{h}{\omega - iJ}\right)$$

and

$$\int \frac{\delta}{\delta U_\omega}\left(\frac{\delta}{\delta U_{-\omega}}\,\frac{h}{(\omega - iJ)} + U_{-\omega}\,(\omega + i\,J)\right)_{d\omega}\langle P\rangle = 0 \tag{3.18}$$

as is checked from (3.11). To get a systematic approximation procedure which will derive the above results exactly when $\vec{\mathscr{F}}$ has a gaussian distribution, particular note must be made of the way

$$\mathscr{L}(t) = \langle U(t)\ U(0)\rangle \tag{3.19}$$

$$\mathscr{L}_\omega = \langle U_\omega U_{-\omega}\rangle \tag{3.20}$$

appears in the functional equation (3.17).

The function $\mathscr{L}(t)$ will be symmetric in t and decaying, with no oscillatory behaviour. Its singularities are at $\omega = \pm\, iJ$ on the imaginary axis and are mirror images. But the response of U to the random force can only involve the behaviour of $\langle U(t)\,U(0)\rangle$ for $t > 0$, i.e. when in (3.17) one has written in effect

$$\mathscr{L}_{-\omega} = \frac{h}{\omega^2 + J^2} = \frac{[h/(i\,\omega + J)]}{-i\,\omega + J} \tag{3.21}$$

one is separating the singularities in the upper half plane from the lower; only the upper singularities get used in the causal response of U to the fluctuations.

To obtain these results by a systematic procedure one may write

$$\int \frac{\delta}{\delta U_\omega}\,(\mathscr{F}_\omega + i\,\omega\,U_\omega - J\,U_\omega)\,d\omega\,P$$

$$= \int \frac{\delta}{\delta U_{-\omega}}\left(\frac{\delta}{\delta U_{\omega}}\,\mathscr{S}_\omega + i\,\omega\,U_\omega - JU_\omega\right)P$$

$$+ \int \frac{\delta}{\delta U_{-\omega}}\,\mathscr{F}_\omega d\omega\,P - \int \frac{\delta}{\delta U_{-\omega}}\left(\frac{\delta}{\delta U_{\omega}}\,\mathscr{S}_\omega d\omega\right)P. \tag{3.22}$$

If now $\mathscr{S}$ is considered to have the nominal order $\mathscr{F}^2$, one can expand

$$P = P_0 + P_1 + \ldots \tag{3.23}$$

as a series in $\mathscr{F}$, where P_0 is given by

$$P_0 = N\exp\left(-\int U_\omega\,\mathscr{L}_\omega^{-1}U_\omega d\omega\right) \tag{3.24}$$

$$\int P_0 \, \delta u = 1 \tag{3.25}$$

and

$$\left\langle \int U_\omega U_{-\omega'} \cdot P \, \delta u \right\rangle = \int U_\omega U_{-\omega'} \cdot P_0 \, \delta u = \mathscr{L}_\omega \delta(\omega - \omega'). \tag{3.26}$$

The expansion gives

$$\int \frac{\delta}{\delta U_{-\omega}} \left(\frac{\delta}{\delta U_\omega} \mathscr{S}_\omega + (i\omega - J) U \right) d\omega \, P_1 \tag{3.27}$$

$$= \int d\omega \, \mathscr{F}_\omega U_\omega \mathscr{L}_\omega^{-1} P_0.$$

If

$$P_1 = P_0 p_1, \tag{3.28}$$

Then

$$\int \left(\frac{\delta}{\delta U_\omega} \mathscr{S}_\omega - (i\omega - J) U_\omega \right) \frac{\delta p_1}{\delta U_\omega} d\omega = \int \mathscr{F}_\omega U_{-\omega} \mathscr{L}_\omega^{-1} d\omega \tag{3.29}$$

and

$$p_1 = \int \mathscr{F}_\omega U_{-\omega} \mathscr{L}_\omega^{-1} (i\omega - J)^{-1} \, d\omega. \tag{3.30}$$

It will be seen that the operator on the left of (3.27) is Hermite's operator or rather a sum of Hermite's operators for each ω, and if the series $P_0 + P_1 + P_2 + \ldots$ is resolved into Hermite functions conditions (3.25) and (3.26) ensure that no second order functions occur and all zeroth order functions are in P_0. Proceeding to the next order

$$P_2 = P_0 \int [(\mathscr{F}_\omega \mathscr{F}_{-\omega} - \mathscr{S}_\omega (i\omega + J)) (U_\omega U_{-\omega} - \mathscr{L}_\omega) \mathscr{L}_\omega^{-1} / 2J] \, d\omega. \tag{3.31}$$

At this point, one may average, $\langle P_1 \rangle = 0$ and from (3.26)

$$\mathscr{S}(i\omega + J) = \langle \mathscr{F}_\omega \mathscr{F}_{-\omega} \rangle = 2h. \tag{3.32}$$

It is readily confirmed that all $\langle P_i \rangle$ $i > 0$ are then made zero and since $\mathscr{L} = \mathscr{S}(i\omega - J)^{-1}$ one has

$$\mathscr{L} = \frac{2h}{\omega^2 + J^2}. \tag{3.33}$$

It will be seen that

$$\left(\frac{\partial}{\partial t}+J\right)\mathscr{L}(t)=\frac{1}{2\pi}\int\frac{2h}{(i\omega+J)}e^{i\omega t}d\omega \tag{3.34}$$

$$=2h\,\Theta_{-}e^{Jt},\ \begin{pmatrix}\Theta_{-}=0\text{ for }t>0,\\ \Theta_{-}=1\text{ for }t<0\end{pmatrix}. \tag{3.35}$$

At $t=0$ the Θ_{-} must be interpreted (as is usual with discontinuous functions defined by Fourier integrals) as having a value of $\frac{1}{2}$. Thus at $t=0$, since

$$\mathscr{L}(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2h\,d\omega}{\omega^2+J^2} \tag{3.36}$$

$$=h/J \tag{3.37}$$

$$\frac{\partial\mathscr{L}}{\partial t}=-J\cdot\frac{h}{J}+h=0. \tag{3.38}$$

If h were not the result of instantaneously fluctuating input $\mathscr{L}$ would curl over smoothly at $t=0$, but since it is very convenient to avoid extraneous time dependences this definition is suitable bearing in mind that at $t=0$, $\partial\mathscr{L}/\partial t$ has to be interpreted as

$$\frac{1}{2}\left.\frac{\partial\mathscr{L}}{\partial t}\right|_{t=0+}+\frac{1}{2}\left.\frac{\partial\mathscr{L}}{\partial t}\right|_{t=0^-}. \tag{3.39}$$

In the next section the full development will be given.

4. THE GENERAL DEVELOPMENT

From the model one can expect that the theory will revolve around the determination of the correlation $\mathscr{L}$, and $\mathscr{L}$ is to be expected in the form

$$\mathscr{L}_k=\frac{D_k}{\Omega_k}=\frac{A_k/\Omega_{-k}}{\Omega_k} \tag{4.1}$$

where Ω_k^{-1} contains all the singularities in the upper half plane, Ω_{-k}^{-1} has all its singularities in the lower half plane, and A_k is an entire function (for further specification of A_k one runs into familiar trouble (CASTILLEJO et al. [6] which will not be dealt with here). The operator Ω is the response function in the sense that if some extra perturbation is applied to the fluid, the response

$$(\delta u)_k=(\Omega^{-1})_k\delta\mathscr{F}_k$$

and from (2.1) directly

$$\Omega_{\vec{k}}(t-t')\Big|_{t\to t'} \to \delta(t-t')\left[\frac{\partial}{\partial t'} + \nu|\vec{k}|^2\right]. \tag{4.2}$$

The basic equation (3.5) will be expanded around the solution $\langle P_0 \rangle$ of

$$\int \frac{\delta}{\delta u_k}\left(\frac{\delta}{\delta u_{-k}} D_k + \Omega_k u_k\right) d^4k \langle P_0 \rangle = 0 \tag{4.3}$$

by writing it as

$$\int\left\{\frac{\delta}{\delta u_k}\left(\frac{\delta}{\delta u_{-k}} D_k + \Omega_k u_k\right) + \frac{\delta}{\delta u_k}\left[\int (\Sigma Muu)_k + \mathscr{F}_k \mathscr{D}_{\vec{k}}\right]\right.$$

$$\left. - \frac{\delta}{\delta u_k}\left[\frac{\delta}{\delta u_k} D_k + \Omega_k u_k + (ik_0 + \nu|\vec{k}|^2 u_k)\right]\right\} d^4k\, P = 0. \tag{4.4}$$

Thus

$$\int \frac{\delta}{\delta u_k}\left(\frac{\delta}{\delta u_{-k}} D_k + \Omega_k u_k\right) d^4k\, P_1 = \int (Muu)_{+k} u_{-k} \mathscr{L}_k^{-1} + \mathscr{F}_k \mathscr{D}_k u_{-k} \mathscr{L}_k^{-1}) P_0 \tag{4.5}$$

having the solution

$$P_1 = P_0\left[\int \frac{M_{kjl}^{\alpha\beta\gamma}\, u_k^{\alpha} u_j^{\beta} u_l^{\gamma} / \mathscr{L}_k}{\Omega_k + \Omega_j + \Omega_l} + \int \frac{\mathscr{F}_k \mathscr{D}_{\vec{k}} u_{-k} \mathscr{L}_k^{-1}}{\Omega_{-k}}\right] \tag{4.6}$$

(which shows that on an average (4.2) holds and confirms the status of Ω).

The next order gives many terms (catalogued in [1] as is the complete expansion) but collecting those containing the second Hermite polynomials, and at this point averaging so that with $\langle \mathscr{F}_k^{\alpha} \mathscr{F}_k^{\beta} \rangle = 2h_k \delta^{\alpha\beta}$ one has

$$o = D_k - \Omega_k \mathscr{L}_k = \int \frac{L_{kjl}}{\Omega_k + \Omega_j + \Omega_l} \mathscr{L}_l(\mathscr{L}_j - \mathscr{L}_k)\, d^4j\, d^4l \tag{4.7}$$

$$+ \frac{2h_k}{\Omega_{-k}} + (ik_0 - \nu|\vec{k}|^2)\mathscr{L}_k,$$

i.e.

$$(ik_0 + \nu|\vec{k}|^2)\mathscr{L}_k + \int \frac{L_{kjl}}{\Omega_k + \Omega_j + \Omega_l} \mathscr{L}_l(\mathscr{L}_k - \mathscr{L}_j)\, d^4j\, d^4l = -\frac{2h_k}{\Omega_{-k}}. \tag{4.8}$$

It is very tempting at this point to identify

$$D_k = \frac{h_k}{\Omega_{-k}} + \int \frac{L_{kjl}}{\Omega_k + \Omega_j + \Omega_l} \mathscr{L}_l \mathscr{L}_j\, d^4j\, d^4l \tag{4.9}$$

and

$$\Omega_k = ik_0 + \nu|\vec{k}|^2 + \mathscr{R}_k \tag{4.10}$$

where

$$\mathscr{R}_k = \int \frac{L_{kjl}\,\mathscr{L}_l}{\Omega_k + \Omega_j + \Omega_l}\, d^4j\, d^4l. \tag{4.11}$$

This is equivalent to taking Ω^{-1} to have a single pole since if one tries $\mathscr{R}_k = R_{\vec{k}}$ then

$$\begin{aligned}\Omega_k + \Omega_j + \Omega_l &= i(k_0 + j_0 + l_0) + \nu|\vec{k}|^2 + \nu|\vec{j}|^2 + \nu|\vec{l}|^2 \\ &\quad + R_{\vec{k}} + R_{\vec{j}} + R_{\vec{l}} \\ &= 0 + \text{function of } \vec{k},\ \vec{j},\ \vec{l} \text{ alone,}\end{aligned} \tag{4.12}$$

and from (4.11) this is self-consistent. As proved in [1] this will hold also to all higher orders of accuracy. This implies that

$$\mathscr{L}_{\vec{k}}(t) = q_{\vec{k}} \exp[-(\nu|k|^2 + R_{\vec{k}})t] \tag{4.13}$$

and though this may often have considerable validity (see TAYLOR [7]) it is easily shown to be inadequate for very short and very long times, as will be discussed below. The identification, however, clearly contradicts (4.8) unless one may approximate $\int L(\Sigma\Omega)^{-1}\mathscr{L}_l\mathscr{L}_j d^4l\, d^4j$ to have a character similar to h_k/Ω_{-k} and to be approximated by a Θ_-. I shall consider two extreme cases of h_k to illustrate (4.8).

The first is that of white noise $h_k = h$, which is very well-behaved, but not a good description of physical situations in which input is normally concentrated at small $\vec{k}$. This suggests the second case in which the input is idealized to $\mathscr{H}\,\delta(\vec{k})$, and will be much more tricky to solve.

I have found (4.8) a very difficult equation to treat without further approximation. The trouble lies in the kernel $(\Sigma\Omega)^{-1}$ and to simplify this the following proposal is made. The function will have form as in Fig. 3.

Suppose that this is approximated by $q_{\vec{k}} \exp|-\omega_{\vec{k}}/t|$) so that

$$\int_0^\infty \mathscr{L}_k(t)dt = \frac{q_{\vec{k}}}{\omega_{\vec{k}}} = \frac{\mathscr{L}_{\vec{k}}(0)}{\omega_k}. \tag{4.14}$$

Then for the kernel $(\Sigma\Omega)^{-1}$ let us approximate by

$$\begin{aligned}\Omega_k &\cong ik_0 + \omega_{\vec{k}} \\ \Omega_k + \Omega_j + \Omega_l &= \omega_{\vec{k}} + \omega_{\vec{j}} + \omega_{\vec{l}}.\end{aligned} \tag{4.15}$$

The actual $\mathscr{L}$ turns out in the first case to be well represented by $\exp(-\omega_{\vec{k}} t)$.

Fig. 3

Analogues of the two processes shown in Fig. 1

In the second case it is more like $\exp(-k_{\vec{k}}t^3)$, but since three Ω are involved it is to be hoped that it will still be adequate there. Adopting the approximation one has now the kernel independent of time variables and so

$$\frac{\partial \mathscr{L}_{\vec{k}}(t)}{\partial t} + \nu |\vec{k}|^2 \mathscr{L}(t) + \int L_{\vec{k}\vec{j}\vec{l}} (\Sigma\omega)^{-1} (\mathscr{L}_{\vec{k}}(t)\, q_l - \mathscr{L}_j(t) \mathscr{L}_l(t)) d^3j\, d^3l = 2\,H(t) \quad (4.16)$$

where

$$H(t) = \frac{1}{2\pi} \int \frac{e^{-ik_0 t}}{\Omega_{-\vec{k}-k_0}} h_{\vec{k}k_0}\, dk_0 . \quad (4.17)$$

Clearly for $t > 0$, $H(t)$ is zero since Ω_{-k} has its singularities in the lower half plane. Thus $H(t)$ contains $\Theta_-(t)$ and as in the model as $t \to 0$ reaches the value $\frac{1}{2} H(t=0-)$ and this is in fact h_k from (2.16), for putting $t = 0$ (and as $\partial q/\partial t = 0$)

$$\nu |\vec{k}|^2 q_{\vec{k}} + \int \frac{L_{\vec{k}\vec{j}\vec{l}}}{(\omega_{\vec{k}} + \omega_{\vec{j}} + \omega_{\vec{l}})} q_{\vec{l}} (q_{\vec{k}} - q_{\vec{j}}) d^3j\, d^3l = h_{\vec{k}} . \quad (4.18)$$

This, of course, has assumed a steady state; for slowly changing external circumstances one will get

$$\frac{1}{2} \frac{\partial q_{\vec{k}}}{\partial t} + \nu |\vec{k}|^2 q_{\vec{k}} + \int \frac{L_{\vec{k}\vec{j}\vec{l}}}{(\omega_{\vec{k}} + \omega_{\vec{j}} + \omega_{\vec{l}})} q_{\vec{l}} (q_{\vec{k}} - q_j) d^3l\, d^3j = h_{\vec{k}}, \quad (4.19)$$

thus confirming that the approximations so far have maintained the conservation of energy. Equation (4.18) acts as a boundary condition upon (4.8), which has now become (4.16).

If one writes

$$\mathscr{L}_{\vec{k}}(t) = q_{\vec{k}} e^{-\phi_{\vec{k}}(t)} \quad (4.20)$$

$$-\frac{\partial \phi_{\vec{k}}}{\partial t} + \nu |\vec{k}|^2 + q_{\vec{k}}^{-1} \int L_{\vec{k}\vec{j}\vec{l}} (\Sigma\omega)^{-1} e^{-\phi_{\vec{l}}} q_{\vec{l}} (q_{\vec{j}} e^{-\phi_{\vec{k}}} - q_{\vec{k}}) d^3j\, d^3l = 2He^{\phi_{\vec{k}}}\, q_{\vec{k}} . \quad (4.21)$$

If (4.18) is multiplied by $e^{-\phi_{\vec{k}}}$ and subtracted from (4.20) one is left, for $t > 0$

$$\frac{\partial \phi}{\partial t} + \int L_{kjl}\, q_{\vec{j}} q_{\vec{l}} q_{\vec{k}}^{-1} (\Sigma\omega)^{-1} (e^{\phi_{\vec{k}} - \phi_{\vec{k}} - \phi_{\vec{l}}} - 1)\, d^3j d^3l = \frac{h_{\vec{k}}}{q_{\vec{k}}} . \quad (4.22)$$

Thus (4.9), (4.12) and

$$\omega_k = \left[\int_0^\infty e^{-\phi_{\vec{k}}(\tau)}\, d\tau \right]^{-1} \tag{4.23}$$

constitute the set of equations to be solved in the next section. It will be observed that $q_k(t) > 0$ from (4.19), for if q_k passed through zero anywhere, at that time $\partial q_k / \partial t > 0$ and it would always increase. If one attempts the apparently simpler approximation of eliminating $\langle uuu \rangle$ to get equations for $\langle uu \rangle$ in terms of $\langle uuuu \rangle$ and then puts $\langle uuuu \rangle \cong \langle uu \rangle \langle uu \rangle$, the positive definite nature of q_k is not preserved (cf. KRAICHNAN [8]). This kind of approximation has been made in quantum field theory and results likewise in negative probabilities and the occurrence of ghost poles in the complex plane. One can also show that $\mathscr{L}_k = \langle |u_k|^2 \rangle$ is also positive.

5. THE SOLUTIONS

(a) White noise $h_{\vec{k}} = h$

The equation (4.19) is perfectly straight forward when ν is large, but the interesting case is $\nu \to 0$, i.e. large Reynolds number. As $\nu \to 0$ the sink of energy recedes to very large $\vec{k}$ and it is possible to start with $\nu = 0$, an interesting state corresponding to infinite coupling constant in quantum field theory. (The usual methods of quantum field theory have applied to turbulence by Wyld (1962), but are different from the present expansion which is more tailored to transport phenomena.) It will be argued that the term

$$\int (\Sigma\omega)^{-1} \mathscr{L}_{\vec{j}}(t) \mathscr{L}_{\vec{l}}(t)\, L_{\vec{k}\vec{j}\vec{l}}\, d^3j\, d^3l \tag{5.1}$$

will to a large extent decrease quite quickly in time, so much so that it will soon resemble an inhomogeneity, and over a large t range, from (4.16), after some time

$$\frac{\partial \mathscr{L}_k}{\partial t} + \nu\, |\vec{k}|^2 \mathscr{L}_k + \mathscr{L}_{\vec{k}}(t) \int q_{\vec{j}} L_{kjl} (\Sigma\omega)^{-1}\, d^3j\, d^3l = 0\,, \tag{5.2}$$

i.e.

$$\mathscr{L}_{\vec{k}}(t) = q_{\vec{k}}\, e^{-\omega_{\vec{k}} t} \tag{5.3}$$

where now

$$\omega_{\vec{k}} = \nu |\vec{k}|^2 + R_{\vec{k}} \tag{5.4}$$

$$R_{\vec{k}} = \int q_{\vec{j}} L_{\vec{k}\vec{j}\vec{l}} (\Sigma\omega)^{-1}\, d^3j\, d^3l. \tag{5.5}$$

In the limit $\nu = 0$ one has

$$R_{\vec{k}} = \int \frac{q_{\vec{j}} L_{\vec{k}\vec{j}\vec{l}}}{R_{\vec{k}} + R_{\vec{j}} + R_{\vec{l}}} d^3 j d^3 l \tag{5.6}$$

and if, as will appear soon,

$$q_k = q |\vec{k}|^{-5/3}, \tag{5.7}$$

clearly

$$R_{\vec{k}} = r \, |\vec{k}|^{5/3} \tag{5.8}$$

where

$$r^2 = q \int \frac{L_{\vec{k}\vec{j}\vec{l}} |\vec{j}|^{-5/3} |\vec{k}|^{-5/3} d^3 j d^3 l}{(|\vec{k}|^{5/3} + |\vec{j}|^{5/3} + |\vec{l}|^{5/3})} . \tag{5.9}$$

From (4.19) one has

$$\int \frac{L_{\vec{k}\vec{j}\vec{l}} (q_{\vec{k}} - q_{\vec{j}}) q_l \, d^3 j d^3 l}{(R_{\vec{k}} + R_{\vec{j}} + R_{\vec{l}})} = h, \tag{5.10}$$

hence putting $q_{\vec{k}} = q |\vec{k}|^{-5/3}$ one gets

$$\frac{q^2}{r} \int \frac{L_{\vec{k}\vec{j}\vec{l}} (|\vec{k}|^{-5/3} - |\vec{j}|^{-5/3}) |\vec{l}|^{-5/3}}{(|\vec{k}|^{5/3} + |\vec{j}|^{5/3} + |\vec{l}|^{5/3})} d^3 j d^3 l = h. \tag{5.11}$$

A more polished solution is obtained by observing that near $t = 0$ one has from (4.16) or (4.22) that

$$\frac{\partial \mathscr{L}}{\partial t} = h, \tag{5.12}$$

which develops into

$$(h + S_{\vec{k}}) = \frac{\partial \mathscr{L}}{\partial t} + \omega_{\vec{k}} \mathscr{L}_{\vec{k}} \tag{5.13}$$

where

$$S_{\vec{k}} = \int \frac{L_{\vec{k}\vec{j}\vec{l}} q_j q_l d^3 j d^3 l}{(R_{\vec{k}} + R_{\vec{j}} + R_{\vec{l}})}, \tag{5.14}$$

i.e.

$$q_k = \frac{h + S_{\vec{k}}}{(\nu |\vec{k}|^2) + R_{\vec{k}}} \tag{5.15}$$

$$\frac{\partial \mathscr{L}_{\vec{k}}}{\partial t} = \omega_{\vec{k}} \mathscr{L}_{\vec{k}}. \tag{5.16}$$

where $\omega_{\vec{k}} q_{\vec{k}} = h + S_{\vec{k}}$.

For very long times there will always be a region of j space such that $\omega_{\vec{j}}+\omega_{\vec{l}}<\omega_{\vec{k}}$ for any $\omega_{\vec{j}}=\vec{j}^n$ $n>1$. It follows that at very long times one must look at the full form (4.22). It is easy to see that as $\nu\to 0$, $\phi_{\vec{k}}(t)=\phi(|\vec{k}|^{5/3}t)$, so that putting $|\vec{k}|^{5/3}t\to|\vec{k}|^{5/3}$ one obtains in dashed variables

$$-\frac{\partial\phi}{\partial|\vec{k}|}=\frac{3}{5}\frac{q}{r}\int\frac{L_{\vec{k}\vec{j}\vec{l}}|\vec{j}|^{-5/3}|\vec{l}|^{-5/3}|\vec{k}|^{2/3}}{(|\vec{k}|^{5/3}+|\vec{j}|^{5/3}+|\vec{l}|^{5/3})}(e^{\phi_{\vec{k}}-\phi_{\vec{j}}-\phi_{\vec{l}}}-1)\,d^3j\,d^3l. \qquad (5.17)$$

Now $t\to\infty$ is $|\vec{k}|\to\infty$ and the vital thing is that $e^{\phi_{\vec{k}}-\phi_{\vec{j}}-\phi_{\vec{l}}}\leq 1$, which can be shown to imply that

$$\phi_{\vec{k}}\sim|\vec{k}|, \qquad (5.18)$$

i.e.

$$\mathscr{L}_k(t)\xrightarrow[t\to\infty]{}q_k e^{-c|\vec{k}|t^{3/5}}. \qquad (5.19)$$

The picture is as in Fig. 4. At very large $|\vec{k}|$, $q_{\vec{k}}$ falls off much faster as

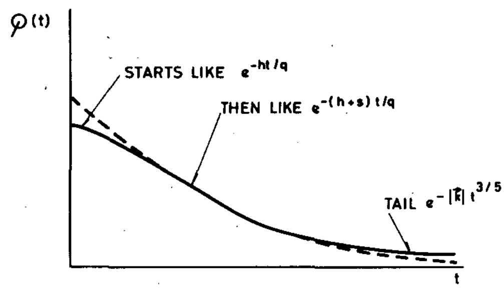

Fig. 4

$Q_k(t)$ as a function of t

$\nu|\vec{k}|^2$ comes into play, but the cascade up to large $|\vec{k}|$ causes no mathematical trouble. Note that if an input with a non-zero fluctuation time had been used $\mathscr{L}(t)$ would have zero derivative at $t\sim 0$ and be something like a gaussian.

(b) Red noise, $h_{\vec{k}}\sim\mathscr{H}\delta(\vec{k})$

It is impossible really to obtain an input $\delta(\vec{k})$ but clean mathematics are obtained only by going to the limit. If the range can be broken up into one in which energy is put in, then a big range with no external input or output, then an output viscous range, it is usual to argue after Kolmogoroff that the intermediate range will depend only on the total rate of input, not on its fine structure and not on the viscosity. The example above shows that one may let $\nu\to 0$ without trouble, i.e. push the sink to infinite $|k|$. To get the Kolmogoroff argument applying <u>everywhere</u> one must squash the source down to the origin. It will be argued then that if source and sink are represented by $\mathscr{H}\delta(\vec{k})-\mathscr{H}\delta(\vec{k}-\vec{\infty})$, $q_{\vec{k}}$ should be given by the $\mathscr{H}^{2/3}|\vec{k}|^{-11/3}$ of Kolmogoroff. (This is usually written as $E(k)=d\epsilon^{2/3}|\vec{k}|^{-5/3}$ where

$$\epsilon = \frac{1}{(2\pi)^3}\mathscr{H},\ 4\pi k^2 E(k) = (2\pi)^{-3} q_{\vec{k}}.)$$

It will also be argued that

$$\phi_{\vec{k}}(t) = \phi(\mathscr{H}^{1/3}\,|\vec{k}|^{2/3}\,t), \tag{5.20}$$

and that $R_{\vec{k}}$ (defined now by (4.23) not by (5.6)) is $\rho|\vec{k}|^{2/3}$. Suppose this is the case, let us solve (4.19) first, i.e.

$$\int \frac{L_{\vec{k}\vec{j}\vec{l}}\, q_{\vec{l}}(q_{\vec{j}} - q_{\vec{k}})\, d^3j\, d^3l}{H^{1/3}\rho\,(|\vec{k}|^{2/3} + |\vec{j}|^{2/3} + |\vec{l}|^{2/3})} = \mathscr{H}[\delta(\vec{k}) - \delta(\vec{k} - \infty)] \ , \tag{5.21}$$

and the solution will be

$$q_{\vec{k}} = q\,|\vec{k}|^{-11/3} H^{2/3}.$$

If this is to be true

$$\frac{q^2}{\rho}\int \frac{L_{\vec{k}\vec{j}\vec{l}}\,|\vec{l}|^{-11/3}(|\vec{k}|^{-11/3} - |\vec{j}|^{-11/3})\, d^3j\, d^3l}{(|\vec{k}|^{2/3} + |\vec{j}|^{2/3} + |\vec{l}|^{2/3})} = \delta(k) - \delta(k - \infty)\ . \tag{5.22}$$

It may seem surprising that the integral can give δ functions so it is worth presenting a simple model. Consider the integral equation

$$\int_0^\infty \frac{\sqrt{xy}}{(x+y)}\ \frac{1}{|x-y|}\,(f(x) - f(y))\,dy = \lambda\,\delta(x) - \lambda\,\delta(x-\infty), \tag{5.23}$$

($\sqrt{xy}$ is the analogue of L, $(x+y)$ of ΣR, $|x-y|$ of $q_{\vec{l}}$). It will be shown that $f(x) = \mu x^{-1}$ is a solution. For consider

$$\int_0^A dx \int_0^\infty dy\, \frac{\sqrt{xy}}{(x+y)} \left(\frac{y-x}{|y-x|}\right) \frac{1}{xy}\,. \tag{5.24}$$

From the symmetry one can subtract

$$\int_0^A dx \int_0^\infty dy\, \frac{1}{(x+y)}\ \frac{(y-x)}{|y-x|}\,\frac{1}{\sqrt{xy}}\ , \tag{5.25}$$

which is zero, to give (5.24) equal to

$$\int_0^A dx \int_A^\infty dy\, \frac{1}{\sqrt{xy}}\ \frac{1}{(x+y)}\,. \tag{5.26}$$

Writing $x' = Ax$, $y' = Ay$, this integral equals

$$\int_0^1 dx' \int_1^\infty dy' \frac{1}{\sqrt{x'y'}\ (x'+y')} \tag{5.27}$$

which is well-behaved equalling C say. This means that

$$\int_0^A dx \int_0^\infty dy\ \frac{\sqrt{xy}}{(x+y)} \left(\frac{1}{x} - \frac{1}{y}\right) \frac{1}{|x-y|}\, dy = C, \tag{5.28}$$

where C is independent of A. It follows that

$$\int_0^\infty dy\ \frac{\sqrt{xy}}{(x+y)} \left(\frac{1}{x} - \frac{1}{y}\right) \frac{1}{|x-y|} = 0, \tag{5.29}$$

unless $x = 0$, which can be readily checked by writing $y \to x^2/y$ in the f(y) part of the integral, thereby transforming it into the f(x) part. Thus the integral over y satisfies the condition for weak convergence, (TEMPLE [9]) is zero except at $x = 0$ but its integral from $x = 0$ is C. Hence

$$\mu C = \lambda \int_0^A \delta(x) dx = \frac{\lambda}{2}\ . \tag{5.30}$$

One can use a similar argument integrating down from infinity to get the other δ function.

Returning to (5.22) all the steps go through. One has to be careful to check that all integrals exist; they do, and so integrating $\vec{k}$ inside (say) the unit sphere, $\vec{j}$ outside

$$\frac{q^2}{\rho} \int_0^{|\vec{k}|<|} d^3k \int_{|\vec{j}|>|\times(|\vec{k}|^{2/3}+|\vec{j}|^{2/3}+|\vec{l}|^{2/3})^{-1}} d^3j \left(|\vec{k}|^{-11/3}\, |\vec{l}|^{-11/3} - |\vec{l}|^{-11/3}\, |\vec{j}|^{-11/3}\right) = \frac{q^2}{\rho} C = 1. \tag{5.31}$$

The numerical value of C is 0.19, and will be used in the next section to get q, ρ.

Now one has to justify this $R_{\vec{k}}$ by solving (4.22, 4.23). This is an equation where dimensional arguments do not help. Put

$$|\vec{k}|\, t^{3/2} \to |\vec{k}| \tag{5.32}$$

giving (since h_k/q_k is negligible except at $t = 0$, $|\vec{k}| = 0$)

$$\frac{d\phi}{d|\vec{k}|} = \frac{2}{3}\frac{q}{\rho}\int \frac{L_{\vec{k}\vec{j}\vec{l}}|\vec{j}|^{-11/3}|\vec{l}|^{-11/3}|\vec{k}|^{8/3}(1 - e^{\phi_{\vec{k}}-\phi_{\vec{j}}-\phi_{\vec{l}}})}{(|\vec{k}|^{2/3} + |\vec{j}|^{2/3} + |\vec{l}|^{2/3})} d^3j\, d^3l. \quad (5.33)$$

Finally one may transform $\vec{j} \to \vec{j}|k|$, $\vec{l} \to \vec{l}|k|$ to get

$$\frac{d\phi}{d|\vec{k}|} = |\vec{k}|^{-1/3}\frac{2q}{3\rho}\int \frac{d^3j\, d^3l\, L\, |\vec{j}|^{-11/3}|\vec{l}|^{-11/3}}{(1 + |j|^{2/3} + |\vec{l}|^{2/3})}(1 - e^{\phi_{\vec{k}}-\phi_{j|\vec{k}|}-\phi_{\vec{l}|\vec{k}|}}). \quad (5.34)$$

Now one may consider the limiting cases firstly of small and then large $\vec{k}$. The highly singular kernel has to be cancelled near $j \sim 0$ and near $l \sim 0$ by $\phi_{\vec{k}} = \phi_{|\vec{k}|\vec{j}} + \phi_{|\vec{k}|\vec{l}}$. Consider $\vec{j} \sim 0$, and suppose (as will be confirmed later) that

$$\phi_{|\vec{k}|\vec{l}} - \phi_k \gg \phi_{|\vec{k}|\vec{j}} \text{ as } \vec{j} \sim 0. \quad (5.35)$$

Then since

$$\begin{aligned} |k|\,|l| &= \vec{k}(1 - 2|\vec{j}|\cos\theta + |\vec{j}|^2)^{1/2} \\ &\cong |\vec{k}| - |\vec{k}|\,|\vec{j}|\cos\theta \end{aligned} \quad (5.36)$$

$$\phi_{|\vec{k}|\vec{l}} - \phi_{\vec{k}} \cong |\vec{k}|\,|\vec{j}|\cos\theta\,\frac{d\phi}{d|\vec{k}|}. \quad (5.37)$$

Crudely speaking then the factor $(1 - \exp(\phi_{\vec{k}} - \phi_{\vec{j}|\vec{k}|} - \phi_{\vec{l}|\vec{k}|}))$ is then zero up to

$$|j| \sim (|\vec{k}|\, d\phi/d|\vec{k}|)^{-1}, \quad (5.38)$$

and allowing a 2 for the other singular region

$$\frac{d\phi}{d|\vec{k}|} \sim \frac{q}{\rho}|\vec{k}|^{-1/3}\int_{(|\vec{k}|\,d\phi/d|\vec{k}|)^{-1}} |\vec{j}|^2 d|\vec{j}|\,|\vec{j}|^{-11/3}\psi \quad (5.39)$$

(ψ = angular contribution), so

$$\frac{d\phi}{d|\vec{k}|} = \psi|\vec{k}|^{-1/3}\left(\frac{p}{\rho}\right)\left(|\vec{k}|\frac{d\phi}{d|\vec{k}|}\right)^{2/3}, \quad (5.40)$$

i.e.

$$\left(\frac{d\phi}{d|k|}\right)^{1/3} = \left(\frac{\psi q}{\rho}\right)|\vec{k}|^{1/3} \quad (5.41)$$

$$\phi = \frac{\psi^3 q^3}{2\rho^3}|\vec{k}|^2 = \gamma|\vec{k}|^2 \quad (5.42)$$

$$\phi_{\vec{k}}(t) = \tfrac{1}{2}\mathscr{H}\psi^3 q^3\rho^{-3}|\vec{k}|^2 t^3. \quad (5.43)$$

(Note this will not be valid at t = 0, since the input will cause a singularity there. This must also be so since $\partial^2 \mathscr{L}/\partial t^2|_{t=0}$ must be positive, being $\langle \int |u_k^2|^2 k_0^2 d^4k \rangle$.) A precise numerical evaluation gives the coefficient γ to be 1.6×10^{-6}. Now turn to the case of large $|\vec{k}|$. It is clear then that $\exp(\phi_{\vec{k}} - \phi_{\vec{j}} - \phi_{\vec{l}})$ must be bounded. If $\phi_{\vec{k}} \sim |\vec{k}|^2$ everywhere this could not be the case since then $|\vec{k}|^2 - |\vec{k}+\vec{j}|^2 - |\vec{j}|^2$ is positive for $\vec{j} = -\vec{k}/2$ and this region would give the integral $\sim\exp(|\vec{k}|^2)$ for large $|\vec{k}|$. In fact any power higher than the first is unacceptable as $|\vec{k}| \to \infty$ since it will give a negative term increasing like $\exp(|\vec{k}|^n)$. In this limit the $d\phi/d|\vec{k}|$ term is unimportant, the vital thing being that the 1 is cancelled in the kernel. This must imply that

$$\mathrm{Max}\,(\phi_{\vec{k}} - \phi_{|\vec{k}|\vec{j}} - \phi_{|\vec{k}|\vec{l}}) \to 0 \text{ as } \vec{k} \to \infty.$$

This implies that ϕ must be linear in $|\vec{k}|$ as in the noise case; clearly this is a quite general result.

$$\text{Thus for small } \vec{k} \begin{bmatrix} \phi \propto |\vec{k}|^2 \\ \phi_{\vec{k}}(t) \propto |k|^2 t^3 \end{bmatrix} \text{for large } \vec{k} \begin{bmatrix} \phi \propto |\vec{k}| \\ \phi_{\vec{k}}(t) \propto |\vec{k}|\, t^{3/2} \end{bmatrix}. \tag{5.44}$$

Suppose that ϕ is approximated by $\gamma |k|^2/(1+\beta|\vec{k}|)$, (5.45)
one may then obtain β from the condition that as $|k| \to \infty$

$$\int L\,q\,q\,(\Sigma R)^{-1}(1 - e^{\phi - \phi - \phi}) \to 0. \tag{5.46}$$

Now the integration is still dominated by the $\vec{j} \sim 0$ region, so one should get

$$\int L_{\vec{k}\vec{j}\vec{l}} |\vec{j}|^{-11/3} d^3j\,(\;\; - \exp(-\frac{\gamma}{\beta}|\vec{k}+\vec{j}| + \frac{\gamma}{\beta}|\vec{k}|)) = 0\,, \tag{5.47}$$

a relation which is independent of $|\vec{k}|$ as it must be if the reasoning is sound. This integral has been evaluated giving $\beta = 3.2 \times 10^{-4}$. (5.48)
Finally then from (5.32), (5.43) and (5.48) one obtains $\rho = 1.2 \times 10^{-2}$, $q = 8.6$, ($\gamma = 1.6 \times 10^{-6}$, $\beta = 3.2 \times 10^{-4}$) so Kolmogoroff's constant α

$$E(k) \cong 1 = 0\,\epsilon^{2/3} |\vec{k}|^{-5/3}. \tag{5.49}$$

This is of the order of magnitude of the experimental value of 1.3 (GIBSON and SCHWARZ [10]), and is surprising since crude approximations have been used. This of course is still not a full solution of the (5.34) equation, but to solve (5.34) directly on a machine is prohibitively lengthy. The final form of ϕ as a function of time is shown in Fig. 5. Physically the slow beginning emanates from the start from a state of "equilibrium" characteristic of all steady stochastic processes. The tail comes from the coupling to the many modes and is a more spectacular phenomenon in the power inputs since it decays very slowly. Here even in the tail the decay is fast and

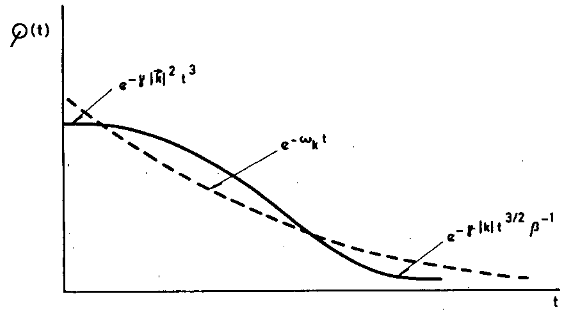

Fig. 5

Final form of Q as a function of time

shows the exponential approximation (dotted line) to be rather crude. One could also argue that rather than fitting $R_{\vec{k}}$ by taking

$$q_{\vec{k}} R_{\vec{k}}^{-1} = \int_0^\infty \mathscr{L}_{\vec{k}}(t)\, dt,$$

one should use

$$\left(\int_0^\infty \mathscr{L}_{\vec{k}}(t)\, \mathscr{L}_{\vec{j}}(t)\, \mathscr{L}_{\vec{l}}(t)\, dt \right)^{-1}$$

in the kernel of the integral. Since q upon γ to the sixth power these alternatives will not matter very much. But it would of course be much better to avoid the exponential form altogether and use Ω self-consistently. (I am working on this at present).

The coefficient $R_{\vec{k}} = \rho|\vec{k}|^{2/3}$ can be checked with experiment this way. Consider a highly turbulent flow with a mean drift which varies very slowly with position. This is basically the same situation as in the molecular chaos present in steady laminar flow, so that if $\langle u \rangle$ is the drift one can argue in the usual phenomenological way that one derives hydrodynamic equations:

$$\frac{\partial\langle\vec{u}\rangle}{\partial t} + R_{\vec{k}}\langle\vec{u}\rangle + \langle\vec{u}\rangle\cdot\vec{\nabla}\langle u\rangle - \vec{\nabla}p = 0$$

with $\qquad \vec{\nabla}\langle\vec{u}\rangle = 0$

and $\qquad \langle\vec{u}\rangle = 0$ on boundaries. (5.50)

Thus for steady channel flow

$$\mathscr{H}\rho\,|\vec{k}|^{2/3} u_{\vec{k}} = (\nabla p)_{\vec{k}}, \qquad (5.51)$$

i.e. if $\qquad u(x) = \sum_m u_m \sin(x\pi m/L)$

(L channel width)

$$\mathscr{H}^{1/3}\ r\ \sum_m{}^{1} u_m\left(\frac{\pi m}{L}\right)^{2/3} \sin(x\pi m/L) = \sum_m (\nabla p)_m\ \sin(x\pi m/L) \tag{5.52}$$

where

$$(\nabla p)_m = (\nabla p)\frac{2}{L}\int_0^L \sin(x\pi m/L)\,dx \tag{5.53}$$

$$= \frac{4}{\pi m}\ (\nabla p)\ (m\ \text{odd}). \tag{5.54}$$

Thus

$$u(x) = 4(\nabla p)\,L^{2/3}\mathscr{H}^{-1/3}p^{-1}\,\pi^{-1}\sum_{m\,\text{odd}} \frac{\sin(\pi x m/L)}{m^{5/3}}\ . \tag{5.55}$$

The transform of m^{-3} is the parabola of Poiseuille, $f\,m^{-1}$ is a step function, $m^{-5/3}$ lies in between, and the profile seems to agree reasonably with experiment (see e.g. Goldstein's Modern Developments in Fluid Mechanics, Volume II).

The rate of input of energy is $\dot{E} = \nabla p\,\vec{U}$, $\vec{U}$ the mean velocity,

i.e.
$$\mathscr{H} = (2\pi)^3\ \nabla p\,\vec{U}, \tag{5.56}$$

but if the sum in (5.56) is called $\psi(x)$

$$\vec{U} = \frac{2}{\pi^2}\,\vec{\psi}\,(\nabla p)\,L^{2/3}\,/(\nabla p\vec{U})^{1/2}\,p \tag{5.57}$$

i.e.
$$\vec{U} = \text{constant}\ \times(\nabla pL)^{1/2}, \tag{5.58}$$

which is D'Arcy's Law. By comparison with experiment $\rho \sim \cdot\ 5\times10^{-3}$ which is something like the value obtained above. The value of α is much improved by using this value of r.

The equations given so far can be extended just as can the field theoretic expansion. Details are given in Reference [1], but since it is so hard to solve those given here, I shall not pursue the n^{th} term.

6. MAGNETOHYDRODYNAMIC TURBULENCE

The equation of incompressible m.h.d. flow in the presence of a constant external field H_0 is

$$\frac{\partial\vec{U}}{\partial t}+\vec{U}\cdot\vec{\nabla}\vec{U} = -\frac{1}{\rho}\,\vec{\nabla}\left(p+\frac{(\vec{H}+\vec{H}_0)^2}{8\pi}\right)+\frac{1}{4\pi\rho}\,(\vec{H}+\vec{H}_0)\cdot\vec{\nabla}\vec{H}+\ \nu\nabla^2\vec{U}+\mathscr{F}_1$$

$$\frac{\partial\vec{H}}{\partial t}+\vec{U}\cdot\vec{\nabla}\vec{H} = (\vec{H}+\vec{H}_0)\cdot\ \vec{\nabla}\vec{U}+\frac{c^2}{4\pi\sigma}\,\nabla^2\vec{H}+\vec{F}_2, \tag{6.1}$$

or in a compressed notation, $h = (H + H_0)\sqrt{4\pi}$, $\rho = 1$,

$$\frac{\partial h}{\partial t} + \Sigma M_2 uh + \eta\, |k|^2 h = \mathscr{F}_2 \mathscr{D}$$

$$\frac{\partial u}{\partial t} + \Sigma M_1 uu - \Sigma M_1 hh + \nu |\vec{k}|^2 u = \mathscr{F}_1 \mathscr{D}. \tag{6.2}$$

(M_1 is the previous M, M_2 analogously.)

If u, h are considered a six vector $\vec{a}$ one may write

$$\frac{\partial \vec{a}}{\partial t} + \vec{\vec{\beta}}\vec{a} = \Sigma\, \vec{\vec{G}}\vec{a}\vec{a} + \vec{\mathscr{F}}\vec{\vec{\mathscr{D}}},$$

where β contains ν, η and H_0, so has off diagonal elements. One can now run through all the analysis again, defining

$$q^{(1)} = \langle UU \rangle, \qquad q^{(2)} = \langle hh \rangle \qquad q^{(3)} = \langle uh \rangle$$

$$h^{(1)} = \langle \mathscr{F}^{(1)} \mathscr{F}^{(2)} \rangle, \quad h^{(2)} = \langle \mathscr{F}^{(2)} \mathscr{F}^{(2)} \rangle \quad h^{(3)} = \langle F^{(1)} \mathscr{F}^{(2)} \rangle.$$

It appears that

$$\langle \mathscr{F}^{(1)} \mathscr{F}^{(2)} \rangle = 0.$$

in actual physical circumstances and this implies $\langle uh \rangle = 0$. Consider $H_0 = 0$ for the moment, then the Boltzmann equations for equilibrium are

$$\nu |\vec{k}|^2 q_{\vec{k}}^{(1)} + \int \frac{L^{(11)}_{\vec{k}\vec{j}\vec{l}} q^{(1)}_{\vec{l}} \left(q^{(1)}_{\vec{k}} - q^{(1)}_{\vec{j}}\right)}{\omega_k^{(1)} + \omega_j^{(1)} + \omega_l^{(1)}} d^3j\, d^3l + \int \frac{L^{(12)}_{\vec{k}\vec{j}\vec{l}} q^{(2)}_{\vec{l}} q^{(1)}_{\vec{k}} - L^{(11)}_{\vec{k}\vec{j}\vec{l}} q^{(2)}_{\vec{j}} q^{(2)}_{\vec{l}}}{\omega_{\vec{k}}^{(1)} + \omega_{\vec{j}}^{(2)} + \omega_{\vec{l}}^{(2)}} d^3j\, d^3l = h^{(1)} \tag{6.3}$$

$$\eta |\vec{k}|^2 q^{(2)} + \int \frac{L^{(22)}_{\vec{k}\vec{j}\vec{l}} q^{(2)}_{\vec{l}} q^{(1)}_{\vec{j}} - L^{(21)}_{\vec{k}\vec{j}\vec{l}} q^{(2)}_{\vec{k}} q^{(2)}_{\vec{l}}}{(\omega_l^{(1)} + \omega_j^{(2)} + \omega_k^{(2)})} d^3j\, d^3l + \int \frac{L^{(22)} q^{(1)}_{l} q^{(2)}_{k}}{(\omega_l^{(1)} + \omega_j^{(2)} + \omega_k^{(2)})} d^3j\, d^3l = h^{(2)}$$

where

$$L^{(11)} = M_1 M_1, \; L^{(12)} = M_1 M_2 \text{ and } L^{(22)} = M_2 M_2. \tag{6.4}$$

It will be noted that the conservation of energy now only holds for both equations, i.e.

$$\int \nu |\vec{k}|^2 q^{(1)}\, d^3k + \int \eta |\vec{k}|^2 q^{(2)}\, d^3k = \int h^{(1)} d^3k + \int h^{(2)}\, d^3k, \tag{6.5}$$

and energy can flow from u to h and back. Clearly if $h^{(2)} \neq 0$ then $q^{(1)}$ and $q^{(2)}$ are non-zero. But if $h^{(2)} = 0$ there is a solution with $q^{(2)}$ zero, but it may not be the only solution. If one lets $\nu \to 0$ and $\eta \to 0$ then the answer to this question would appear definite in the two extreme cases considered above since $h^{(1)}$ can be extracted right out of the calculation, but we have not progressed far enough to answer the question yet. This question is a special case of the problem of the partition of energy among $q^{(1)}$, $q^{(2)}$ for given $h^{(1)}$, $h^{(2)}$ (CHANDRASEKHAR). The discussion in the literature is restricted mostly to the case of large ν and η, but one knows that these can be made zero and cannot influence the limiting case. Calculations are continuing to study this problem.

Turning now to an external field one can see that if $H_0 \to \infty$ the equations have non-linear terms which are purely two-dimensional, so that one has "sheets" of two-dimensional turbulence linked by Alfvén waves. Culham experiments on Zeta* suggest that the energy in these Alfvén waves is rather small so that while one should do a full treatment and calculate just how much energy there is in the field direction, one can start by considering two-dimensional fluid turbulence. Now in two dimensions the non-linear terms not only conserve energy but also conserve vorticity. Thus in two dimensions not only

$$\int \Lambda_{\vec{k}\vec{j}\vec{l}}\, q_{\vec{l}}(q_{\vec{k}} - q_{\vec{j}})\, d^2j\, d^2l\, d^2k = 0, \tag{6.6}$$

but also

$$\int \vec{k}^2 \Lambda_{\vec{k}\vec{j}\vec{l}}\, q_{\vec{l}}(q_{\vec{k}} - q_{\vec{j}})\, d^2j\, d^2l\, d^2k = 0. \tag{6.7}$$

The form of the equation has been set up so that the first is obvious: one only needs Λ symmetric between $\vec{k}, \vec{j}$, and it is. The second is not quite so easy but if one checks carefully it is true in two dimensions. If

$$S_{\vec{j}\vec{l}} = \tfrac{1}{2}\left(|\vec{j}|^2 - |\vec{l}|^2\right)(\vec{l}\times\vec{j})$$

so that

$$\begin{aligned} S_{jl} &= (\vec{k}-\vec{j})(\vec{k}\times\vec{j}) \\ &= -S_{\vec{k}\vec{l}} - S_{\vec{k}\vec{j}} \end{aligned} \tag{6.8}$$

then the two-dimensional kernel is

$$\int \frac{S_{\vec{j}\vec{l}}}{\vec{k}^2\vec{j}^2\vec{l}^2}\, \frac{d^2j\, d^2l}{(\omega_{\vec{k}} + \omega_{\vec{j}} + \omega_{\vec{l}})}\, q_{\vec{j}} q_{\vec{l}} q_{\vec{k}} \left(\frac{S_{\vec{j}\vec{l}}}{q_{\vec{k}}} + \frac{S_{\vec{k}\vec{l}}}{q_{\vec{j}}} + \frac{S_{\vec{k}\vec{l}}}{q_{\vec{l}}}\right) \delta(\vec{k}+\vec{j}+\vec{l}). \tag{6.9}$$

The integral involving $S_{\vec{k}\vec{l}}$ is easily shown to be equal to $-\frac{1}{2}$ the value with $S_{\vec{k}\vec{l}}$ replaced by $S_{\vec{j}\vec{l}}$ so that the kernel can be written

$$\int \frac{S_{jl}^2\, d^2j\, d^2l\, \delta(\vec{k}+\vec{j}+\vec{l})\, q_{\vec{l}}(q_{\vec{k}} - q_{\vec{j}})}{\vec{k}^2\vec{j}^2\vec{l}^2(\omega_{\vec{k}} + \omega_{\vec{j}} + \omega_{\vec{l}})}\, d^2j\, d^2l \tag{6.10}$$

* Experimental work at Culham has appeared only in reports by Rushbridge and Saunders, and Rushbridge and Robinson. Papers discussing experiment and theory are in preparation by these authors.

and satisfies the conservation of energy. But one also sees that it equals

$$\int S_{\vec{j}\vec{l}}\, Z_{\vec{k}\vec{j}\vec{l}}\, \delta(\vec{k}+\vec{j}+\vec{l})\, d^2j\, d^2l, \tag{6.11}$$

where $Z_{\vec{k}\vec{j}\vec{l}}$ is completely symmetric. Multiplying by $\vec{k}^2$

$$S_{jl}\, |\vec{k}|^2 = |\vec{k}|^2 (|\vec{j}|^2 - |\vec{l}|^2)(\vec{l}\times\vec{j}),$$

which is zero when integrated against a symmetric function, so vorticity is also conserved. The solution of the Boltzmann equation is complicated by this invariant. If one tries the Kolmogoroff solution which in two dimensions is $q_{\vec{k}} \propto |\vec{k}|^{-8/3}$ one finds not only $\mathscr{H}\delta(\vec{k}) - \mathscr{H}\delta(\vec{k}-\vec{\infty})$ on the right but also another term. The term $\delta(\vec{k}-\vec{\infty})$ is short for $(1/4\pi K^2)\delta(|\vec{k}|-K)\ (K\to\infty)$, and the new term looks like $(K^2/4\pi k)\,\delta^{11}(|\vec{k}|-K)\mathscr{H}$, so that

$$\int_0^{\text{past } K} \left[\delta(\vec{k}) - \frac{1}{2\pi K}\,\delta(|k|-K) + \frac{K^2}{4\pi k}\,\delta^{11}(|k|-K)\right]|\vec{k}|\, d|\vec{k}|$$

$$= 1 - 1 - 0 = 0 \tag{6.12}$$

(the new term giving zero), and

$$\int k^2 \left[\delta(\vec{k}) - \frac{1}{2\pi K^2}\,\delta(|\vec{k}|-K) + \frac{K^2}{4\pi|\vec{k}|}\,\delta^{11}(|\vec{k}|-K)\right] k\,dk$$

$$= 0 - K^2 + K^2 = 0. \tag{6.13}$$

But the doublet of vorticity at infinity required to sustain a Kolmogoroff spectrum is quite unphysical (though not nonsense: it could happen). This problem also we are actively considering.

I must emphasize that all the discussion in this paper is very highly idealized and may well be of little value in realistic situations. Until fundamental questions are cleared up however it is difficult to assess the value of approximations designed for particular situations.

7. TRANSPORT IN PLASMAS

I shall now show how the ideas developed above can be used as a basis for kinetic theory in a plasma. This work was done in collaboration with G. Lewak and draws heavily on his thesis.

The standard Fokker-Planck equation for a plasma is discussed in detail by BALESCU [13]. It is derived on the assumption that the mean free path is small compared to the Debye length and there is also the assumption that the mean free path is small compared to the Debye length and there is also the assumption that the plasma resonance gives only a small contribution to the transport coefficients. In near equilibrium situations these are very good assumptions but the plasma offers the chance of getting far away from equilibrium when all conventional methods fail. For example, what happens if you put up a strong current through a plasma and keep it steady so that the whole distribution function changes, or what happens if strong fields fluctuating at the plasma frequency are applied? I feel that the traditional time scale argument will fail, and though one can try to develop more general transport equations from Liouville's equation, here the Lagrangian method will be used. Since in hydrodynamic turbulence one uses $\vec{u}(\vec{r}) = \sum_i \vec{v}_i\,\delta(\vec{r}-\vec{r}_i)$ and more generally ρ, T and so on, it is natural to cast the density in configuration and velocity space

$$f(\vec{r},\vec{v},t) = \sum_i \delta(\vec{r}-\vec{r}_i)\,\delta(\vec{v}-\vec{r}_i) \tag{7.1}$$

in a central role. This satisfies

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{r}} + \int \vec{F}(\vec{r}-\vec{s})\,f(\vec{s},\vec{u},t)\,d^3u\,d^3s\,\frac{\partial f}{\partial \vec{v}} = 0 \tag{7.2}$$

exactly (not just in the Vlasov approximation), but (7.2) does not specify f since this from (7.1) vanishes except at N points, N being the number of particles. This condition can be expressed as a set of sum rules which are true irrespective of the motion. They are

$$
\begin{aligned}
&1. \quad \int f\,d^3x\,d^3v = N\\
&2. \quad f(\vec{x},\vec{v},t)f(\vec{x}^1,\vec{v}^1,t) = N\delta(\vec{x}-\vec{x}^1)\,\delta(\vec{v}-\vec{v}^1)f(\vec{x}\vec{v}t)\\
&\qquad + N(N-1)g(\vec{x}\vec{x}'\vec{v}\vec{v}'t) \qquad \text{(g bounded)}\\
&3. \quad f(\vec{x},\vec{v},t)f(\vec{x}^1,\vec{v}^1,t)f(\vec{x}^{11},\vec{v}^{11},t) = N\,\delta(\vec{x}-\vec{x}^1)\delta(\vec{x}-\vec{x}^{11})\delta(\vec{v}-\vec{v}^1)\delta(v-\vec{v}^{11})f_{(xvt)}\\
&\qquad + \sum_{\text{perm}} N(N-1)g(\vec{x}\vec{x}^1\vec{v}\vec{v}^1t)\,f(\vec{x}^{11}\vec{v}^{11}t) + N(N-1)(N-2)h(\vec{x}\vec{x}^1\vec{x}^{11}\vec{v}\vec{v}^1\vec{v}^{11}t)
\end{aligned}
\tag{7.3}
$$

Call these rules $Z_i = 0$.

When one changes from the N particles to f one changes from 6N variables to a continuum, and the constraint that f vanishes except at the points appears in the fact that when the probability distribution $P([f])$ is developed, the solutions at each order must satisfy the sum rules. The great simplification is that to each order P_i only the first i rules need be satisfied. In this way collective co-ordinates need never be redundant. If the mean $\langle f \rangle = f_0$ is removed so that f now means the previous f minus f_0 then

$$\frac{\partial f_0}{\partial t} + \vec{v}\,\frac{\partial f_0}{\partial \vec{r}} + \int \vec{F}\,f_0\,\frac{\partial f_0}{\partial \vec{v}} + \int \vec{F}\,\frac{\partial \mathcal{L}}{\partial \vec{v}} = 0 \tag{7.4}$$

$$\frac{\partial f}{\partial t} + \vec{v}\,\frac{\partial f}{\partial \vec{r}} + \int \vec{F}\,f_0\,\frac{\partial f}{\partial \vec{v}} + \int \vec{F}\,f\,\frac{\partial f_0}{\partial \vec{v}} + \int F\,f\,\frac{\partial f}{\partial \vec{v}} = 0 \tag{7.5}$$

i.e.

$$\left(\frac{\partial}{\partial t}+J\right)f+\int \vec{F} f \frac{\partial f}{\partial \vec{v}} = 0$$

(where $J = \vec{F} f_0 \frac{\partial}{\partial v} + \vec{F}\frac{\partial f_0}{\partial v} + \vec{v}\frac{\partial}{\partial \vec{r}}$ and $\mathscr{L} = \langle f f \rangle = \mathscr{L}(\vec{r}\,\vec{r}^1;\ \vec{v}, \vec{v}^1; t\,t^1)$)

are the set to be solved. Call (7.6) X as before, so that

$$XP = 0 \tag{7.6}$$

$$Z_i P = 0 \tag{7.7}$$

(Equation (7.4) fixes f_0 so the probability need not be discussed.) As before we shall use the statistical equation

$$\int \frac{\delta}{\delta f} X P\, d^3x\, d^3v\, dt = 0 \tag{7.8}$$

and look for a solution ensuring that (7.7) is satisfied or rather it will be enough to satisfy the mean

$$\int Z_i P\, \delta f = 0 \tag{7.9}$$

and of course

$$\int P\, \delta f = 1. \tag{7.10}$$

The expansion will be about

$$P_0 = N \exp\left(-\int f \mathscr{L}^{-1} f\, d(\text{all})\right) \tag{7.11}$$

but now has the added complication that the solutions for P_1, P_2, ... are not unique unless (7.9) is invoked. Using the Fourier space time transform $f_{\vec{k}}(\vec{v}, W)$, one uses

$$\int \frac{\delta}{\delta f}\left(\frac{\delta}{\delta f} D + \Omega f\right) P_0\, d(\text{all}) = 0 \tag{7.12}$$

and needs to solve

$$(\Omega_k + \Omega_j + \Omega_l)\, P_1 = \Sigma F f \frac{\partial f}{\partial v} f / \mathscr{L} P_0 \tag{7.13}$$

in a compressed notation.

In the absence of interaction this still has the solution

$$P_1^{(0)} = N\int f^3(x\,v\,t)\,d(\text{all})P_0, \tag{7.14}$$

since unlike the turbulence problem $\Sigma\Omega$ can have zeros for invariants of the motion, and here this means the total number of particles. But it can be argued that the number of particles will be the only invariant of consequence when f's "collide" so that

$$P_1 = \int \frac{F f \frac{\partial f}{\partial v} f/\mathscr{L}}{\Sigma\Omega} P_0 + \int N f f f P_0 \tag{7.15}$$

where the singularity in $\Sigma\Omega$ is taken in the causal sense).

Proceeding as before one reaches (keeping a highly symbolic notation as the full expressions are very involved)

$$\frac{\partial\mathscr{L}}{\partial t} + J\mathscr{L} + \int \frac{\left(F\frac{\partial}{\partial v} F\frac{\partial}{\partial v}\right)}{\Sigma\Omega} \mathscr{L}(\mathscr{L}-\mathscr{L})\,d\,(\text{all}) = 0, \tag{7.16}$$

which defines the time correlation, and as before can be simplified to

$$\frac{\partial\mathscr{L}}{\partial t} + J\mathscr{L} + \int \frac{\left(F\frac{\partial}{\partial v} F\frac{\partial}{\partial v}\right)}{\Sigma\omega} (\mathscr{L}_k^{(t)}\mathscr{L}_1^{(0)} - \mathscr{L}_j(t)\mathscr{L}_1(t)\,d\,\text{all} \tag{7.17}$$

where Ω is simulated by $ik_0 + J + \omega_{\vec{k}}(v)$. (7.18)

When the two terms in L are put equal one has a slightly different situation to the turbulence case as now J is complex, and it is the imaginary part which comes in, i.e. if

$$\frac{\partial\mathscr{L}}{\partial t}(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t,t^1) + K(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t,t^1) = 0 \tag{7.19}$$

$$\frac{\partial\mathscr{L}}{\partial t'}(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t,t^1) + K^*(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t,t^1) = 0. \tag{7.20}$$

If

$$\mathscr{L}(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t,t) = q(\vec{r},\vec{r}^{\,1};\vec{v},\vec{v}^{\,1};t) \tag{7.21}$$

$$\frac{\partial q}{\partial t} + (K + K^*) = 0. \tag{7.22}$$

This and (7.4) will reduce to the normal Fokker Planck equation if Ω_k is approximated by $ik_0 + J$ alone. This method gives for equilibrium the correct Debye Huckel value for q, and indeed gives the distribution for q, i.e.

for f all at the same time exactly. This is because the Gibbs distribution is exact and is quadratic in the f. The whole expansion $P_1 + P_2 + . -$ in that case turns out independent of F and is simply the Jacobian of the transformation from the co-ordinates to the continuous variables

$$J = Ne^{-\int ff/N} \; (1 + N\,fff + . -.) \qquad (7.23)$$

The basic equations (4.8) and (7.16) have been derived on what is essentially the assumption that the interaction represents a random non-linearity in function space (and is not the random phase approximation which assumes $\vec{f}\vec{f} = 0$ and makes (7.4) the Vlasov equation). It is possible quite easily to write down corrections to them, but this is rather pointless since one hopes they will work in the kind of situation described earlier. Work is at present proceeding on these applications.

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ASYMPTOTIC METHODS IN THE HYDRODYNAMIC THEORY OF STABILITY*

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I. INTRODUCTION

The use of asymptotic methods in the linear hydrodynamic theory of stability is well known, e.g. in connection with the problem of stability of Poiseuille flow (for a more detailed account seen reference [1]). The main point is that it is necessary to reach solutions and find the eigenvalues $\omega = \omega(k)$ for the given boundary conditions of the following equation:

$$\alpha \frac{d^4\varphi}{d\xi^4} - \overline{U}_2(\xi, k, \omega)\frac{d^2\varphi}{d\xi^2} + \overline{U}_1(\xi, k, \omega)\varphi = 0, \tag{1}$$

where α is a small parameter, ξ is a co-ordinate (in the case of Poiseuille flow α is proportional to the viscosity). The presence of the small parameter α makes it possible to construct a formal asymptotic series which will give a solution for a correctly chosen power of α.

Recently, a large number of studies have appeared on the subject of the stability of a slightly non-uniform plasma. In those cases where a detailed analysis was made, the problem reduced to the following equation:

$$\frac{d^2\varphi}{d\xi^2} - \overline{U}(\xi, k, \omega)\varphi = 0. \tag{2}$$

In order that (2) may include an explicitly small parameter β, characterizing a slight non-uniformity, we introduce a non-dimensional co-ordinate $x = \xi/L$ (L being the characteristic dimension of the problem). We assume $\overline{U} = k_0^2 U$, where U is always approximately unity, except near point x_0, where $U(x_0) = 0$. We then have instead of Eq. (2):

$$\beta\frac{d^2\varphi}{dx^2} - U(x, k, \omega)\varphi = 0, \tag{3}$$

$$\beta = \frac{1}{k_0^2 L^2} \ll 1. \tag{4}$$

In [2] it was proposed that a small β should be used in finding the asymptotic solutions, which are well known in quantum mechanics under the name

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* Work performed by ZASLAVSKY, G. M., MOISEEV, S. S. and SAGDEEV, R. Z.

"quasi-classical". (For a detailed survey of work in this direction, see [10] and [11].)

In a number of cases the following situation arises: in the region considered there exists a point at which U becomes ∞. This fact has been studied in connection with the problem of wave transformation in a plasma [3]. In the cases examined in Ref. [3] the pole U was imaginary and vanished when account was taken of the higher derivative with the small α-type parameter*. In problems on wave transformation in a plasma, we used the method of successive approximations [4].

An asymptotic method similar to that used in both Refs. [4] and [1] was employed in Ref. [5] for an equation of the type (1) in a study of the stability of a non-uniform plasma, with account being taken of finite conductivity. It will be clear from what follows that this method is of very limited applicability.

The present study aims to show that there is a simple asymptotic approach to the analysis of the equation:

$$\alpha\beta^2 \varphi^{IV} - \beta U_2(x,k,\omega)\varphi'' + U_1(x,k,\omega)\varphi = 0. \tag{5}$$

Equation (5) models the above - mentioned problems for the conditions of a slightly non-uniform medium.

II. STATEMENT OF THE PROBLEM

The physical considerations discussed in the introduction make necessary an analysis of the following equation:

$$\alpha\beta^2 \frac{d^4\varphi}{dx^4} - \beta U_2(x,k,\omega)\frac{d^2\varphi}{dx^2} + U_1(x,k,\omega)\varphi = 0, \tag{6}$$

where x is a non-dimensional co-ordinate; k and ω are the parameters of the problem; α and β are small parameters:

$$\alpha, \beta \ll 1. \tag{7}$$

Usually, in the physical statement of the problem, the parameter α is involved in calculation of a slight dissipative process, and β is a "quasi-classical" parameter, equal to the ratio between the characteristic length of change of φ and the characteristic length of change of U_1, U_2. In (6) U_1 and U_2 are non-dimensional parameters and

$$U_1,\ U_2 \simeq 1, \tag{8}$$

except for the points where they become zero.

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* These remarks are a rather rough representation of the situation studied in Ref. [3].

Solutions tending to zero at $\pm\infty$ will be referred to below as finite or local, otherwise they will be termed non-local.

For $\beta = 1$ in (6), an analysis of the equation has been made in studies by C.C. Lin [1] and Wasow [6] in connection has been made in studies stability of Poiseuille flow. For $\alpha = 0$ the equation becomes a second-order equation, which has been the subject of detailed study in numerous works, especially in connection with the quasi-classical approximation in quantum mechanics (e.g. see reference [7]).

We are looking for a solution to Eq. (6) in the following form:

$$\varphi = C \exp\left\{\frac{1}{\sqrt{\beta}}\int^{x} q(x)\,dx\right\}, \tag{9}$$

$$q(x) = q^{(0)}(x) + \sqrt{\beta}\, q^{(1)}(x) + \dots . \tag{10}$$

Substituting (9) and (10) in (6), and taking into account (7) and (8), we get:

$$q^{(0)4} - \frac{U_2}{\alpha} q^{(0)2} + \frac{U_1}{\alpha} = 0, \tag{11}$$

$$4\, q^{(1)} q^{(0)3} + 6\, q^{(0)2} \frac{dq^{(0)}}{dx} - \frac{U_2}{\alpha}\left(\frac{dq^{(0)}}{dx} + 2\, q^{(1)} q^{(0)}\right) = 0. \tag{12}$$

From equation (11) we find $q^{(0)}$:

$$q^{(0)} = \pm\left[\frac{U_2}{2\alpha} \pm \left(\frac{U_2^2}{4\alpha^2} - \frac{U_1}{\alpha}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}},$$

or, taking into account small α, we get the following two pairs of values:

$$\begin{aligned} q_i^{(0)} &= \pm\left(\frac{U_1}{U_2}\right)^{\frac{1}{2}} \qquad (i = 1, 2),\\ q_i^{(0)} &= \pm\left(\frac{U_2}{\alpha}\right)^{\frac{1}{2}} \qquad (i = 3, 4). \end{aligned} \tag{13}$$

Similarly, from (12) we get:

$$\begin{aligned} q_i^{(1)} &= -\frac{1}{2}\frac{dq_i^{(0)}}{dx}\frac{1}{q_i^{(0)}} \qquad (i = 1, 2),\\ q_i^{(1)} &= -\frac{5}{2}\frac{dq_i^{(0)}}{dx}\frac{1}{q_i^{(0)}} \qquad (i = 3, 4). \end{aligned} \tag{14}$$

Formulae (13) and (14) allow us to write the solution (9) in the following form (to within the following terms in the expansion (10)):

$$\varphi_i = \frac{C}{\sqrt{p_i}} \exp \int^x p_i \, dx \qquad (i = 1, 2), \tag{15}$$

$$\varphi_i = \frac{C}{\sqrt{p_i^5}} \exp \int^x p_i \, dx \qquad (i = 3, 4), \tag{16}$$

where $p_i = q_i^{(0)}/\sqrt{\beta}$.

The solution obtained in (15) is asymptotic and its accuracy is limited to the region of applicability of the expansion (10), which region we refer to below as external. Clearly, the solution (15), (16) is not applicable to regions near the points where U_1 and U_2 become zero, referred to below as internal regions. The solution for the internal regions can be sought separately. Accordingly, the solution of Eq. (6) for the given boundary conditions reduces to the following three procedures: (1) finding solutions in the internal and external regions; (2) showing what each of the solutions for a particular region becomes in some other region (this question arises owing to the presence of Stokes lines when an asymptotic expression is used); and (3) satisfying the boundary conditions (in addition to everything else, this also gives an equation for the eigenvalues of the problem).

It should be noted that the point where $U_2 = 0$ has no special importance, since in the vicinity of this point the role of the term with φ^{IV} in (6) is unimportant and the behaviour of the solution in the vicinity of this point is determined by the theory developed for Eq. (6) with $\alpha = 0$.

In what follows, without limiting the generality of the method developed below, and for the sake of convenience, we shall select the specific form $U_1(x)$ and $U_2(x)$ (Fig. 1). In Fig. 2 regions I, II are external, and region III is internal.

The above considerations conclude the statement of the problem whose solution will be worked out in sections III-V.

III. WEAK CASE

For the selected form of $U_1(x)$ and $U_2(x)$ (Fig. 1) the values U_1, U_2 become zero at the points A, B and 0_1, 0_2*, respectively. We will assume that the distance between B and 0_2 is greater than unity. In the vicinity of the point 0_2 we may represent

$$U_2 = Ux, \; x < 1, \; U \simeq 1, \tag{17}$$

and regard U_1 as of constant value. For purposes of visualization, the regions in which various approximations are applicable are shown in Fig. 3. The expansion (17) holds good in section $(0_2, 1)$; solutions (15), (16) hold good for sections 1 and 3 respectively.

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* The statements made will be symmetrical relative to the substitution $(0_1, A) \rightleftarrows (0_2, B)$; we shall refer only to points $(0_2, B)$.

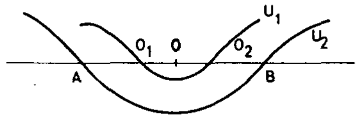

Fig. 1

Specific forms for $U_1(x)$ and $U_2(x)$.

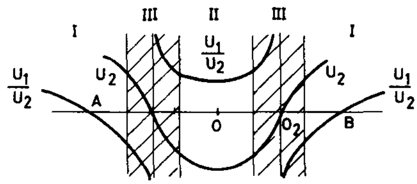

Fig. 2

Case for which regions I and II are external and region III is internal.

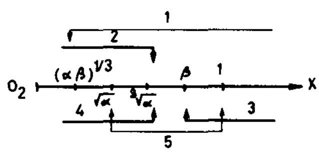

Fig. 3

Possibility of coupling due to the fact that sections 1 and 2 have a common part.

For $x < 1$ equation (6) takes the form:

$$\alpha\beta^2\varphi^{IV} - \beta Ux\varphi'' + U_1\varphi = 0. \tag{18}$$

As in [1] we make the substitution

$$x = \alpha^{1/3}y \tag{19}$$

and consider the solution of the equation obtained:

$$\beta\frac{d^4\varphi}{dy^4} - Uy\frac{d^2\varphi}{dy^2} + \frac{\alpha^{1/3}}{\beta}U_1\varphi = 0, \tag{20}$$

in the vicinity of $y \approx 1$.

In this section of the paper we shall limit ourselves to the case

$$\frac{\alpha^{1/3}}{\beta} < 1, \tag{21}$$

where a solution can be found similar to that of Ref. [1] in the form of an asymptotic series:

$$\varphi = \varphi^{(0)} + \frac{\alpha^{1/3}}{\beta}\varphi^{(1)} + \dots . \tag{22}$$

Substituting (22) in (20) we get:

$$\beta \frac{d^4\varphi^{(0)}}{dy^4} - Uy\frac{d^2\varphi^{(0)}}{dy^2} = 0. \tag{23}$$

The region in which the solutions of Eq. (23) are applicable is determined on the right-hand side by the values $x \simeq \alpha^{1/3}$ and is indicated in Fig. 3 by section 2 (or 4). Equation (23) has the following four solutions [1]:

$$\varphi_1 = 1; \qquad \varphi_2 = x; \tag{23a}$$

$$\begin{aligned} \varphi_3 &= \int^y dy \int^y dy\sqrt{y}\, H^{(1)}_{1/3}\left[\frac{2}{3}(iy)^{3/2}\left(\frac{U}{\beta}\right)^{\frac{1}{2}}\right]; \\ \varphi_4 &= \int^y dy \int^y dy\sqrt{y}\, H^{(2)}_{1/3}\left[\frac{2}{3}(iy)^{3/2}\left(\frac{U}{\beta}\right)^{\frac{1}{2}}\right], \end{aligned} \tag{24}$$

where $H^{(1)}$, $H^{(2)}$ are Hankel functions of the first and second type, respectively. Considering that the argument of the Hankel functions in (24) is large, we can write for the solutions of φ_3 and φ_4, which become zero at $+\infty$,

$$\begin{aligned} \varphi &\simeq x^{-5/4}\exp\left[-\frac{2}{3}\left(\frac{U}{\alpha\beta}\right)^{\frac{1}{2}} x^{3/2}\right] && (x > 0) \\ \varphi &\simeq |x|^{-5/4}\sin\left[\frac{2}{3}\left(\frac{U}{\alpha\beta}\right)^{\frac{1}{2}} |x|^{3/2} + \frac{\pi}{4}\right] && (x < 0). \end{aligned} \tag{25}$$

If φ may not become zero for $+\infty$, then for $x > 0$ the solution also consists of a growing exponential and the solution for $x < 0$ is determined by the normal rules, which take into account that $x = 0$ is a turning point [7]. The solutions of (25) become the solutions determined by Eq. (16) and they can therefore be coupled. This possibility of coupling is due to the fact that sections 1 and 2 in Fig. 3 have a common part within which coupling in fact takes place. The picture is entirely different in the case of the solutions of (23a) and (15), which do not coincide with one another and cannot therefore be directly coupled. This is due to the following circumstance. The pair of solutions in (23a) have, in principle, no quasi-classical form and for them we have

$$k_y^2 \equiv \frac{1}{\varphi}\frac{d^2\varphi}{dy^2} = 0. \tag{26}$$

The equality (26) is in fact determined to an accuracy of:

$$\frac{\alpha^{1/3}}{\beta} \ll 1. \tag{27}$$

The inequality (27) means that the regions, where the solutions of (15) and (23a) are fit (corresponding to sections 3 and 4 in Fig. 3), do not overlap.

To overcome this difficulty let us consider the equation:

$$\beta U x \varphi'' - U_1 \varphi = 0, \tag{28}$$

which is true as a zero approximation in the region

$$\sqrt{\alpha} < x < 1, \tag{29}$$

(section 5 in Fig. 3). The solution of this equation is:

$$\varphi = \sqrt{x}\, Z_1\left[-2\left(\frac{U_1 x}{U\beta}\right)^{\frac{1}{2}}\right], \tag{30}$$

where Z_1 is one of two linearly independent cylindrical functions (for example, I_1 and N_1). For small values of the argument in (30) we have:

$$\varphi_1 = 1;\ \varphi_2 = x, \tag{31}$$

i.e. Eq. (30) becomes Eq. (23a). For large values of the asymptotic argument Z_1 coincides with (15). This completes the first procedure mentioned in section II of this paper. The answer to the second part of the problem, also mentioned in that section, is given by the theorems of Wasow [6] which retain their force in the present case.

Equations for eigenvalues in the case of local solutions can be written down immediately, proceeding as in the quasi-classical approximation for a second-order equation [7] in deriving the "rules of quantization":

$$\int_{0_1}^{0_2}\left(\frac{U_2}{\alpha\beta}\right)^{\frac{1}{2}} dx = (n + \tfrac{1}{2})\pi, \tag{32}$$

$$\int_{0_2}^{A}\left(\frac{U_1}{\beta U_2}\right)^{\frac{1}{2}} dx = (n + \tfrac{1}{2})\pi. \tag{33}$$

The expressions (32) and (33) give two independent solutions for the eigenvalues. This corresponds to the fact that for $+\infty$ (or $-\infty$) we have two linearly independent solutions, determined by (15) or (16), which are then separately "extended" at $-\infty$ (or $+\infty$) (a connection only arises for α/β^2).

IV. CLASSIFICATION

In Eq. (18), which is true for $x < 1$, we make the substitution

$$x = \beta y, \tag{34}$$

which gives

$$\frac{\alpha}{\beta^2}\frac{d^4\varphi}{dy^4} - Uy\frac{d^2\varphi}{dy^2} + U_1\varphi = 0. \tag{35}$$

The solution of this equation is obtained by the method of Laplace:

$$\varphi(y) = \int \frac{1}{t^2}\exp\left(yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right)dt, \tag{36}$$

where the integration is performed in the plane of the complex variable t along a contour at the ends of which the function:

$$\exp\left(yt - \frac{\alpha}{\beta^2}\frac{t^3}{3U} + \frac{1}{t}\frac{U_1}{U}\right)$$

becomes zero. The solution (36), in accordance with (34) and (17), is true in the region $y < 1/\beta \gg 1$. We will limit our discussion to the following region:

$$1 < y < \frac{1}{\beta} \gg 1, \tag{37}$$

or

$$\beta < x < 1. \tag{37a}$$

Since $y > 1$ in the range considered, the "saddle-point" method can be made use of in determining the integral in (36). We have the following four "saddle points":

$$t_0 \equiv \bar{q}_i(y) = \pm\left(\frac{\beta^2}{2\alpha}U\right)^{\frac{1}{2}}\sqrt{y \pm \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}}. \tag{38}$$

This determines four contours, integration along which gives four linearly independent solutions. By appropriately selecting the contours we get the following solutions:

$$\varphi_i(y) \simeq \left[\frac{\pi}{y\left(\frac{U_1}{U}\frac{1}{\bar{q}_i^3} - \frac{\alpha}{\beta^2}\frac{\bar{q}_i}{U}\right)}\right]^{\frac{1}{2}}\frac{1}{\bar{q}_i^2}\exp\int^y \bar{q}_i(y)\,dy \qquad (i = 1, 2, 3, 4). \tag{39}$$

Determining:

$$\begin{aligned} \bar{q}_i &= \pm\left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\sqrt{y - \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}} \qquad (i = 1, 2), \\ \bar{q}_i &= \pm\left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\sqrt{y + \left(y^2 - \frac{\alpha}{\beta^2}\frac{4U_1}{U^2}\right)^{\frac{1}{2}}} \qquad (i = 3, 4), \end{aligned} \tag{40}$$

we get from (39):

$$\varphi_i\ (y) \simeq (\overline{q}_i)^{-1/2} \exp \int^y \overline{q}_i\ (y)\, dy \qquad (i = 1, 2),$$
$$\varphi_i\ (y) \simeq (\overline{q}_i)^{-5/2} \exp \int^y \overline{q}_i\ (y)\, dy \qquad (i = 3, 4). \tag{41}$$

For large values of y, it is not difficult to see that the solutions of (41) become the corresponding solutions in the internal region (15), (16).

Let us consider the value of y, for which the inner root of (40) becomes zero:

$$y_0 \equiv -ia = \pm \left(\frac{\alpha}{\beta^2} \frac{4U_1}{U^2}\right)^{\frac{1}{2}}. \tag{42}$$

(For the type of functions considered $U_1(x)$ and $U_2(x)$, at the points y_0, the value $U_1 < 0$ and y_0 is purely imaginary.) The points y_0 will be referred to below as branching points. Taking (8) and (34) into account, we see that the value at the branching points is $x \simeq \sqrt{\alpha}$, and the distance between the branching points is approximately $\sqrt{\alpha}$. From (42) it immediately follows that:

$$a < 1 \qquad (\alpha/\beta^2 < 1), \tag{43}$$

$$a > 1 \qquad (\alpha/\beta^2 > 1). \tag{44}$$

In the case (43) the branching points do not fall within the region of (37), where solution (41) holds good, and they may be disregarded.

In case (44) the situation is different and, as we shall see from what follows, by taking the branching points into account, we make an essential change in the entire treatment and this may lead to a qualitatively different physical picture of the process. We shall call case (43) the weak case and (44) the strong case.

Since the solution of the problem stated in section III was true for $\alpha/\beta^2 < \beta < 1$, it is correct to consider it as the weak case.

It should be noted that if we introduce the concept of wavelength, $\lambda \simeq \varphi(d\varphi/dx)^{-1}$, then, in the strong case, many wavelengths can be "fitted in" between branching points, which is not so in weak case. Thus, classification is made in accordance with the number of wavelengths lying between the branching points (i.e. in accordance with the ratio between the parameters α and β), although the distance between the branching points is the same in both cases ($\simeq \sqrt{\alpha}$).

V. STRONG CASE

As has already been noted, the need to take account of the branching points a_1, a_2, b_1 and b_2 (Fig. 4) entirely changes the rules for the transition from, say, the region $x < 0_1$ to the region $x > 0_1$.

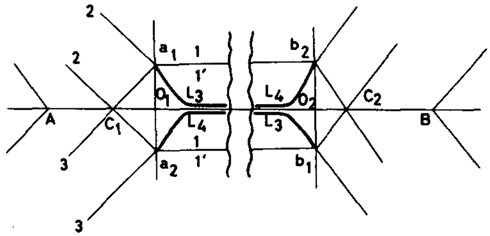

Fig. 4

Pattern for the level lines with respect to each of the branching points separately.

To obtain these rules for (41) we shall use the following form of the roots in (40) (in the vicinity of the point 0_1):

$$\pm\sqrt{y-(y^2+a^2)^{\frac{1}{2}}} = \pm\frac{1}{\sqrt{2}}(\sqrt{y+ia}-\sqrt{y-ia}); \qquad (y>0)$$

$$= \pm\frac{i}{\sqrt{2}}(\sqrt{|y|+ia}-\sqrt{|y|-ia}); \qquad (y<0), \tag{45}$$

$$\pm\sqrt{y+(y^2+a^2)^{\frac{1}{2}}} = \pm\frac{1}{\sqrt{2}}(\sqrt{y+ia}+\sqrt{y-ia}); \qquad (y>0)$$

$$= \pm\frac{i}{\sqrt{2}}(\sqrt{|y|+ia}+\sqrt{|y|-ia}); \qquad (y<0) \tag{46}$$

We then write down the solutions of (41) in the form:

$$\varphi_{1,2} = (\bar{q}_1)^{-1/2}\exp\left[\pm\int^y (w_1(y)-w_2(y))dy\right]; \qquad (y>0)$$

$$= (\bar{q}_1)^{-1/2}\exp\left[\pm i\int^y (w_1(|y|)-w_2(|y|))dy\right]; \qquad (y<0), \tag{47}$$

$$\varphi_{3,4} = (\bar{q}_3)^{-5/2}\exp\left[\pm\int^y (w_1(y)+w_2(y))\,dy\right]; \qquad (y>0)$$

$$= (\bar{q}_3)^{-5/2}\exp\left[\pm\int^y (w_1(|y|)+w_2(|y|))dy\right]; \qquad (y<0). \tag{48}$$

where

$$w_1 = \left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\times\sqrt{y-ia},$$

$$w_2 = \left(\frac{\beta^2 U}{2\alpha}\right)^{\frac{1}{2}}\times\sqrt{y+ia}. \tag{49}$$

In expressions (47) and (48) the exponential is factorized for both branching points, which then makes it possible to employ rules of the type in [8].

On the left-hand side of point A* we write down an arbitrary solution, which becomes zero at $-\infty$,

$$\varphi = |\bar{q}_1|^{-1/2} \exp\left[-i\int_A^y w_1(y)dy + i\int_A^y w_2(y)dy\right] + D\,|q_3|^{-5/2}\left[\exp\int^y w_1(y)dy + \int^y w_2(y)dy\right]. \tag{50}$$

On the right-hand side of point A, the second term of (50) does not change, but the first is transformed in accordance with the rules of (45). This gives:

$$\varphi(y) = |\bar{q}_1|^{-1/2}\left[\exp\left(-\frac{i\pi}{4} + i\int_A^y \bar{q}_1 dy\right) + \exp\left(\frac{i\pi}{4} - i\int_A^y \bar{q}_1\, dy\right)\right]$$
$$+ D\,|\bar{q}_3|^{-5/2}\left[\exp\int^y w_1(y)dy + \int^y w_2(y)\,dy\right]. \tag{51}$$

By taking into account formulae (47) to (49) we can establish a pattern for the level lines with respect to each of the branching points separately (Fig. 4). The level lines from two adjacent branching points intersect on the real axis at points C_1, C_2. The solution of (51) can then be written in the following form, for $A < y < C_1$:

$$\varphi(y) = \frac{1}{|\bar{q}_1|^{1/2}} e^{i\phi_1} \exp\left[-\int_y^{a_1} w_1(y)dy + \int_y^{a_2} w_2(y)\,dy\right]$$
$$+ \frac{1}{|\bar{q}_1|^{1/2}} e^{-i\phi_1} \exp\left[\int_y^{a_1} w_1(y)\,dy - \int_y^{a_2} w_2(y)\,dy\right]$$
$$+ D\,|\bar{q}_3|^{-5/2} \exp\left[\int_y^{a_1} w_1(y)\,dy + \int_y^{a_2} w_2(y)\,dy\right], \tag{52}$$

where

$$i\phi_1 = \left(\frac{\beta^2}{2\alpha}\right)^{\frac{1}{2}}\left[\int_{L_1}\sqrt{U_2(z) - (4\,U_1(z)\alpha/\beta^2)^{\frac{1}{2}}}\,dz - \int_{L_2}\sqrt{U_2(z) + (4U_1(z)\alpha/\beta^2)^{\frac{1}{2}}}dz\right] - i\frac{\pi}{4}. \tag{53}$$

The integrals in (52) are taken from the branching points a_1 (a_2) along a line where ω_1 (ω_2) is purely imaginary, to the point C_1 and then along the real axis. The contours L_1 and L_2 in (53) start at point A, go along the real axis to point C_1 and then along the lines, where ω_1 and ω_2, respectively, are purely imaginary, up to the branching points a_1, a_2. In writing (53)

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* The meaning of all the symbols and letters used in this section and not discussed in the text is clear from Fig. 4.

account is also taken of the fact that in the vicinity of point 0_1, where $U_2 = 0$, the expressions under the integral sign are transformed into

$$\overline{p}_1 = \left[\frac{\beta^2 U}{2\alpha}(z - ia)\right]^{\frac{1}{2}}$$

and

$$\overline{p}_2 = \left[\frac{\beta^2 U}{2\alpha}(z + ia)\right]^{\frac{1}{2}},$$

respectively. Sections are selected along the lines 1. It is not difficult to see that the value of ϕ_1 determined by expression (53) is purely real.

In order to write the solution of (52) (determined on the left-hand side of point 0_1) for the right-hand side of 0_1, use is made of the rules of rotation given in the Appendix. It is convenient to introduce the following designations:

$$p_1 = \sqrt{\frac{\beta^2}{2\alpha}[U_2 - (4U_1\alpha/\beta^2)^{\frac{1}{2}}]},$$
$$p_2 = \sqrt{\frac{\beta^2}{2\alpha}[U_2 + (4U_1\alpha/\beta^2)^{\frac{1}{2}}]}. \quad (54)$$

Then, for $0_1 < y < 0_2$, and to within the constant factor, we have:

$$\varphi(j) = ie^{i\phi_2}\left|p_1 + p_2\right|^{-5/2}\exp\left[i\int_{a_1}^{y} p_1(z)\,dz + i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$-(ie^{i\phi_1} + D)\left|p_1 + p_2\right|^{-5/2}\exp\left[-i\int_{a_1}^{y} p_1(z)\,dz - i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$+e^{i\phi_1}\left|p_1 - p_2\right|^{-1/2}\exp\left[i\int_{a_1}^{y} p_1(z)\,dz - i\int_{a_2}^{y} p_2(z)\,dz\right]$$
$$+(2\cos\phi_1 - iD)\left|p_1 - p_2\right|^{-1/2}\exp\left[-i\int_{a_1}^{y} p_1(z)dz + i\int_{a_2}^{y} p_2(z)dt\right] \quad (55)$$

where integration is performed from a_1, a_2 respectively along the lines L_3, L_4 (Fig. 4), continuing down to the real axis and proceeding along it to the point y. Transferring the solution of (55) to point 0_2, we can re-write it in the form:

$$\varphi(y) = ie^{i\varphi_1} \frac{\Phi}{|p_1 + p_2|^{5/2}} \exp\left[-i\int_y^{b_1} p_1(z)dz - i\int_y^{b_2} p_2(z)dz\right]$$

$$- \frac{ie^{i\varphi_1} + D}{\Phi} |p_1 + p_2|^{-5/2} \exp\left[i\int_y^{b_1} p_1(z)\,dz + \int_y^{b_2} p_2(z)\,dz\right]$$

$$+ \frac{e^{i\varphi_1}\,\Psi}{|p_1 - p_2|^{1/2}} \exp\left[-i\int_y^{b_1} p_1(z)\,dz + i\int_y^{b_2} p_2(z)\,dz\right]$$

$$+ \frac{2\cos\phi_1 - iD}{\Psi} |p_1 - p_2|^{-1/2} \exp\left[i\int_y^{b_1} p_1(z)\,dz - i\int_y^{b_2} p_2(z)dz\right] \tag{56}$$

The integrals from y to b are interpreted in the same way as those from y to a. In addition we have assumed:

$$\Phi \equiv e^{i\varphi_2} = \exp\left[i\int_{L_3} p_1(z)dz + i\int_{L_4} p_2(z)dz\right], \tag{57}$$

$$\Psi = \exp\left[-i\int_{L_3} p_1(z)\,dz + i\int_{L_4} p_2(z)\,dz\right]. \tag{58}$$

The contours L_3 and L_4 are shown in Fig. 4. It can easily be seen that the argument of the exponential in (58) is purely real, and ϕ_2 in (57) is purely imaginary. For this, contour L_3 is curved in such a way that it goes from a_1 to 0_1, thence along the real axis and subsequently from 0_2 to b_1. We follow a similar procedure with L_4. Then in accordance with (54), on the real axis, where L_3 and L_4 coincide, we have $\int_{0_1}^{0_2}(p_1 - p_2)dy$, which is purely imaginary; $\int_{0_1}^{0_2}(p_1 + p_2)dy$, which is purely real. Taking into account the fact that:

$$\int_{a_2}^{0_1} (z + ia)^{1/2} dz = (ia)^{3/2},$$

$$\int_{a_1}^{0_1} (z - ia)\ \ dz = (-ia)^{3/2},$$

we come directly to the above-mentioned statement.

To the right of point 0_2, using again the rules of rotation indicated in the Appendix, we get:

$$\varphi(y) = |p_1 + p_2|^{-5/2}\left(i\Phi e^{i\varphi_1} - i\,\frac{2\cos\phi_1 - iD}{\Psi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz + \int_{b_2}^{y} p_2(z)dz\right]$$

$$+\,|p_1 + p_2|^{-5/2}\left(\frac{2\cos\phi_1 - iD}{\Psi} - \frac{ie^{i\varphi_1} + D}{\Phi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz + \int_{b_2}^{y} p_2(z)dz\right]$$

$$+\,|p_1 - p_2|^{-1/2}\left(-e^{i\varphi_1}\Phi + e^{i\varphi_1}\Psi - \frac{e^{i\varphi_1} - iD}{\Phi} + \frac{2\cos\phi_1 - iD}{\Psi}\right)$$

$$\times \exp\left[-\int_{b_1}^{y} p_1(z)\,dz + \int_{b_2}^{y} p_2(z)\,dz\right]$$

$$+\,|p_1 - p_2|^{-1/2}\left(\frac{2\cos\phi_1 - iD}{\Psi}\right)\exp\left[\int_{b_1}^{y} p_1(z)dz - \int_{b_2}^{y} p_2(z)dz\right]. \qquad (59)$$

The condition of the absence of increasing solutions at $+\infty$ is given by

$$e^{i\varphi_1}\Phi\Psi = 2\cos\phi_1 - iD$$

and

$$(\Psi - \Phi)\,e^{i\varphi_1} - \frac{e^{i\varphi_1} - iD}{\Phi} + \frac{2\cos\phi_1 - iD}{\Psi} = \frac{e^{2i\varphi_3}(2\cos\phi_1 - iD)}{\Psi}, \qquad (60)$$

where

$$i\phi_3 = -\int_{L_1'} p_1(z)dz + \int_{L_2'} p_2(z)dz, \qquad (61)$$

and contours L_1' and L_2' are similar to contours L_1, L_2 and are obtained, respectively, from b_1, b_2 through point C_2 to B. It should be noted that ϕ_3, like ϕ_2, is purely real and positive.

Solving the system (60), we find:

$$e^{i(\varphi_1 + \varphi_2 + \varphi_3)} = \pm 1 \qquad (62)$$

from which

$$\phi_1 + \phi_2 + \phi_3 = n\pi,$$

or

$$i\int_{L_1} p_1\,dz - i\int_{L_2} p_2 dz + \int_{L_3} p_1\,dz + \int_{L_4} p_2 dz - i\int_{L_1'} p_1 dz + i\int_{L_2'} p_2 dz = \left(n + \frac{1}{2}\right)\pi \qquad (63)$$

Equation (63) is a generalized "quantization rule" for the strong case. The left-hand side of (63) represents the total incidence of a phase consisting of three parts:

(1) Phase incidence in the region $A0_1$, where the wavelength

$$\lambda_x \simeq \left[\int (U_1/\beta U_2)^{\frac{1}{2}}\, dx\right]^{-1};$$

(2) Phase incidence in the region 0_1, 0_2, where the wavelength

$$\lambda_x \simeq \left[\int (U_2/\alpha\beta)^{\frac{1}{2}}\, dx\right]^{-1};$$

(3) Phase incidence in the region 0_2B which is of the same type as that in region $A0_1$.

This "strong coupling" of the oscillations is characteristic of the strong case and to that extent condition (63) expresses this fact.

VI. REMARKS

1. In accordance with the classification given in section IV, for $\alpha/\beta^2 < 1$ we have the weak case. The solution given in section III is true for $\alpha/\beta^2 < \beta$. Thus, for the weak case there remains a region $\beta \leqslant d/\beta^2 < 1$ not yet considered. The solution given in section III, as has already been stated, is a generalization of a known solution [1] for the quasi-classical case. However, a solution can be found for $\alpha/\beta^2 < 1$, including $\alpha/\beta^2 < \beta$ as a particular case. For this, we refer to the formulas (39) - (42). The solutions of (41) are unknown for $\alpha/\beta^2 < 1$. The coupling rules are the same for them as in section III, since the branching points of (42) do not lie in the region in which (41) is valid. The "quantization rules" of (32) and (33) remain the same.

The statements which have been made represent a unification of the method developed in sections IV and V. A purely technical difference arises in connection with the fact that the asymptotic solutions found in (41) have a different structure for the Stokes lines, depending on whether or not the branching points of (42) fall in the region in which solution (41) is valid.

2. The asymptotic method presented can be easily generalized for the case in which the behaviour of $U_2(x)$, in the vicinity of U_2 = zero, has the form: $U_2 \simeq U x^m$, it being quite natural that the conditions for coupling the solutions should change, although equations (32), (33) and (62) remain the same. The case of m = 2 for weak coupling was investigated in [5]. For example, the "gravitational mode" found in [5] can be obtained directly from condition (33).

3. It is known [3, 4, 9] that the existence of a non-uniformity in the medium results in one type of oscillation in a certain range giving rise to another type of oscillation (wave "transformation" effect).

A detailed physical picture of this phenomenon is given in [3]. The method developed above can be applied to this phenomenon. The transformation effect is already contained in the solution. Thus, for example, in the strong case (section V), the presence of an oscillating solution $\varphi_{3,4}$ in the region $0_1 0_2$ leads to the appearance of an oscillatory solution $\varphi_{1,2}$ in the region $A0_1$. It can be said that the points leading to transformation are branching points. The coefficient of transformation is obtained as the ratio of the amplitudes φ_1 and φ_3. Of course, in the weak case the transformation effect is small, since the "birth" of a new solution takes place successively, according to the small parameter α/β^2. The essential factor in the strong case is the strong transformation, where the coefficient of transformation may be approximately unity.

VII. SOME CHARACTERISTIC FEATURES OF A PLASMA INSTABILITY IN THE FIELD OF GRAVITY

By way of illustrating the theory set out above, let us consider the question of the characteristic features of the stabilization of a so-called "flute" instability of a plasma, taking into account the finiteness of the ion Larmor radius [12]. The differential equation for the perturbed values in the case with which we are concerned has, as we know from [10], the form:

$$\beta \varphi'' - \left[1 - \frac{G}{r(r-1)}\right] \varphi = 0 \tag{64}$$

where $\beta = (k_y L)^{-2}$ (L is a characteristic dimension), $r = \omega/\omega_i$; $\omega_i = (cT/eH_0) \times k_y n_0'/n_0$; $G = (g/\omega_i^2)\, n_0'/n_0$; [g is the gravity acceleration, n_0 is the density, T is the temperature, H_0 is the intensity of the magnetic field, $n_0' = (g/\omega_i^2)\, n_0'/n_0$]. The instability becomes stabilized if $G \lesssim 1$, and the treatment may be considered correct if finite solutions can be shown to exist. However, as seen from (64), the coefficient of φ breaks down at the point where $r = 1$ and the existence of finite solutions has to be substantiated. We shall therefore proceed on the basis of broader assumptions in deriving the equation for the perturbed values and shall take into account the perturbation of the temperature T, which in a quasi-classical approximation satisfies the equation:

$$\begin{gathered} \frac{3}{2} n_0\left(\frac{\partial T}{\partial t} + (\vec{V}_{0i} \cdot \vec{\nabla})T\right) + n_0 T_0 \operatorname{div} \vec{V}_i = -\operatorname{div} \vec{q}_i \\ \vec{q}_i = \frac{5}{2}\, \frac{c n_0 T_0}{e H_0} (\vec{h} \times \vec{\nabla} T), \quad \left(\vec{h} = \frac{\vec{H}_0}{H}\right). \end{gathered} \tag{65}$$

$\vec{V}_{0i}$ and T_0 are the unperturbed ion velocity and temperature, respectively. For simplicity, we shall consider the electrons to be cold. Selecting the perturbations in the form $\varphi(x) \exp(iyk_y + i\omega t)$ and making standard, simple calculations, differing from the derivation of (64) only in that account is taken of temperature perturbation, we obtain the following equation for φ in a system of co-ordinates in which the ions are at rest:

$$\begin{aligned} \varphi^{IV} &- \left\{2\beta^{-1} + (r-1)\left[3rR^2 - \frac{G}{\beta(r-1)^2}\right]\right\} \varphi'' \\ &+ \left\{\beta^{-2} + \beta^{-1}(r-1)\left[3rR^2 - \frac{G}{\beta(r-1)^2}\right] - 3\beta^{-1}R^2 G\right\} \varphi = 0, \end{aligned} \tag{66}$$

$R = L/r_i$ (r_i being the Larmor radius of the ions).

For simplicity, let us consider the case of a weak connection, corresponding to two separate equations for finding the eigenfrequencies (32) and (33). Here, (33) corresponds to the case of ordinary flute perturbations, the role of the second "turning point" being played by the point where $U_2 = 0$, and finite solutions exist if, outside the interval between the "turning points", the potential U_1/U_2 leads to damping solutions. If

$$rR^2 \gg \frac{G}{\beta(r-1)^2} \qquad (G \lesssim 1),$$

we obtain from (66) a result corresponding to (64), i.e. a stabilization of the instability.

Let us now consider what is the result of the second equation for eigenfrequencies, i.e.

$$\int_{0_1}^{0_2} \sqrt{U_2}\, dx = \left(n + \frac{1}{2}\right)\pi .$$

A qualitatively correct result can already be obtained from the condition $U_2 \approx 0$. Using the form of U_2 from (66) in the case where $r = 1 + r_1$ ($r_1 \ll 1$) we find that

$$r_1 = \frac{2}{3}(k_y r_i)^2 \pm \left[\frac{4}{9}(k_y r_i)^4 + (k_y r_i)^2 G\right]^{\frac{1}{2}}. \tag{67}$$

From (67) we see that taking account of the temperature perturbations leads to an instability if

$$|G| > k_y^2 r_i^2. \tag{68}$$

As we can see, the stabilization of this instability imposes more rigid conditions on the ion Larmor radius than would be required according to analysis of equation (64). The distance between the points of "intersection" of the solutions in this case is:

$$x \simeq k_y r_i L \tag{69}$$

and the treatment used is correct if

$$\frac{L}{r_i}(k_y r_i)^{3/2} \ll 1 \tag{70}$$

APPENDIX

In this Appendix we shall derive the rules for coupling the solutions of φ_i in the case of rotation around the branching points (more specifically, let us take the points a_1 and a_2 in Fig. 4). We start with line 1, proceeding from point a_1. On this line, we write the solution in the form:

$$\varphi(y) = A_1\Pi_1 \exp\left\{i\int^{y}[w_1(z)-w_2(z)]\,dz\right\} + B_1\Pi_2 \exp\left\{i\int^{y}[w_1(z)-w_2(z)]\,dz\right\}$$
$$+ C_1\Pi_3 \exp\left\{i\int^{y}[w_1(z)+w_2(z)]dz\right\} + D_1\Pi_4 \exp\left\{i\int^{y}[w_1(z)+w_2(z)]dz\right\} \quad \text{(A1)}$$

where the quantity Π_i is determined from (39). We get:

$$\Pi_1 = \Pi_2 = \exp\left[-\frac{1}{2}\ln(w_1 - w_2)\right],$$
$$\Pi_3 = \Pi_4 = \exp\left[-\frac{5}{2}\ln(w_1 + w_2)\right], \quad \text{(A2)}$$

for rotation around point a_1, and

$$\Pi_1 = \Pi_2 = \exp\left[-\frac{1}{2}\ln(w_2 - w_1)\right],$$
$$\Pi_3 = \Pi_4 = \exp\left[-\frac{5}{2}\ln(w_2 + w_1)\right], \quad \text{(A3)}$$

for rotation around point a_2.

It will be seen from (A1) and (A4) that rotation can take place around points a_1 and a_2 separately, the same coupling rules obtaining in the case of rotation around each branching point separately as in the case of [8]. It should be noted that the pair of solutions at A_1 and D_1 rotate around point a_1 independently. Similarly, the pair of solutions at A_1 and C_1 and the pair at B_1 and D_1 rotate around point a_2 independently. A_i, B_i, C_i and D_i define a system of coefficients for a solution in the vicinity of the lines with index i, from points a_1 and a_2. The result of simultaneous rotation around a_1 and a_2 then leads to the following:

$$\begin{cases} A_2 = A_1 + iD_1 \\ B_2 = B_1 + iD_1 \\ C_2 = iA_1 + iB_1 + C_1 - D_1 \\ D_2 = D_1 \end{cases} \qquad \begin{cases} A_3 = A_2 + iC_2 \\ B_3 = B_2 + iC_2 \\ C_3 = C_2 \\ D_3 = iA_2 + iB_2 - C_2 + D_2 \end{cases}$$
$$\text{(A4)}$$
$$\begin{cases} A_1' = A_3 + iD_3 \\ B_1' = B_3 + iD_3 \\ C_1' = iA_3 + iB_3 + C_3 - D_3 \\ D_1' = D_3 \end{cases} \qquad \begin{cases} A_1' = -B_1 \\ B_1' = -A_1 \\ C_1' = -D_1 \\ D_1' = -C_1 \end{cases}$$

These are the coupling formulas which are being sought. When we write the last column in (A4) we take account of the fact that the solution of Eq. (6) must be analytical in the complex plane y.

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* A translation or transliteration of each Cyrillic reference is given, set in italics, to aid the reader.

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